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비국소 경계 조건들을 가진 상미분 방정식들의 근의 존재성에 음함수 정리들의 응용 I

Application of Implicit Function Theorem to Existence of Solutions to Ordinary Differential Equations with Nonlocal Boundary Conditions, I

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<Abstract>

We consider the problem

$$y''=a(x,y)(y-b), \quad y(0)=0, \quad y'(1)=g(y(\xi), y'(\xi)),$$

($0<x<1$, ξ fixed in(0,1)) as a model of steady-state heat conduction in a rod when the heat flux at the end x=1 is determined by observation of the temperature and heat flux at some interior point ξ. We establish conditions sufficient for existence, uniqueness.

Key words : Nonlocal Boundary Conditions, General Linear, Ordinary Differential Equation, A Priori Bounds, Boundary Value Problem

Introduction

Let a slender heat-conduction rod of normalized length one occupy the interval $0<x<1$. Suppose that the lateral surface dissipates or absorbs heat according to Newton's law of cooling into a constant ambient temperature b. Let the surface coefficient of heat transfer and the heat transfer surface area per unit length depend on location x and rod temperature $y(x)$; both of these functions are positive. Then the steady-state heat distribution in the rod satisfies the differential equation

$y''(x)=a(x,y(x))(y(x)-b)$, where a is the product of the two functions mentioned above and so positive. We are interested in the possibility of determining the temperature distribution in the rod when the heat flux at one end is determined by observation of the temperature and heat flux at some point(s) in the interior of the rod; for example, by observation of a thermocouple reading at a known interior point. Thus the boundary condition at, say, $x=1$ will have the form $y'(1)=g(y(\xi), y'(\xi))$ for some $\xi \in (0,1)$ and suitable function g. At the other boundary point $x=0$ we suppose for simplicity that the temperature has a specified value; by choosing the zero of the temperature scale to be this value we may assume that the boundary condition at $x=0$ is homogeneous. Thus the problem proposed has the form

$$y''(x)=a(x,y(x))(y(x)-b) \quad (0<x<1),$$
$$y(0)=0, \quad y'(1)=g(y(\xi), y'(\xi)) \qquad (1)$$

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for some fixed $\xi\in(0, 1)$.

A related problem but with Dirichlet-type boundary conditions is studied in [4]. In [8] rather similar problems of our basic form are considered, but the right-hand sides of the differential equations are allowed to grow only sublinearly or at a sufficiently slow linear rate. [9,10] are devoted to three-point boundary value problems for nonlinear equations with the boundary conditions having the special linear form $u(0)= c_1, u(1-)=u(\xi)+c_2$ for constants c_1, c_2. In [11] a general linear boundary condition $u(-1)=\int_0^1 u(x)d\mu(x)+c_1$ is considered for linear equations. [2,3] study nonlinear equations with linear boundary conditions. Three-point boundary conditions of the general form $u(0)= c_1, u(1/2)= c_2, u(1)= c_3$ have been studied for nonlinear [12, 13] or linear [7] second-order differential equations with parameters. The Neumann-type multipoint boundary condition in (1) seems not to have been studied before. A comprehensive introduction to the question of existence of positive solutions in general is provided in [1].

In the following we first establish one rather specialized implicit function theorem which does not require - or guarantee - differentiability, but only continuity; this theorem will be utilized in establishing certain a priori estimates in the following section. We treat existence and uniqueness questions for g decreasing in both arguments.

Implicit Function Theorems

The following lemma is form of the implicit function theorem that does not require differentiability, only continuity. It was used without detailed proof in [4] and will be exploited systematically here in the sequel.

Lemma 1. *Let $\mu\in R^n$, let $x\in R$, and let $g(x,\mu)$ be continuous in all variables and nonincreasing in x for each $\mu\in R^n$. Then the equation*

$$x=g(x,\mu) \tag{4}$$

has a unique solution $x(\mu)$, and $x(\cdot)$ is a continuous function of μ.

