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On the Lower and Upper Percentile Values of the Sphericity Decision Rule

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Introduction and Preliminaries

Let $\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_N}$ be a random sample drawn from a p-variate normal distribution with mean μ and covariance matrix $\mathbf{\Sigma}$, where the underlying parameters are unknown. A problem of considerable interest is to test the null hypothesis $H_o : \mathbf{\Sigma} = \sigma^2\mathbf{I}_p$ against the alternative hypothesis $H_1 : \mathbf{\Sigma} \neq \sigma^2\mathbf{I}_p$, where σ^2 is unspecified. This null hypothesis is known as the hypothesis of sphericity.

Let $\mathbf{S}$ be the sample covariance matrix and $\mathbf{A}$ be the matrix of sum of squares and cross product, where $\mathbf{A} = \sum_{\alpha=1}^{N}(\mathbf{X}_\alpha - \bar{\mathbf{X}})(\mathbf{X}_\alpha - \bar{\mathbf{X}})' = (N-1)\mathbf{S} = n\mathbf{S}$, or $\mathbf{S} = \frac{1}{N-1}\mathbf{A}$, and $\mathbf{X}_\alpha,\ \alpha = 1, 2, \ldots, N$, are independently and identically distributed as MVN$(\mu, \mathbf{\Sigma})$.

It is known that the criterion for testing the null hypothesis H_o was first derived by Mauchly (1940) and is given by

$$W = \frac{det\ \mathbf{S}}{(\frac{1}{p}tr\ \mathbf{S})^p}. \tag{1}$$

Alternatively,

$$W = \frac{det\mathbf{A}}{(\frac{1}{p}tr\mathbf{A})^p}. \tag{2}$$

The decision rule given in (1.1) or (1.2) is known as the sphericity decision rule. The distribution of the sphericity decision rule either in the null case or in the non-null case were studied, among others, by Mauchly (1940), Box (1949), Nair (1938), Consul (1967, 1969), and Khatri and Srivastava (1971). For $p = 2$ Mauchly obtained the

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exact pdf of the criterion W. Mauchly also obtained the approximate percentage values for $p = 3$ and for various values of N. Consul (1967) obtained the exact density of the criterion in the null case for small p. For large p Consul (1969) obtained the densities in closed form in terms of the well known Meijer's G-function. Consul (1969) was able to use the expansion of the G-function in terms of hypergeometric function to simplify the result for the cases $p = 2$ and $p = 3$.

In this paper, we study the pdf of the sphericity decision rule W as obtained by Consul which is stated in Theorem (3.1). We also obtained the cdf using equation (3.2). To assess the accuracy of the result we computed the total probability bounded by the cdf using equation (3.2) for various values of N. The percentage values at various levels of significance are also obtained and presented.

The densities obtained by Consul (1967), Mathai and Rathie (1970), and Khatri and Srivastava (1971) are not quite suitable for computations for large values of p. Nagarsenkar and Pillai (1974) derived the density function of W in the null case using methods similar to those of Box (1949) and Nair (1938). Because of these difficulties, some authors studied the asymptotic distribution in the null case. According to Muirhead (1982), the asymptotic distribution of the criterion $-2\rho \log W$ has a chi-square distribution with $(p+1)(p-1)/2$ degrees of freedom. However, the results are of limited use since the sample size N must be large enough.

In this paper, we present the derivation of the sphericity rule W for special values of p using Nagarsenker (1972) technique and we carry out some numerical comparisons based on the findings. The methods used in this paper are based on the central moments of W and then applying Mellin transform, inverse Mellin transform and some complex analysis.

Moments of Sphericity Criterion W

To obtain some information about the exact or asymptotic distribution of W we need the central moments of W. The moments of the criterion are used to obtain exact expressions of the density function by employing Mellin transform approach and the inverse Mellin transform. We now provide the following result as given in Muirhead (1982).

