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Reference Priors in a Two-Way Mixed-Effects Analysis of Variance Model

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Abstract

We first derive group ordering reference priors in a two-way mixed-effects analysis of variance(ANOVA) model. We show that posterior distributions are proper and provide marginal posterior distributions under reference priors. We also examine whether the reference priors satisfy the probability matching criterion. Finally, the reference prior satisfying the probability matching criterion is shown to be good in the sense of frequentist coverage probability of the posterior quantile.

Keywords : Error Variance, Frequentist coverage probability, Jeffreys' prior, Matching prior, Reference prior, Two-way mixed-effects ANOVA model.

1. Introduction

The problem of estimating variance in analysis of variance model has been investigated by many statisticians. Especially in Bayesian point of view, the inference of variance components in random effects model has been treated by many statisticians for a long time. First, consider the balanced one-way random effect model : $y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$, $i = 1,2,\cdots,I$, $j = 1,2,\cdots,J$, where μ is an known constants, and the α_i and ε_{ij} are independent normal variables with 0 means

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and variance σ_α^2 and σ^2, respectively. Box and Tiao(1973) chose a prior distribution $\pi(\mu, \sigma^2, \sigma_\alpha^2) \propto \sigma^{-2}(\sigma^2 + J\sigma_\alpha^2)^{-1}$ and calculated the posterior distributions. They observed the relationship between the posterior of $\phi = J\sigma_\alpha^2/\sigma^2$ and a frequentist result in a hypothesis testing problem. Ye(1994) developed reference priors for ϕ, examined frequentist coverage probabilities of posterior quantiles for various ϕ and compared of the Bayes estimators for reference priors. The ordered group reference prior algorithm of Berger and Bernard(1989) is applied to the balanced one-way random effect model by Berger and Bernardo(1992a). Also, Chung and Dey(1998) derived reference priors and first order probability matching priors for intraclass correlation $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma^2)$ and examined the frequentist coverage probabilities of posterior quantiles for various ρ. Recently Kim, Kang, and Lee(2001) derived the second order probability matching criterion for the ratio of the variance components. They shows that among all of the reference priors given in Ye(1994), the only one reference prior satisfies the second order matching criterion.

Now, consider a two-way mixed-effects analysis of variance(ANOVA) model:

$$y_{ij} = \mu_i + \beta_j + \varepsilon_{ij},\ i = 1, 2, \cdots, p(>1), j = 1, 2, \cdots, q(>1), \qquad (1.1)$$

where μ_i is the ith mean effect of fixed effects, random effect β_j's independent and identically distributed as $N(0, \sigma_\beta^2)$, ε_{ij}'s are assumed to be independent and identically distributed as $N(0, \sigma^2)$. Further, the β_j's are also assumed to be independent of ε_{ij}'s. Vounatsu and Smith(1997) studied Baysian approach for variance component model and hierarchical model with 2-variance component for balanced and unbalanced case in this model. Recently Chang and Kim(2002) consider the problem of estimating σ^2 in this model (1.1) using Jeffreys' prior, reference prior, and matching priors. They then compare quantiles of marginal posterior densities of σ^2 in two real data set.

In this paper, we consider a Bayesian analysis of error variance in the model (1.1) using reference priors. Since our focus is fully Bayesian, choice of priors is very important. The determination of reasonable noninformative priors in multiparameter problem is not easy; common noninformative priors, such as Jefferys' prior can have features that have dramatic effect on the posterior. More specifically, Bernardo(1979) pointed out that if we are interested in a subset of parameters, the rest being nuisance parameters, then Jeffreys' prior may be inappropriate for representing vague of little prior information. In order to overcome this problem, Bernardo(1979) proposed the reference prior approach for the development of the noninformative prior. Berger and Bernardo(1989, 1992b) extended their algorithm to multiparameter problem.

The purpose in this paper is to obtain reference priors for $(\sigma^2, \sigma_\beta^2, \boldsymbol{\mu})$ where σ^2 is the parameter of interest. The paper is arranged as follows. Section 2 derives the reference priors. In Section 3, we show that posterior distributions are proper and provide marginal posterior distributions under reference priors. In Section 4, we examine whether the reference priors satisfy the probability matching criterion. Finally, Section 5 provides frequentist coverage probabilities of the posterior credible sets using the reference priors.