Proof : Uniqueness is obvious since the left side of (4) is increasing and the right side is nonincreasing. Consider the function $h(x,\mu)\equiv x-g(x,\mu)$, which is increasing in x, and let $x_2 > x_1$.
Then

$$\begin{aligned} &h(x_2,\mu)-h(x_1,\mu)\\ &= x_2-x_1-[g(x_2,\mu)-g(x_1,\mu)]\\ &\ge x_2-x_1. \end{aligned}$$

Fixing x_1 and letting $x_2\to+\infty$, we see that $h(x,\mu)\to+\infty$ as $x\to+\infty$ for any μ; similarly fixing x_2 and letting $x_2\to-\infty$, we see that $h(x,\mu)\to-\infty$ as $x\to-\infty$. Existence of a solution to $h(x,\mu)=0$ follows from the intermediate value theorem.
There remains only to prove continuity. Let $\varepsilon>0$ be given. Suppose first that δ is such that $x(\mu+\delta)\ge x(\mu)$; then

$$\begin{aligned} &0\le x(\mu+\delta)-x(\mu)\\ &=g(x(\mu+\delta),\mu+\delta)-g(x(\mu),\mu)\\ &=[g(x(\mu+\delta),\mu+\delta)-g(x(\mu),\mu+\delta)]\\ &\quad+[g(x(\mu),\mu+\delta)-g(x(\mu),\mu)]\\ &\le g(x(\mu),\mu+\delta)-g(x(\mu),\mu) \end{aligned}$$

since g is nonincreasing in x. The last expression can be made less than ε by choosing $||\delta||$ small, from the continuity of g in its second argument. Suppose next that $x(\mu+\delta)\le x(\mu)$, so that

$$\begin{aligned} &0\ge x(\mu+\delta)-x(\mu)\\ &=[g(x(\mu+\delta),\mu+\delta)-g(x(\mu),\mu+\delta)]\\ &\quad+[g(x(\mu),\mu+\delta)-g(x(\mu),\mu)]\\ &\ge g(x(\mu),\mu+\delta)-g(x(\mu),\mu)\ge-\varepsilon \end{aligned}$$

if $||\delta||$ is small enough, as before.

Existence and Uniqueness of Solutions

Theorem 1. *Let $\xi\in(0,1)$; let $g\in C(R^2)$ be nonincreasing in each of its arguments; let $a(x, y)$ be continuous and positive on $[0,1]\times(-\infty, \infty)$. Then (1) has a classical solution.*

Proof : We first obtain a priori bounds, uniform in $\lambda \in [0,1]$, on any solution y_λ of the one parameter family of problems

$$y_\lambda''(x) = \lambda a(x, y_\lambda(x))(y_\lambda(x) - b) \quad (0<x<1), \quad y_\lambda(0) = 0$$
$$y_\lambda'(1) = g(y_\lambda(\xi), y_\lambda'(\xi)). \quad (5)$$

We treat the case $\lambda = 0$ separately. Obviously $y_0 = \alpha x$ is then the solution of (5) if and only if $\alpha = g(\alpha\xi, \alpha)$. Since ξ is fixed, from Lemma 1 we see that this equation has a unique solution for α, and thus that (5) has a unique solution for this case.

We assume throughout the remainder of the derivation that $\lambda > 0$. Suppose that y_λ has an interior local maximum or minimum at some $\hat{x}$. Since $y_\lambda''(x) > 0$ for $y_\lambda(x) > b$ and $y_\lambda''(x) < 0$ for $y_\lambda(x) < b$, we have that $|y_\lambda(\hat{x})| \le |b|$, providing the desired a priori bound. There remains to consider the possibilities that y_λ achieves its maximum or its minimum at x=1. We examine first the case that y_λ achieves an absolute maximum at x=1 ; without loss of generality we may assume that $y_\lambda(1) > b$.