Theorem 1 *If* $W = \frac{det\,\mathbf{S}}{(\frac{1}{p}tr\,\mathbf{S})^p}$, *where* $\mathbf{S}$ *is the sample covariance matrix as based on a random sample of size* N *from a p-variate normal population with parameters* μ, $\mathbf{\Sigma}$, *then the moments of order* h *of the criterion is given by*

$$\mu_h = E(W^h) \quad = \quad p^{ph}\frac{\Gamma_p(\frac{pn}{2})\Gamma_p(\frac{n}{2}+h)}{\Gamma_p(\frac{pn}{2}+ph)\Gamma_p(\frac{n}{2})}$$

$$\cdot \sum_{k=0}^{\infty} \frac{(pn/2)_k}{(pn/2+ph)_k k!} \sum_K (\frac{n}{2}+h)_K \tilde{C}_K(I-\sigma^2\mathbf{\Sigma}), \tag{3}$$

where $\sigma^2 > 0,\ n = N-1$ *and* $(a)_k = \prod_{i=1}^{p}(a-\frac{i-1}{2})_{ki}$, $(x)_k = x(x+1)\dots(x+k-1)$ *which is known as Pochhammer formula. Also* $\Gamma_p(a) = \pi^{p(p-1)} \prod_{i=1}^{p} \Gamma(a-\frac{i-1}{2})$. $\Gamma_p(a)$ *is known as the multivariate Gamma function and* $\Gamma(\cdot)$ *is the classical Gamma function. The second partition in the formula (2.1) is over all the partitions* $K = (k_1, k_2, \dots, k_p),\ k_1 \geq k_2 \geq \dots \geq k_p \geq 0$ *of the integer* k ($\sum_{i=1}^{p} k_i = k$) *and* $\tilde{C}_K$ *is the Zonal Polynomial (James (1964)) corresponding to* k.

A considerable simplification of formula (2.1) occurs when the null hypothesis is true, i.e., $H_o : \mathbf{\Sigma} = \sigma^2\mathbf{I}_p$.

Corollary2.1: When $H_o : \mathbf{\Sigma} = \sigma^2\mathbf{I}_p$, the moments of the sphericity decision criterion is obtained as special case from the noncentral moments as obtained in the above theorem. Hence we have

$$\mu_h = E(W^h) = p^{ph} \frac{\Gamma_p(\frac{pn}{2})\Gamma_p(\frac{1}{2}n+h)}{\Gamma_p(\frac{pn}{2}+ph)\Gamma_p(\frac{n}{2})}. \tag{4}$$

Upon using the expression of the multivariate Gamma function we may write μ_h as follows.

$$\mu_h = p^{ph} \frac{\Gamma[(\frac{1}{2}p(N-1)]}{\Gamma[(\frac{1}{2}p(N-1)+ph]} \prod_{i=1}^{p} \left\{ \frac{\Gamma[(\frac{1}{2}(N-i)+h]}{\Gamma(\frac{N-i}{2})} \right\}, \tag{5}$$

where $N = n+1$. We may point out that (2.3) was first obtained by Mauchly (1940).

Exact Distribution of W for p = 3.

Now we present Consul's (1967) result in connection with the density of W for $p = 3$. By using the density function as stated in the following theorem, we obtain the cdf of W. The total probability bounded by the cdf is computed for various values of N. Also we carried out the lower percentage values of $p = 3$ at different level of significance $\alpha = 0.01, 0.05, 0.005$ and for various values of N. Table 1 provides these percentage values.

We now state Consul's theorem.

Theorem 2 *If* $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_N$ *is a random sample from MVN(*$\mu, \mathbf{\Sigma}$*), then the pdf of* W *for testing the null hypothesis is given by*

$$f(w) = \frac{3}{2}K(n)w^{\frac{n}{2}-2} \sum_{r=0}^{\infty} (\frac{4}{27})^r \frac{\Gamma(3r+\frac{3}{2})}{\Gamma(2r+1)\Gamma(r+\frac{5}{2})} (1-w)^{\frac{3}{2}+r}. \tag{6}$$

Table 1: The Lower Percentage Values of W For Testing Sphericity Criterion for $p = 3$ Based on Consul's Result.