2. The Grouped Ordering Reference Priors

In Berger and Bernardo's(1992a) reference prior approach to the one-way random-effect model, the ordered group is very important. That is, the form of reference priors can be changed by the ordered grouping. This divides the parameters into two subgroups, called parameter of interest and nuisance parameters. Notation such as $\{\sigma^2, (\boldsymbol{\mu}, \sigma_\beta^2)\}$ will be used to specify the group and the importance of parameters; $\{\sigma^2, (\boldsymbol{\mu}, \sigma_\beta^2)\}$ means that there are two groups, with σ^2 being more important than the group $(\boldsymbol{\mu}, \sigma_\beta^2)$. A group such as $\{\sigma^2, \boldsymbol{\mu}, \sigma_\beta^2\}$ means that σ^2 is the most important parameter and σ_β^2 is the least important. Using Berger and Bernardo's algorithm of computing the reference priors, the reference prior distributions for different groups of ordering of $(\sigma^2, \boldsymbol{\mu}, \sigma_\beta^2)$ are obtained as follows:

For model (1.1), the likelihood function of parameters $(\sigma^2, \boldsymbol{\mu}, \sigma_\beta^2)$ is given by

$$l(\sigma^2, \sigma_\beta^2, \boldsymbol{\mu} | \boldsymbol{y}) \propto (\sigma^2)^{-\frac{q(p-1)}{2}} (\sigma^2 + p\sigma_\beta^2)^{-\frac{q}{2}}$$
$$\times \exp\Big[-\frac{1}{2}\Big(\frac{p\sum_{j=1}^{q}(y_{.j}-y_{..})^2}{\sigma^2+p\sigma_\beta^2} + \frac{\sum_{i=1}^{p}\sum_{j=1}^{q}(y_{ij}-y_{i.}-y_{.j}+y_{..})^2}{\sigma^2}$$
$$+ \frac{q\sum_{i=1}^{p}(y_{i.}-y_{..}-(\mu_i-\mu_.))^2}{\sigma^2} + \frac{pq(y_{..}-\mu_.)^2}{\sigma^2+p\sigma_\beta^2}\Big)\Big], \tag{2.1}$$

where $y_{i.} = \frac{1}{q}\sum_{j=1}^{q} y_{ij}$, $y_{.j} = \frac{1}{p}\sum_{i=1}^{p} y_{ij}$, $y_{..} = \frac{1}{pq}\sum_{i=1}^{p}\sum_{j=1}^{q} y_{ij}$, and $\mu_. = \frac{1}{p}\sum_{i=1}^{p}\mu_i$. Now using notations $\boldsymbol{y} = (y_{11}, \cdots, y_{pq})^T$ and $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)^T$, (2.1) becomes

$$l(\sigma^2, \sigma_\beta^2, \boldsymbol{\mu} | \boldsymbol{y}) \propto (\sigma^2)^{-\frac{q(p-1)}{2}} (\sigma^2 + p\sigma_\beta^2)^{-\frac{q}{2}}$$
$$\times \exp[-\frac{1}{2}(\frac{p\sum_{j=1}^{q}(y_{.j} - y_{..})^2}{\sigma^2 + p\sigma_\beta^2} + \frac{\sum_{i=1}^{p}\sum_{j=1}^{q}(y_{ij} - y_{i.} - y_{.j} + y_{..})^2}{\sigma^2})]$$
$$\cdot \exp[-\frac{q}{2\sigma^2}((\boldsymbol{\mu} - \boldsymbol{y})'(I_p - \frac{\sigma_\beta^2}{\sigma^2 + p\sigma_\beta^2} J_p)(\boldsymbol{\mu} - \boldsymbol{y}))]. \tag{2.2}$$

From (2.2), Fisher information matrix is

$$I_1(\sigma^2, \sigma_\beta^2, \boldsymbol{\mu}) = \begin{pmatrix} \frac{q(p-1)}{2\sigma^4} + \frac{q}{2(\sigma^2 + p\sigma_\beta^2)^2} & \frac{pq}{2(\sigma^2 + p\sigma_\beta^2)^2} & \mathbf{0}' \\ \frac{pq}{2(\sigma^2 + p\sigma_\beta^2)^2} & \frac{p^2 q}{2(\sigma^2 + p\sigma_\beta^2)^2} & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \frac{q}{\sigma^2} I_p - \frac{q\sigma_\beta^2}{\sigma^2(p\sigma_\beta^2 + \sigma^2)} J_p \end{pmatrix}, \tag{2.3}$$

where I_p is the $p \times p$ identity matrix and J_p is the $p \times p$ matrix with 1 as all elements.