Case 1: $b \ge 0$. In this case $y_\lambda(1) > 0$. Were $y_\lambda(\xi) < 0$ to hold, y_λ would have a negative minimum at some $\tau \in (0,1)$, where $y_\lambda(\tau) < b$, implying in turn that $y_\lambda''(\tau) < 0$. This contradiction shows that $y_\lambda(\xi) \ge 0$ must be valid. From the monotonicity of g we get that

$$y_\lambda'(1) = g(y_\lambda(\xi), y_\lambda'(\xi))$$
$$\le g(0, y_\lambda'(\xi)) \quad (6)$$

Since the maximum of y_λ occurs at 1, we must have $y_\lambda'(1) \ge 0$. Were $y_\lambda'(\xi) < 0$ and $y_\lambda(\xi) \ge b$, there would exist $\tau \in (0, \xi)$ such that $y_\lambda(\tau) > b$ and y_λ has a local maximum at τ, contradiction. Were $y_\lambda'(\xi) < 0$ and $y_\lambda(\xi) < b$, there would exist $\tau \in (\xi, 1)$ such that $y_\lambda(\tau) < b$ and y_λ has a local minimum at τ. again a contradiction. Therefore it must be that $y_\lambda'(\xi) \ge 0$, whence (6) yields $y_\lambda'(1) \le g(0,0)$. Since the graph of y_λ is concave up whenever $y_\lambda(x) > b$, it follows that $y_\lambda'(x) \le y_\lambda'(1)$ whenever $y_\lambda(x) > b$ and therefore that

$$y_\lambda(x) \le y_\lambda(1) \le b + y_\lambda'(1) \le b + g(0,0),$$

a suitable a priori upper bound.

Case 2: $b < 0$. There is no loss of generality in assuming $y_\lambda(1) > 0$. If $y_\lambda \ge 0$ on [0,1], then $y_\lambda'' > 0$ since $y_\lambda > b$ there. But then the graph of y_λ is concave up, so $y_\lambda'(\xi) \ge 0$. It follows that $y_\lambda'(x) < y_\lambda'(1) \le g(0,0)$ on [0, 1], supplying the a priori upper bound $y_\lambda(x) \le g(0,0)$.

There remains to consider the subcase in which y_λ has a negative minimum at some $\tau \in (0,1)$. Then $y_\lambda(\tau) \ge b$, and hence $y_\lambda(x) \ge b$, must hold, implying that $y_\lambda(\xi) \ge b$ and thus that $y_\lambda'(1) \le g(b, y_\lambda'(\xi))$. If $y_\lambda'(\xi) \ge 0$ holds, then from this we have the bound $y_\lambda(x) \le y_\lambda(1) \le y_\lambda'(1) \le g(b,0)$. Suppose finally that $y_\lambda'(\xi) < 0$. By the mean value theorem for the derivative, there is a $\zeta \in (0, \xi)$ such that

$$\frac{b}{\xi} \le \frac{y_\lambda(\xi)}{\xi} = y_\lambda'(\zeta) \le y_\lambda'(\xi)$$

since the graph of y_λ is concave up. From this we have that

$$y_\lambda(1) \le y_\lambda'(1) \le g(b, b/\xi).$$

Combining all the arguments above yields the existence of a constant k_1, independent of $\lambda \in [0,1]$, such that $y_\lambda(x) \le k_1$ for all $x \in [0,1]$. Since the change of variables $y_\lambda \to -y_\lambda$, $b \to -b$, $g(s, t) \to -g(-s,-t)$ leaves the form and properties of the problem (5) invariant, there is a constant k_2 such that $y_\lambda(x) \ge -k_2$. Thus there is a constant C_0, independent of $\lambda \in [0,1]$, such that $|y_\lambda(x)| \le C_0$ for all $x \in [0, 1]$ and any solution y_λ of (5). From the differential equation we have that

$|y_\lambda''(x)| \le C_2$ for $C_2 = \max_{x\in[0,1],|y|\le C_0} a(x,y)|y-b|$. By the mean value theorem,

there exists $\tau \in (0,1)$ such that $|y_\lambda'(\tau)| = |y_\lambda(1)| \le C_0$; thus

$$|y_\lambda'(x)| \le \left|\int_\tau^x |y_\lambda''(t)|dt\right| + |y_\lambda'(\tau)| \le C_2 + C_0 \equiv C_1$$

on [0, 1]. This completes the derivation of a priori bounds.