	α				α				α		
N	.05	.01	.005	N	.05	.01	.005	N	.05	.01	.005
5	0.006			34	0.705	0.622	0.592	63	0.833	0.781	0.761
6	0.040	0.007		35	0.713	0.681	0.601	64	0.836	0.784	0.764
7	0.088	0.034	0.019	36	0.720	0.640	0.610	65	0.838	0.787	0.767
8	0.139	0.066	0.047	37	0.727	0.648	0.619	66	0.841	0.790	0.771
9	0.189	0.102	0.078	38	0.734	0.656	0.628	67	0.843	0.793	0.774
10	0.235	0.139	0.111	39	0.740	0.664	0.636	68	0.845	0.796	0.777
11	0.279	0.175	0.144	40	0.746	0.671	0.643	69	0.847	0.798	0.780
12	0.318	0.210	0.177	41	0.751	0.678	0.651	70	0.849	0.801	0.783
13	0.355	0.243	0.208	42	0.757	0.685	0.658	71	0.851	0.804	0.785
14	0.388	0.275	0.239	43	0.762	0.691	0.665	72	0.853	0.806	0.788
15	0.418	0.305	0.268	44	0.767	0.697	0.671	73	0.855	0.809	0.791
16	0.445	0.332	0.295	45	0.772	0.703	0.678	74	0.857	0.811	0.793
17	0.471	0.358	0.321	46	0.776	0.709	0.684	75	0.859	0.813	0.796
18	0.494	0.383	0.345	47	0.781	0.714	0.690	76	0.861	0.816	0.798
19	0.515	0.406	0.368	48	0.785	0.720	0.695	77	0.862	0.818	0.801
20	0.535	0.427	0.389	49	0.789	0.725	0.701	78	0.864	0.820	0.803
21	0.553	0.447	0.410	50	0.793	0.730	0.706	79	0.866	0.822	0.806
22	0.570	0.466	0.429	51	0.797	0.735	0.711	80	0.867	0.824	0.808
23	0.586	0.482	0.447	52	0.800	0.739	0.716	82	0.870	0.828	0.812
24	0.601	0.500	0.464	53	0.804	0.744	0.721	84	0.873	0.832	0.816
25	0.614	0.515	0.480	54	0.807	0.748	0.725	86	0.876	0.836	0.820
26	0.627	0.530	0.495	55	0.811	0.752	0.730	88	0.879	0.839	0.824
27	0.639	0.544	0.509	56	0.814	0.756	0.734	90	0.882	0.843	0.828
28	0.650	0.557	0.523	57	0.817	0.760	0.738	92	0.884	0.846	0.831
29	0.661	0.569	0.536	58	0.820	0.764	0.742	94	0.886	0.849	0.835
30	0.671	0.581	0.548	59	0.823	0.767	0.746	96	0.889	0.852	0.838
31	0.680	0.592	0.560	60	0.826	0.771	0.750	98	0.891	0.855	0.841
32	0.689	0.601	0.571	61	0.828	0.774	0.754	100	0.893	0.864	0.851
33	0.697	0.613	0.582	62	0.831	0.778	0.757	105	0.898	0.864	0.851

Upon integrating the density (3.1) the cdf is given as follows.

$$F(w) = \begin{cases} 0, & w < 0 \\ K(n)\Gamma(\frac{n}{2}-1)\sum_{r=0}^{\infty}\left[\frac{\Gamma(3r+\frac{3}{2})}{\Gamma(2r+1)\Gamma(\frac{n}{2}+r+\frac{3}{2})}\right](\frac{4}{27})^r & \\ \cdot I_w(\frac{n}{2}-1, r+\frac{5}{2}), & 0 \le w < 1 \\ 1, & 1 \le w \end{cases} \quad (7)$$

where

$$K(n) = 2^{n+1}\Gamma(\frac{3}{2}n)\left[\Gamma(n-1)\Gamma(\frac{n}{2}-1)3^{\frac{1}{2}(3n+1)}\right]^{-1}, \quad (8)$$

and $I_w(p,q) = B(p,q)\int_0^w x^{p-1}(1-x)^{q-1}dx$ is the incomplete beta distribution.

Using the result of this theorem we provide the lower percentage values of the criterion at different levels of significance and various values of N and the results are presented in Table 1.