From (2.3), we have the following theorem of the reference prior of the ordering group $(\sigma^2, \boldsymbol{\mu}, \sigma_\beta^2)$.

Theorem 2.1. For the balanced two-way mixed-effects ANOVA model if σ^2 is the parameter of interest, then the reference prior distributions for different groups of ordering for $(\sigma^2, \boldsymbol{\mu}, \sigma_\beta^2)$ are;

Group ordering	*Reference prior*
$\{\sigma^2, (\boldsymbol{\mu}, \sigma_\beta^2)\}$	$\pi_1 \propto (\sigma^2)^{-\frac{1}{2}} (\sigma^2 + p\sigma_\beta^2)^{-\frac{3}{2}}$
$\{(\sigma^2, \sigma_\beta^2), \boldsymbol{\mu}\}\{\sigma^2, \sigma_\beta^2, \boldsymbol{\mu}\}\{\sigma^2, \boldsymbol{\mu}, \sigma_\beta^2\}$	$\pi_2 \propto (\sigma^2)^{-1} (\sigma^2 + p\sigma_\beta^2)^{-1}$
$\{(\sigma^2, \boldsymbol{\mu}), \sigma_\beta^2\}$	$\pi_3 \propto (\sigma^2)^{-\frac{p}{2} - \frac{3}{4}} (\sigma^2 + p\sigma_\beta^2)^{-1}$
$\{(\sigma^2, \boldsymbol{\mu}, \sigma_\beta^2)\}$	$\pi_4 \propto (\sigma^2)^{-\frac{p+1}{2}} (\sigma^2 + p\sigma_\beta^2)^{-\frac{3}{2}}$

Proof. We apply an algorithm by Berger and Bernardo(1989) to (2.3). Since the derivations of other reference priors are similar, we consider only the reference prior for the group $\{(\sigma^2, \boldsymbol{\mu}), \sigma_\beta^2\}$.

Step 1 : The usual reference prior for σ_β^2 with $(\sigma^2, \boldsymbol{\mu})$ given becomes,

$$\pi(\sigma_\beta^2 | (\sigma^2, \boldsymbol{\mu})) = \left(\frac{p^2 q}{2(\sigma^2 + p\sigma_\beta^2)^2}\right)^{\frac{1}{2}}$$

Step 2 : Choose an increasing sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets of the parameter space Ω for $((\sigma^2, \boldsymbol{\mu}), \sigma_\beta^2)$ such that $\bigcup_i K_i = \Omega$. Take $K_i = [\frac{1}{i}, i] \times [-i, i]^p \times [\frac{1}{i}, i]$. Then normalize $\pi(\sigma_\beta^2 | (\sigma^2, \boldsymbol{\mu}))$ on $\Omega_{i,(\sigma^2, \boldsymbol{\mu})}$

$= \{\sigma^2_\beta : ((\sigma^2, \boldsymbol{\mu}), \sigma^2_\beta) \in K_i\}$ obtaining $p_i(\sigma^2_\beta | (\sigma^2, \boldsymbol{\mu})) = C_i((\sigma^2, \boldsymbol{\mu})) \cdot \pi(\sigma^2_\beta | (\sigma^2, \boldsymbol{\mu}))$ where

$$\begin{aligned} C_i^{-1}((\sigma^2, \boldsymbol{\mu})) &= \int_{\frac{1}{i}}^{i} \left(\frac{p^2 q}{2}\right)^{\frac{1}{2}} (\sigma^2 + p\sigma^2_\beta)^{-1} d\sigma^2_\beta \\ &= \left(\frac{p^2 q}{2}\right)^{\frac{1}{2}} \frac{1}{p} [\log(\sigma^2 + p_i) - \log(\sigma^2 + \frac{p}{i})] \end{aligned}$$