Let C^2 denote the Banach space $\{y \in C^2[0,1] : y(0)=0\}$ with the norm

$$\|y\|_2 = \max\{\max_{[0,1]}|y(x)|/C_0,\ \max_{[0,1]}|y'(x)|/C_1,\ \max_{[0,1]}|y''(x)|/C_2\}$$

and C^1 the space $\{y \in C^1[0,1] : y(0)=0\}$ with the norm

$$\|y\|_1 = \max\{\max_{[0,1]}|y(x)|/C_0,\ \max_{[0,1]}|y'(x)|/C_1\};$$

let C^0 denote the space $C[0,1]$ with the usual norm $\|y\|_0 = \max\{\max_{[0,1]}|y(x)|$. By the Ascoli-Arzela theorem, the injection map $j: C^2 \to C^1$ defined by $jy = y$ is completely continuous. The map $F : C^1 \to C^0$ defined by $F(y)(x) = a(x, y(x))(y(x)-b)$ is continuous.

We want now to define a map L_λ that is essentially the map $u \to u_\lambda$ where

$$u_\lambda'' = \lambda u, \quad u_\lambda(0) = 0$$
$$u_\lambda'(1) = g(u_\lambda(\xi),\ u_\lambda'(\xi));$$

solving this problem, we see that u_λ must satisfy

$$u_\lambda(x) = xg(u_\lambda(\xi),\ u_\lambda'(\xi)) - \lambda\int_0^x su(s)ds - \lambda x\int_x^1 u(s)ds$$

Motivated by this in our search for fixed points, we define

$$(L_\lambda u)(x) = x\,C_\lambda(u) - \lambda\int_0^x su(s)ds - \lambda x\int_x^1 u(s)ds \tag{7}$$

on C^0, where the constant $C_\lambda(u)$ must satisfy the equation

$$C_\lambda(u) = g\Big(C_\lambda(u)\xi - \lambda\int_0^\xi su(s)ds - \lambda\xi\int_\xi^1 u(s)ds,\ C_\lambda(u) - \lambda\int_\xi^1 u(s)ds\Big). \tag{8}$$

We must, of course, show that (8) has a unique solution. But the right-hand side of (8) is a nonincreasing function of $C_\lambda(u)$ and the left-hand side is linear with positive slope, so by Lemma 1 (8) does indeed have a unique solution. Moreover, this solution obviously depends continuously on $\lambda \in [0,1]$ and $u \in C^0$ since the coefficients in (8) do. Thus L_λ is well-defined on C^0. In addition.

$$(L_\lambda u)'(x) = C_\lambda(u) - \lambda\int_x^1 u(s)ds,$$
$$(L_\lambda u)''(x) = \lambda u(x),$$

so $L_\lambda(\cdot)$ is continuous into C^2.

We define a homotopy H by $H_\lambda = L_\lambda \circ F \circ j$; $[0,1]\times C^2 \to C^2$. If u is a fixed point of H_λ for some λ, then

$$u(x) = x\,C_\lambda(a(\cdot, u(\cdot))(u(\cdot)-b)) - \lambda\int_0^x sa(s,u(s))(u(s)-b)ds - \lambda x\int_x^1 a(s,u(s))(u(s)-b)ds \tag{9}$$

where C_λ satisfies (8) with $u(\cdot)$ replaced by $a(\cdot, u(\cdot))(u(\cdot)-b)$. It follows that

$$u'(x) = C_\lambda(a(\cdot, u(\cdot))(u(\cdot)-b)) - \lambda\int_x^1 a(s,u(s))(u(s)-b)ds,$$
$$u''(x) = \lambda a(x,u(x))(u(x)-b).$$

From the first of these equations we get that $u'(1) = C_\lambda(a(\cdot, u(\cdot))(u(\cdot)-b)$ which from (9) and (8) with u replaced by

$a(\cdot, u)(u-b)$ gives

$$u'(1) = g(u(\xi), u'(\xi));$$

i.e., u is a solution of the problem (5). Conversely, solutions of (5) are fixed points of H_λ. Our *a* priori estimates now show that any fixed point u of H_λ $(\lambda \in [0,1])$ satisfies $\| u \|_2 \le 1$.