1 Exact Distribution of W For p = 2 and p = 3 Using Contour Integration

In this section we consider the problem of the distribution for W when the number of variables is equal to 2 and 3. The method used is based on the central moments as given in Corollary 2.1 and then applying the inverse Mellin transform. It is noted that if $\mu_h = E(W^h)$ exists and under some regularity assumptions, the density function can be obtained as follows.

$$f(w) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} w^{-h-1}\mu_h dh. \quad (9)$$

This is known by the inverse Mellin transform. Later the need will arise for some contour integration and calculations of the residues at the poles of Gamma function. Further the results were simplified in a suitable form for computations. Using the densities obtained we have derived the cdf for each case separately, i.e., for $p = 2$ and $p = 3$. Some extensive computer work is carried out to calculate the total probability bounded by the cdf obtained for $p = 2$ and $p = 3$. We have also carried out enormous computations to evaluate the percentage values at levels of significance $\alpha = 0.05, 0.01, 0.005$ and for different values of N. Tabulations of these percentage values are presented later.

Further some comparisons have been made between the percentage values computed using Consul's result and that obtained by applying the result developed in this section. Next we outline the derivation in detail.

Pdf and Cdf of W for p = 2

Upon employing inverse Mellin transformation and using μ_h in (2.3), the pdf of W is obtained as

$$f(w) = K(p,n)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{w^{-(h+1)}p^{ph}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(N-i)+h]}{\Gamma(\frac{1}{2}pn+ph)}dh, \tag{10}$$

where $n = N-1$ and $K(p,n) = \frac{\Gamma(\frac{1}{2}pn)}{\prod_{i=1}^{p}\Gamma(\frac{N-i}{2})}$. Letting $s = \frac{1}{2}(N-p)+h$, (4.2) can be written as

$$f(w) = K(p,n)p^{-(\frac{1}{2}p(N-p)}w^{\frac{1}{2}(N-p)-1}P(w), \tag{11}$$

where

$$P(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\frac{w}{p^p})^{-s}\frac{\prod_{i=1}^{p}\Gamma(s+\frac{p-i}{2})}{\Gamma[p(s+\frac{p-1}{2})]}ds, \tag{12}$$

where $c = \frac{1}{2}(N-p)$. Now letting $p = 2$ in (4.4), we get

$$P(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(w/4)^{-s}\frac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s+1)}ds. \tag{13}$$

Now, using the duplication formula

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(2s)}{2^{2s-1}} \tag{14}$$

in each of the Gamma function in (4.5), we get

$$P(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sqrt{\pi}(\frac{w^{-s}}{s})ds. \tag{15}$$

The pole of the integrand is at $s = 0$ and its residue is $\sqrt{\pi}$. Hence,

$$f(w) = \sqrt{\pi}2^{N-2}K(2,n)w^{\frac{1}{2}(N-4)}, \tag{16}$$

where, $K(2,n) = \frac{\Gamma(n)}{\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})}$. Thus, $f(w)$ is given by

$$f(w) = (\frac{1}{2})^{n-1}\frac{\sqrt{\pi}\Gamma(n)}{\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})}w^{\frac{1}{2}(n-3)}. \tag{17}$$

Using the duplication formula again, we get

$$f(w) = \frac{1}{2}(n-1)w^{\frac{1}{2}(n-3)},\ 0 < w < 1, \tag{18}$$

which agrees with the result of Mauchly (1940). The corresponding cdf is given by

$$F(w) = \begin{cases} 0, & w < 0, \\ w^{\frac{n-1}{2}}, & 0 \leq w < 1, \\ 1, & 1 \leq w. \end{cases} \tag{19}$$

The lower percentage points are also computed and presented in Table 2. Hence, we may summarize the results in the following theorem.

Theorem 3 *Let* $\mathbf{S}$ *be the sample covariance matrix based on a random sample of size* N *from a* $MVN(\mu, \mathbf{\Sigma})$*. The pdf and the cdf of* $W = \frac{det\mathbf{S}}{(\frac{1}{p}tr\mathbf{S})^p}$, *for* $p = 2$ *are given by (4.10) and (4.11), respectively.*

Table 2: The Lower Percentage Values of W For Testing Sphericity Criterion for $p = 2$ Based on Contour Integration.