Step 3 : The marginal reference prior for $(\sigma^2, \boldsymbol{\mu})$ with respect to $p_i(\sigma^2_\beta | (\sigma^2, \boldsymbol{\mu}))$ is

$$\begin{aligned} &\pi_i((\sigma^2, \boldsymbol{\mu})) \\ &= \exp\left\{\frac{1}{2} \int_{\frac{1}{i}}^{i} \frac{(\sigma^2 + p\sigma^2_\beta)^{-1}}{\frac{1}{p}[\log(\sigma^2 + p_i) - \log(\sigma^2 + \frac{p}{i})]} \log \frac{q^{p+1}(p-1)}{2} (\sigma^2)^{-p-1} (\sigma^2 + p\sigma^2_\beta)^{-1} d\sigma^2_\beta \right\} \\ &= \left[\frac{q^{(p+1)}(p-1)}{2}\right]^{\frac{1}{2}} (\sigma^2)^{-\frac{p+1}{2}} e^{-\frac{1}{4}[\log(\sigma^2 + p_i) + \log(\sigma^2 + \frac{p}{i})]} \end{aligned}$$

Step 4 : The reference prior for $((\sigma^2, \boldsymbol{\mu}), \sigma^2_\beta)$ is for fixed $((\sigma^2_0, \boldsymbol{\mu_0}), \sigma^2_{\beta_0})$,

$$\begin{aligned} \pi((\sigma^2, \boldsymbol{\mu}), \sigma^2_\beta) &= \lim_{i \to \infty} \frac{C_i(\sigma^2, \boldsymbol{\mu}) \pi_i(\sigma^2, \boldsymbol{\mu})}{C_i(\sigma^2_0, \boldsymbol{\mu_0}) \pi_i(\sigma^2_0, \boldsymbol{\mu_0})} \pi(\sigma^2_\beta | (\sigma^2, \boldsymbol{\mu})) \\ &= \frac{(\sigma^2)^{-\frac{p+1}{2}}}{(\sigma^2_0)^{-\frac{p+1}{2}}} \cdot e^{-\frac{1}{4} \log \frac{\sigma^2}{\sigma^2_0}} \cdot \left(\frac{p^2 q}{2(\sigma^2 + p\sigma^2_\beta)^2}\right)^{\frac{1}{2}} \\ &\propto (\sigma^2)^{-\frac{p+1}{2} - \frac{1}{4}} (\sigma^2 + p\sigma^2_\beta)^{-1} \\ &= (\sigma^2)^{-\frac{p}{2} - \frac{3}{4}} (\sigma^2 + p\sigma^2_\beta)^{-1}. \end{aligned}$$

Note that the prior distributions are all free of the location parameter $\boldsymbol{\mu}$. All the reference prior distributions are proportional to a negative powers of σ^2 and $(\sigma^2 + p\sigma^2_\beta)$. Therefore, a general form of the prior can be written as

$$\pi(\boldsymbol{\mu}, \sigma^2, \sigma^2_\beta) \propto (\sigma^2)^{-a} (\sigma^2 + p\sigma^2_\beta)^{-b}, \tag{2.4}$$

where a and b are positive numbers.

Also, Jeffreys' prior which is the square root of the determinant of the expected Fisher information matrix is given by

$$\pi_J(\boldsymbol{\mu}, \sigma^2, \sigma^2_\beta) \propto (\sigma^2)^{-\frac{p+1}{2}} (\sigma^2 + p\sigma^2_\beta)^{-\frac{3}{2}}. \tag{2.5}$$

Therefore, the Jeffreys' prior is same as the reference prior π_4 for $(\sigma^2, \boldsymbol{\mu}, \sigma^2_\beta)$.

It has been argued the reference prior distribution depend on the sample size and design and therefore violate the likelihood principle. However, there is no satisfactory method of obtaining a noninformative prior in this scenario.