Pick any $\varepsilon > 0$ and define $U = \{y \in C^2 : \| y \|_2 < 1+\varepsilon\}$, so that U is an open bounded subset of C^2.
Since j is completely continuous, $H_\lambda y \mid \bar{U}$ is compact. For $\lambda = 0$ we have

$$(H_0 y)(x) = C_0(a(\cdot, y(\cdot))(y(\cdot)-b))x$$

whence

$$\begin{aligned} &C_0(a(\cdot, y(\cdot))(y(\cdot)-b)) \\ &= g(\xi\, C_0(a(\cdot, y(\cdot))(y(\cdot)-b)), \\ &\quad C_0(a(\cdot, y(\cdot))(y(\cdot)-b))). \end{aligned}$$

But the equation $\alpha = g(\alpha\xi, \alpha)$ has a unique solution for α by Lemma. 1. So $C_0(a(\cdot, y(\cdot))(y(\cdot)-b))$ is independent of y. The a priori estimates show that the compact homotopy H_λ has no fixed points on ∂U, since any solution of (5) lies in U. Because H_0 is a constant map into U, it is essential [5,6]. By the topological transversality theorem [5,6] so is H_1, which therefore has a fixed point in U. This fixed point provides the solution of (1).

We turn now to establishing uniqueness of the solution.

Theorem 2. *Let $\xi \in (0,1)$; let g be nonincreasing in both arguments; let $a(x,y)$ be continuous and locally Lipschitz in y on $[0,1] \times (-\infty, \infty)$; and let $a(x,y)(y-b)$ be nondecreasing in y for each fixed $x \in [0,1]$. Then any solution of (1) is unique.*

Proof: Let u and v be two distinct solutions of (1). By uniqueness for the initial value problem for $y'' = a(x,y)(y-b)$, it must be that $u'(0) \ne v'(0)$. Suppose then without loss of generality that $v'(0) > u'(0)$, and consider the following all-inclusive three cases:

Case 1: $v(x) > u(x)$ on (0, 1] and $v'(x) > u'(x)$ on [0,1]. In this case

$v'(1) = g(v(\xi), v'(\xi)) \le g(u(\xi), u'(\xi)) = u'(1)$ a contradiction.

Case 2: $v(x) > u(x)$ on (0, 1] and $v'(\eta) = u'(\eta)$ for some $\eta \in (0,1]$. Then

$$\begin{aligned} v'(\eta) &= v'(0) + \int_0^\eta a(s, v(s))(v(s)-b)ds \\ &> u'(0) + \int_0^\eta a(s, u(s))(u(s)-b)ds = u'(\eta), \end{aligned}$$

a contradiction.

Case 3 : There exist $\zeta \in (0, 1]$ such that $v(x) > u(x)$ on $(0, \zeta)$ and $v(\zeta) = u(\zeta)$. But this implies that

$$\begin{aligned} v(\zeta) &= v'(0)\zeta + \int_0^\zeta (\zeta - s)a(s, v(s))(v(s)-b)ds \\ &> u'(0)\zeta + \int_0^\zeta (\zeta - s)a(s, u(s))(u(s)-b)ds \\ &= u(\zeta), \end{aligned}$$

again a contradiction.

Remark. Theorems 1 and 2 extend easily to problems with the more general boundary condition at x=1

$$y'(1) = g(y(\xi_1), y'(\xi_1), \cdots, y(\xi_n), y'(\xi_n)), \quad \xi_1, \cdots \xi_n \in (0,1)$$

provided g is nonincreasing in all its arguments. The maps F and L_λ are unchanged except that in the latter $C_\lambda(u)$ must now satisfy

$$C_\lambda(u) = g\Big(C_\lambda(u)\ \xi_1 - \lambda \int_0^{\xi_1} su(s)ds - \lambda\, \xi_1 \int_{\xi_1}^1 u(s)ds,\ C_\lambda(u) - \lambda \int_{\xi_1}^1 u(s)ds, \cdots \Big). \tag{10}$$

instead of (8). Eq. (10) still has a unique solution by Lemma 1. The remainder of the proofs require little adjustment.

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