	α				α				α		
N	.05	.01	.005	N	.05	.01	.005	N	.05	.01	.005
4	0.050	0.010	0.005	34	0.829	0.750	0.718	64	0.908	0.862	0.843
5	0.136	0.046	0.029	35	0.834	0.756	0.725	65	0.909	0.864	0.845
6	0.224	0.100	0.071	36	0.838	0.763	0.732	66	0.911	0.866	0.845
7	0.302	0.158	0.120	37	0.843	0.769	0.739	67	0.912	0.868	0.850
8	0.368	0.215	0.171	38	0.847	0.774	0.745	68	0.913	0.870	0.852
9	0.425	0.268	0.220	39	0.850	0.780	0.751	69	0.914	0.872	0.854
10	0.473	0.316	0.226	40	0.854	0.785	0.757	70	0.916	0.873	0.856
11	0.514	0.359	0.308	41	0.858	0.790	0.762	71	0.917	0.875	0.858
12	0.549	0.398	0.347	42	0.861	0.794	0.767	72	0.918	0.877	0.860
13	0.580	0.433	0.382	43	0.864	0.799	0.772	73	0.919	0.878	0.861
14	0.607	0.464	0.414	44	0.867	0.803	0.777	74	0.920	0.880	0.863
15	0.631	0.492	0.443	45	0.870	0.807	0.782	75	0.921	0.881	0.865
16	0.652	0.518	0.469	46	0.873	0.811	0.786	76	0.922	0.883	0.867
17	0.671	0.541	0.493	47	0.875	0.815	0.790	77	0.923	0.884	0.868
18	0.688	0.562	0.516	48	0.878	0.819	0.794	78	0.924	0.886	0.870
19	0.703	0.582	0.536	49	0.880	0.822	0.798	79	0.925	0.887	0.871
20	0.717	0.599	0.555	50	0.883	0.825	0.802	80	0.926	0.889	0.873
21	0.730	0.616	0.573	51	0.885	0.829	0.806	85	0.930	0.895	0.880
22	0.741	0.631	0.589	52	0.887	0.832	0.809	90	0.934	0.901	0.887
23	0.752	0.645	0.604	53	0.889	0.835	0.812	95	0.938	0.906	0.892
24	0.762	0.658	0.618	54	0.891	0.838	0.816	100	0.941	0.910	0.898
25	0.771	0.670	0.631	55	0.893	0.840	0.819	120	0.950	0.925	0.915
26	0.779	0.681	0.643	56	0.895	0.843	0.822	140	0.958	0.935	0.926
27	0.787	0.692	0.655	57	0.897	0.846	0.825	160	0.963	0.943	0.935
28	0.794	0.702	0.665	58	0.899	0.848	0.828	180	0.967	0.950	0.942
29	0.801	0.711	0.675	59	0.900	0.851	0.830	200	0.970	0.955	0.948
30	0.807	0.720	0.685	60	0.902	0.853	0.833	300	0.980	0.970	0.965
31	0.813	0.728	0.694	61	0.903	0.855	0.836	350	0.983	0.974	0.970
32	0.813	0.728	0.694	62	0.905	0.858	0.838	400	0.985	0.977	0.974
33	0.824	0.743	0.710	63	0.906	0.860	0.841	500	0.988	0.982	0.979

The Pdf and the Cdf of W for p = 3:

By letting $p = 3$ in (4.4), $P(w)$ takes the form

$$P(w) = \frac{1}{2\pi i} 2\sqrt{\pi} \int_{c-i\infty}^{c+i\infty} (\frac{4w}{27})^{-s} \frac{\Gamma(s+1)\Gamma(s+\frac{1}{2})\Gamma(s)}{\Gamma(3s+3)} ds. \tag{20}$$

Again, using the Duplication formula we get

$$P(w) = \frac{1}{2\pi i} 2\sqrt{\pi} \int_{c-i\infty}^{c+i\infty} (\frac{4w}{27})^{-s} \frac{\Gamma(2s)\Gamma(s+1)}{\Gamma(2s+3)} ds. \tag{21}$$

The integral in (4.13) will be evaluated by calculating the residues at the poles and then applying Cauchy residue theorem. The poles of the integrand are at the points $s = -d/2,\ d = 0, 1, 2, \ldots$ and the residue at these points may be found by putting $s = t - d/2$ in the integrand and then take the residue at $t = 0$. Upon substituting $s = t - d/2$ in the last equation, we get

$$I = 2\sqrt{\pi}(\frac{4w}{27})^{\frac{d}{2}-t} \frac{\Gamma(2t-d)\Gamma(t+1-\frac{d}{2})}{\Gamma(3t-\frac{3}{2}d+3)}. \tag{22}$$

It is noted that d may assume even or odd integral values. We must handle each case separately as follows.