3. Marginal Posterior Distributions

According to Bayes theorem, the posterior distribution of $(\sigma^2, \boldsymbol{\mu}, \sigma^2_\beta)$ with respect to the priors in (2.4) is given by

$$\pi(\sigma^2, \boldsymbol{\mu}, \sigma^2_\beta | \boldsymbol{y}) \propto (\sigma^2)^{-a-\frac{q(p-1)}{2}} (\sigma^2 + p\sigma^2_\beta)^{-b-\frac{q}{2}}$$
$$\times \exp[-\frac{1}{2}(\frac{p\sum_{j=1}^{q}(y_{.j}-y_{..})^2}{\sigma^2+p\sigma^2_\beta} + \frac{\sum_{i=1}^{p}\sum_{j=1}^{q}(y_{ij}-y_{i.}-y_{.j}+y_{..})^2}{\sigma^2}$$
$$+ \frac{q\sum_{i=1}^{p}(y_{i.}-y_{..}-(\mu_i-\mu_.))^2}{\sigma^2} + \frac{pq(y_{..}-\mu_.)^2}{\sigma^2+p\sigma^2_\beta})]. \quad (3.1)$$

Integrating out $\boldsymbol{\mu}$ and σ^2_β, the resulting marginal posterior distribution of σ^2 is

$$\pi(\sigma^2 | \boldsymbol{y}) \propto (\sigma^2)^{-\frac{p(q-1)}{2}-a-b+1} e^{-\frac{S}{2\sigma^2}} (\int_0^1 w^{\frac{q-1}{2}+b-2} e^{-\frac{SS_\beta}{2\sigma^2}w} dw), \quad (3.2)$$

where $w = \frac{\sigma^2}{\sigma^2 + p\sigma^2}$. We can see posterior distribution is proper from the following theorem.

Theorem 3.1. The posterior distributions (3.1) is proper if $a > 1 - \frac{(p-1)(q-1)}{2}$ and $b > \frac{3-q}{2}$.

Proof. We prove the result for reference prior (2.4). Full posterior distributions is given by

$$\pi_R(\sigma^2, \boldsymbol{\mu}, \sigma^2_\beta | \boldsymbol{y}) \propto [C(\boldsymbol{y})](\sigma^2)^{-a-\frac{q(p-1)}{2}} (\sigma^2 + p\sigma^2_\beta)^{-b-\frac{q}{2}}$$
$$\times \exp[-\frac{1}{2}(\frac{p\sum_{j=1}^{q}(y_{.j}-y_{..})^2}{\sigma^2+p\sigma^2_\beta} + \frac{\sum_{i=1}^{p}\sum_{j=1}^{q}(y_{ij}-y_{i.}-y_{.j}+y_{..})^2}{\sigma^2}$$
$$+ \frac{q\sum_{i=1}^{p}(y_{i.}-y_{..}-(\mu_i-\mu_.))^2}{\sigma^2} + \frac{pq(y_{..}-\mu_.)^2}{\sigma^2+p\sigma^2_\beta})],$$

where

$$[C(\mathbf{y})]^{-1} = \int_0^\infty \int_0^\infty \int_{R^p} (\sigma^2)^{-a-\frac{q(p-1)}{2}} (\sigma^2 + p\sigma_\beta^2)^{-b-\frac{q}{2}}$$

$$\times \exp[-\frac{1}{2}(\frac{p\sum_{j=1}^{q}(y_{.j}-y_{..})^2}{\sigma^2+p\sigma_\beta^2} + \frac{\sum_{i=1}^{p}\sum_{j=1}^{q}(y_{ij}-y_{i.}-y_{.j}+y_{..})^2}{\sigma^2}$$

$$+ \frac{q\sum_{i=1}^{p}(y_{i.}-y_{..}-(\mu_i-\mu_.))^2}{\sigma^2} + \frac{pq(y_{..}-\mu_.)^2}{\sigma^2+p\sigma_\beta^2})]\, d\boldsymbol{\mu}\, d\sigma_\beta^2\, d\sigma^2. \quad (3.3)$$

Integrating out $\boldsymbol{\mu}$ in (3.3), we obtain following result

$$\int_{R^p} \exp[-\frac{q}{2\sigma^2}(\sum_{i=1}^{p}(y_{i.}-y_{..}-(\mu_i-\mu_.))^2 + \frac{p\sigma^2}{\sigma^2+p\sigma_\beta^2}(y_{..}-\mu_.)^2)]\, d\boldsymbol{\mu}$$

$$= \int_{R^p} \exp[-\frac{q}{2\sigma^2}((\boldsymbol{\mu}-\mathbf{y})'(I_p - \frac{\sigma_\beta^2}{\sigma^2+p\sigma_\beta^2}J_p)(\boldsymbol{\mu}-\mathbf{y}))]\, d\boldsymbol{\mu}$$

$$= (2\pi)^{\frac{p}{2}} q^{-\frac{p}{2}} (\sigma^2)^{\frac{p-1}{2}} (\sigma^2+p\sigma_\beta^2)^{\frac{1}{2}}. \quad (3.4)$$

Then, using the result of (3.4), (3.3) is given by

$$(3.3) = (2\pi)^{\frac{p}{2}} q^{-\frac{p}{2}} \int_0^\infty \int_0^\infty (\sigma^2)^{-\frac{q(p-1)-(p-1)}{2}-a} (\sigma^2 + p\,\sigma_\beta^2)^{-\frac{q-1}{2}-b}$$

$$\times \exp[-\frac{1}{2\sigma^2}(S + \frac{\sigma^2}{\sigma^2+p\sigma_\beta^2} SS_\beta)]\, d\sigma_\beta^2\, d\sigma^2, \quad (3.5)$$

where $S = \sum_{i=1}^{p}\sum_{j=1}^{q}(y_{ij}-y_{i.}-y_{.j}+y_{..})^2$ is error sum of square, $SS_\beta = p\sum_{j=1}^{q}(y_{.j}-y_{..})^2$ is sum of squares due to random effects. Let $r = \frac{1}{\sigma^2}$, $w = \frac{\sigma^2}{\sigma^2+p\sigma_\beta^2}$, then $\sigma_\beta^2 = \frac{1-w}{prw}$ and $|J| = \frac{1}{pr^3w^2}$. Therefore,

$$(3.5) \propto \int_0^\infty \int_0^1 r^{\frac{pq-p}{2}+a+b-3} w^{\frac{q-1}{2}+b-2} \cdot \exp[-\frac{r}{2}(S + w\,SS_\beta)]\, dw\, dr$$

$$< \int_0^\infty r^{\frac{pq-p}{2}+a+b-3} e^{-\frac{r}{2}s} \int_0^\infty w^{\frac{q-1}{2}+b-2} e^{-\frac{rSS_\beta}{2}w}\, dw$$

$$\propto \int_0^\infty r^{\frac{(p-1)(q-1)}{2}+a-2} e^{-\frac{r}{2}s}\, dr$$

$$< \infty \quad \text{if} \quad \frac{(p-1)(q-1)}{2}+a-1>0 \quad \text{and} \quad \frac{q-3}{2}+b>0.$$

Remark 3.1. The prior distribution with $a = 1/2$ in (2.4) produces an improper posterior distribution when $p = 2$ and $q = 2$. This means that the prior π_i for the grouped ordering $\{\sigma^2, (\boldsymbol{\mu}, \sigma_\beta^2)\}$ produces an improper posterior distribution when $p = 2$ and $q = 2$.

4. Probability Matching Prior

Now we obtain the probability matching priors and see whether they include same of the reference priors developed in Section 2. First, we briefly review the probability matching prior as follows.

For a prior π, let $\theta_1^{1-\alpha}(\pi;\boldsymbol{y})$ denote the $(1-\alpha)th$ percentile of the posterior distribution of θ_1, that is

$$P^{\pi}[\theta_1 \le \theta_1^{1-\alpha}(\pi;\boldsymbol{y}) \mid \boldsymbol{y}] = 1-\alpha, \tag{4.1}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$ and θ_1 is the parameter of interest and θ_2 and θ_3 are nuisance parameters. We want to find priors π for which

$$P[\theta_1 \le \theta_1^{1-\alpha}(\pi;\boldsymbol{y}) | \theta] = 1-\alpha + o(n^{-u}) \tag{4.2}$$

for some $u>0$, as n goes to infinity. Priors π satisfying (4.2) are called probability matching priors. If $u=\frac{1}{2}$, then π is referred to as a first order matching prior, while if $u=1$, π is referred to as a second order matching prior.