The First Case - d Even:

If d is even, say $d = 2m$, the integrand in (4.14) takes the form

$$I_m(w) = 2\sqrt{\pi}(\frac{4w}{27})^{m-t} \frac{\Gamma(2t-2m)\Gamma(t-m+1)}{\Gamma(3t-3m+3)}, \tag{23}$$

and by expanding each Gamma function in (4.15) we have

$$I_m(w) = (\frac{3}{2}) 2\sqrt{\pi}(\frac{4w}{27})^{m-t} \frac{\Gamma(2t+1)\Gamma(t+1)\prod_{i=1}^{3m-3}(3t-i)}{t\Gamma(3t+1)\prod_{i=1}^{2m}(2t-i)\prod_{i=1}^{m-1}(t-i)}. \tag{24}$$

Expression (4.16) is valid only for $3m - 3 > 0$ or $(m > 1)$. The cases $m = 0$ and $m = 1$ will be treated later. Now the integrand in (4.16) has a simple pole of first order at $t = 0$ and its corresponding residue is

$$(\frac{3}{2}) 2\sqrt{\pi}(\frac{4w}{27})^{m} \frac{\prod_{i=1}^{3m-3}(-i)}{\prod_{i=1}^{2m}(-i)\prod_{i=1}^{m-1}(-i)} \tag{25}$$

and after some simplifications the residue can be written as

$$I_m = (\frac{3}{2}) 2\sqrt{\pi}(\frac{4w}{27})^{m} \frac{(3m-3)!}{(2m)!(m-1)!}. \tag{26}$$

To obtain the density for $m = 0$ and $m = 1$ we go back to the formula (4.15). Hence for $m = 0$, the integrand (4.15) becomes

$$I_0(w) = 2\sqrt{\pi}(\frac{4w}{27})^{-t}\frac{\Gamma(2t+1)\Gamma(t+1)}{2t\Gamma(3t+3)}, \tag{27}$$

and has a simple pole at $t = 0$, its residue is $\frac{1}{4}$. For $m = 1$, the integrand in (4.15) is given by

$$I_1(w) = 2\sqrt{\pi}\frac{3}{2}(\frac{4w}{27})^{1-t}\frac{\Gamma(2t+1)\Gamma(t+1)}{t(2t-1)\Gamma(3t+1)}, \tag{28}$$

which has a simple pole at $t = 0$ and its residue is given by $2\sqrt{\pi}(\frac{3}{4})(\frac{4w}{27})$. Now by using the residues, the integrand I_m may be written as $I_m(w) = 2\pi i$[sum of residuals]. Hence,

$$I_m(w) = 2\sqrt{\pi}\left\{\frac{1}{4} + (\frac{3}{4}) + \frac{3}{4}(\frac{4w}{27}) + \frac{3}{2}(\frac{4w}{27})^m\frac{(3m-3)!}{(2m)!(m-1)!}\right\}, \tag{29}$$

for $m = 0, 1, 2, 3, \ldots$.

The Second case - d Odd

When d is odd, say $d = 2q + 1$, where q is a nonnegative integer, the integrand in (4.14) becomes

$$J_q(w) = 2\sqrt{\pi}(\frac{4w}{27})^{(\frac{2q+1}{2}-t)}\frac{\Gamma(2t-2q-1)\Gamma(t+\frac{1}{2}-q)}{\Gamma(3t+\frac{3}{2}-3q)}, \tag{30}$$

and by expanding each gamma function as before we may write the integrand as follows

$$J_q(w) = 2\sqrt{\pi}(\frac{4w}{27})^{(\frac{2q+1}{2}-t)}\frac{\Gamma(2t+1)\Gamma(t+\frac{1}{2})\prod_{i=1}^{3q-1}(3t+\frac{1}{2}-i)}{2t\Gamma(3t+\frac{1}{2})\prod_{i=1}^{2q+1}(2t-i)\prod_{i=1}^{q}(t+\frac{1}{2}-i)}. \tag{31}$$