In order to find such matching priors π, it is convenient to introduce orthogonal parametrization. To this end, let

$$\sigma^2 = \psi, \quad \sigma_\beta^2 = \frac{1}{p}(\lambda - \psi), \quad \boldsymbol{\mu} = \boldsymbol{\eta}. \tag{4.3}$$

With this parametrization, based on the likelihood function (2.2) and Fisher information matrix (2.3) of parameters $(\sigma^2, \sigma^2_\beta, \boldsymbol{\mu})$, the Fisher information matrix of parameters $(\psi, \lambda, \boldsymbol{\eta})$ is given by

$$I_2(\psi, \lambda, \boldsymbol{\eta}) = \begin{pmatrix} \frac{q(p-1)}{2\psi^2} & 0 & \mathbf{0}' \\ 0 & \frac{q}{2\lambda^2} & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \frac{q}{\psi} I_p - \frac{q(\lambda-\psi)}{p\psi\lambda} J_p \end{pmatrix}. \tag{4.4}$$

Thus ψ is orthogonal to λ and $\boldsymbol{\eta}$ in the sense of Cox and Reid(1987). By Tibshirani(1989), the class of first order probability matching prior is characterized by

$$\pi_M^{(1)}(\psi, \lambda, \boldsymbol{\eta}) \propto \psi^{-1} \cdot g(\lambda, \boldsymbol{\eta})$$

and the first order probability matching prior of original parameters $(\sigma^2, \sigma^2_\beta, \boldsymbol{\mu})$ is given by

$$\pi_M^{(1)}(\sigma^2, \sigma^2_\beta, \boldsymbol{\mu}) \propto (\sigma^2)^{-1} \cdot g(\sigma^2 + p\sigma^2_\beta, \boldsymbol{\mu}), \tag{4.5}$$

where $g(\sigma^2 + p\sigma^2_\beta, \boldsymbol{\mu})$ is arbitrary differentiable function.

Clearly the class of prior given in (4.5) is quite large, and it is important to narrow down this class of priors. Therefore we derive the class of second order probability matching priors by Mukerjee and Ghosh (1997).

Theorem 4.1. The second order probability matching priors are given by

$$\pi_M^{(2)}(\sigma^2, \sigma_\beta^2, \boldsymbol{\mu}) \propto (\sigma^2)^{-1} \cdot g(\sigma^2 + p\sigma_\beta^2, \boldsymbol{\mu}), \tag{4.6}$$

where $g(\sigma^2 + p\sigma_\beta^2, \boldsymbol{\mu})$ is arbitrary differentiable function.

Proof. Let $\psi = \theta_1, \lambda = \theta_2, \eta_{i-2} = \theta_i, i = 3, \cdots, p+2$, then second order probability matching prior is of the form (4.5), and also g must satisfy an additional differential equation as follows:

$$\begin{aligned} \Delta(\pi_M^{(1)}, \boldsymbol{\theta}) \;=\; & \frac{\partial^2}{\partial\theta_1^2}(I_{11}^{-\frac{1}{2}} g(\theta_2, \cdots, \theta_{p+2})) \\ & - \sum_{r=2}^{p+2}\sum_{s=2}^{p+2} \frac{\partial}{\partial\theta_s}(L_{11r} I^{11} I^{rs} I_{11}^{\frac{1}{2}} g(\theta_2, \cdots, \theta_{p+2})) \\ & - \frac{1}{3}\frac{\partial}{\partial\theta_1}(L_{111}(I^{11})^2 I_{11}^{\frac{1}{2}} g(\theta_2, \cdots, \theta_{p+2})) = 0, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} L_{111} &= E(\frac{\partial^3 L}{\partial\theta_1^3}) = -\frac{q(p-1)}{\theta_1^3} + \frac{3(p-1)(q-1)}{\theta_1^3} - \frac{3(p-1)}{\theta_1^3}, \\ L_{112} &= L_{113} = \cdots = L_{11(p+2)} = 0, \end{aligned}$$

and I^{ij} is (i,j) element of

$$I_2^{-1} = \begin{pmatrix} \frac{2\theta_1^2}{q(p-1)} & 0 & \mathbf{0}' \\ 0 & \frac{2\theta_2^2}{q} & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \frac{\theta_1}{q} I_p + \frac{\theta_2}{q} J_p \end{pmatrix}.$$