The expression in (4.23) is valid only for $q > 0$, and the case when $q = 0$ has to be treated separately. Now for $q > 0$, the integrand in (4.23) has a simple pole of first order at $t = 0$ and its residue is $2\sqrt{\pi}(\frac{4w}{27})^{(\frac{2q+1}{2})}\frac{\prod_{i=1}^{3q-1}(\frac{1}{2}-i)}{2\prod_{i=1}^{2q+1}(-i)\prod_{i=1}^{q}(\frac{1}{2}-i)}$ which can be written as $(\frac{4w}{27})^{(\frac{2q+1}{2})}\frac{(3q-\frac{3}{2})!}{2(2q+1)(q-\frac{1}{2})!}$. Therefore, the integrand in (4.23) for $q = 0$ becomes

$$J_0(w) = 2\sqrt{\pi}(\frac{4w}{27})^{(\frac{1}{2}-t)}\frac{\Gamma(2t+1)\Gamma(t+\frac{1}{2})}{2t(2t-1)\Gamma(3t+\frac{3}{2})}. \tag{32}$$

The quantity $J_0(w)$ has a simple pole of first order at $t = 0$, its corresponding residue is $-2\sqrt{\pi}\sqrt{\frac{4w}{27}}$. Hence the integrand $J_q(w)$ becomes $J_q(w) = 2\sqrt{\pi}i$[Sum of residuals] and is given by

$$J_q(w) = 2\sqrt{\pi}\left\{-\sqrt{\frac{4w}{27}} + (\frac{4w}{27})^{(\frac{2q+1}{2})}\frac{(3q-\frac{3}{2})!}{2(2q+1)!(q-\frac{1}{2})!}\right\}, \quad (33)$$

for $q = 0, 1, 2, 3, \ldots$. Finally by using Cauchy's residue theorem we get $P(w) = 2\pi i$[Sum of residues],

$$P(w) = 2\sqrt{\pi}\left\{\frac{1}{4} - \sqrt{\frac{4w}{27}} + \sum_{m=1}^{\infty} I_m(w) + \sum_{q=1}^{\infty} J_q(w)\right\}, \quad (34)$$

where $I_m(w)$ and $J_q(w)$ are given in (4.21) and (4.25), respectively. Now the density of W when $p = 3$ is given by

$$f(w) = \frac{K(3,n)}{3^{\frac{3}{2}(N-p)}} w^{\frac{1}{2}(N-p)-1} P(w), \quad (35)$$

where $P(w)$ is as in (4.26) and $K(3,n) = \frac{\Gamma(\frac{3n}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{N-2}{2}))\Gamma(\frac{N-3}{2})}$. By using this expression we easily obtain the cdf as follows.

$$F(w) = \begin{cases} 0, & w < 0 \\ \frac{K(3,n)}{3^{\frac{3}{2}(N-p)}\frac{1}{2}(N-p)} w^{\frac{1}{2}(N-p)} P(w), & 0 \le w < 1 \\ 1, & 1 \le w. \end{cases} \quad (36)$$

Now we may state the following theorem.

Theorem 4 *Let* $\mathbf{S}$ *be the sample covariance matrix based on a random sample of size* N *from a MVN*$(\mu, \mathbf{\Sigma})$*. The pdf and the cdf of the criterion* $W = \frac{det\,\mathbf{S}}{(\frac{1}{p}tr\mathbf{S})^p}$*, for* $p = 3$ *is given by (4.27) and (4.28), respectively.*

Computations of Lower Percentage Values of W.

To illustrate the advantages of the density and the cdf derived in this paper we have computed the lower percentage values of the sphericity criterion at different significance levels and for different values of N. It is known that the lower and upper percentage values are of particular interest, particularly in testing of hypotheses. To carry out the computation of the lower tail values we have to compute the total probability bounded by the cdf of the criterion. For this purpose we have computed the total probability using the formula of the cdf as given in (3.2) and (4.28) for

$p = 3$. By observing the total probability bounded by each of the underlying formula we may have some judgement about the rate of convergence of each of the expressions available. Table 3 includes the lower percentage points for different values of $\alpha = 0.01, 0.05$ and for various values of N using (4.28). A comparison has been made in Table 4 between the percentage points as computed using Theorem 3.1 and that of Theorem 4.2.

Table 3: The Lower Percentage Values of W For Testing Sphericity Criterion For $p = 3$ Using (4.28).

N	α		N	α	
	.05	.01		.05	.01
5	0.013	0.003	19	0.496	0.349
6	0.045	0.014	20	0.513	0.355
7	0.090	0.036	21	0.529	0.402
8	0.139	0.066	22	0.543	0.0413
9	0.187	0.098	23	0.555	0.420
10	0.233	0.131	24	0.566	0.422
11	0.275	0.162	25	0.576	0.434
12	0.313	0.192	26	0.585	0.444
13	0.348	0.218	27	0.592	0.453
14	0.379	0.240	28	0.599	0.462
15	0.408	0.259	29	0.603	0.469
16	0.433	0.271	30	0.607	0.476
17	0.456	0.275	31	0.608	0.481
18	0.477	0.253	32	0.651	0.484

Table 4: The Comparison of Percentage Values for Sphericity Criterion for $p = 3$ Using Three Criteria.

5%				1%			
N	Nagarsenker & Pillai	Consul	Mauchly	N	Nagarsenker & Pillai	Consul	Mauchly
				5	0.003		
6	0.045	0.040		6	0.014	0.007	
7	0.90	0.088		7	0.036	0.034	
9	0.187	0.189		9	0.098	0.102	
10	0.233	0.235	0.278	11	0.162	0.175	
11	0.275	0.279		13	0.162	0.175	
20	0.513	0.535	0.580	15	0.259	0.305	
25	0.576	0.614		20	0.355	0.427	0.466
26	0.585	0.627	0.667	22	0.413	0.466	0.504
27	0.592	0.639		24	0.422	0.500	0.538
28	0.599	0.650	0.689	28	0.462	0.557	0.593
30	0.607	0.671	0.708	30	0.476	0.581	0.616
31	0.608	0.680		31	0.481	0.592	
32	0.651	0.689	0.724	32	0.484	0.601	0.637

2 Conclusions

Based on the numerical findings in this paper, we have found that the pdf as obtained by Consul (1967) is not convenient for large N. On the other hand, the pdf and the cdf that are given in (4.27) and (4.28) are more practical for computational purposes. Consequently, we were able to compute the lower tail values of the sphericity criterion W for large values of N by using the results of the last theorem. The results are displayed in Table 3. Table 4 on the next page shows some comparisons between the tail values obtained by Consul, Nagarsenker and Pillai, and Mauchly.

References

1. Box, G.E.P. (1949). A general distribution theory for a class of likelihood criterion, *Biometrika*, 36, 317-346.

2. Consul, P.C. (1967). On the exact distribution of the criterion W for testing sphericity in a p-variate normal distribution, *Annals of Mathematical Statistics*, 38, 1170-1174.

3. Consul, P.C. (1969). The exact distributions of likelihood criteria for different hypotheses, *Multivariate Analysis*, 2, Academic Press, New York.

4. James, A. T. (1964). Distribution of matrix variates and latent roots derived from normal samples, *Annals of Mathematical Statistics*, 35, 475-501.

5. Khatri, C. and Srivastava, M.S. (1971). On exact non-null distribution of the likelihood ratio criterion for sphericity test and equality of two covariance matrices, *Sankhyā*, 71(A), 201-206.

6. Mathai, A.M. and Rathie, P.N. (1970). The exact distributions for the sphericity test, *Journal of Statistical Research*, 4, 140-159.

7. Mauchly, J.W. (1940). Significance test for sphericity of a normal p-variate distribution, *Annals of Mathematical Statistics*, 11, 204-209.

8. Muirhead, R.J. (1982). *Aspects of Multivariate Statistical Theory.* John Wiley.

9. Nagarsenkar, B.N. (1972). The distribution of the sphericity test criterion, *Journal of Multivariate Analysis*, 3, 226-235.

10. Nagarsenkar, B. N. and Pillai, K. C. S. (1974). Distribution of the likelihood ratio criterion for testing $\Sigma = \Sigma_o,\ \mu = \mu_o$, *Journal of Multivariate Analysis*, 4, 114-122.

11. Nair, U.S. (1938). The application of the moment function in the study of distribution laws in statistics, *Biometrika*, 30, 274-294.

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