Then every differentiable function g satisfies (4.7). That is, all the first order probability matching priors are the second order probability matching priors. Thus the resulting second order probability matching priors for ψ are given by

$$\pi_M^{(2)}(\psi, \lambda, \boldsymbol{\eta}) \propto \psi^{-1} \cdot g(\lambda, \boldsymbol{\eta})$$

and the second order probability matching priors for σ^2 are

$$\pi_M^{(2)}(\sigma^2, \sigma_\beta^2, \boldsymbol{\mu}) \propto (\sigma^2)^{-1} \cdot g(\sigma^2 + p\sigma_\beta^2, \boldsymbol{\mu})$$

This completes the proof.

Remark 4.1. There are infinitely many matching priors for σ^2 up to $o(n^{-1})$. Among the reference priors developed in Section 2 π_2 is the only matching prior.

5. Simulation and Discussion

Welch and Peers(1963) proved that, using Jeffreys' prior the difference between the frequentist coverage of the αth posterior quantile and α is the order of $o(1/n)$ when n gets large. A popular property is that the frequentist coverage probability of $(1-\alpha)th$ posterior quantile showed be close to $1-\alpha$.

Table 5.1 provides frequentist coverage probabilities of 0.05(0.95) posterior quantile for σ^2 under four reference priors for different value of p and q. The computation of those numerical values is based on the following algorithm for any fixed true σ^2 and prespecified probabilty value α. Here α is 0.05(0.95). Let $(\sigma^2)^{\pi}(\alpha|\boldsymbol{y})$ be the posterior α-quantile of σ^2 given $\boldsymbol{y}$. That is to say, $F((\sigma^2)^{\pi}(\alpha|\boldsymbol{y})|\boldsymbol{y}) = \alpha$, where $F(\cdot|\boldsymbol{y})$ is the marginal posterior distribution of σ^2. Then the frequentist coverage probability of this one sided credible interval of σ^2 is

$$P_{\sigma^2}(\alpha:\sigma^2) = P_{\sigma^2}(0<\sigma^2 \le (\sigma^2)^{\pi}(\alpha|\boldsymbol{y})).$$

Table 5.1. Frequentist coverage probabilities of 0.05(0.95) **posterior quantiles for** σ^2

(p, q)	π_1	π_2	π_3	π_4
(2, 8)	0.02(0.92)	0.03(0.93)	0.02(0.92)	0.01(0.75)
(3, 4)	0.03(0.94)	0.05(0.95)	0.05(0.93)	0.03(0.89)
(4, 3)	0.05(0.97)	0.04(0.97)	0.01(0.80)	0.01(0.76)
(5, 5)	0.05(0.96)	0.05(0.95)	0.02(0.85)	0.01(0.73)
(5, 6)	0.03(0.93)	0.04(0.94)	0.03(0.90)	0.01(0.74)
(6, 5)	0.04(0.95)	0.05(0.95)	0.03(0.87)	0.01(0.77)
(5, 10)	0.03(0.90)	0.05(0.95)	0.02(0.85)	0.01(0.78)
(8, 2)	0.03(0.93)	0.06(0.98)	0.02(0.80)	0.01(0.70)
(10,10)	0.03(0.95)	0.04(0.95)	0.01(0.85)	0.02(0.85)

The estimated $P_{\sigma^2}(\alpha:\sigma^2)$ when $\alpha = 0.05(0.95)$ is shown in Table 5.1. Actually Table 5.1 was computed in the following way. For fixed σ^2, we take 10.000 independent random samples of $\boldsymbol{y}$ from the model (1.1). Note that under the prior

π, for fixed $\boldsymbol{y}$, $\sigma^2 \leq (\sigma^2)^{\pi}(\alpha|\boldsymbol{y})$ if and only if $F((\sigma^2)^{\pi}(\alpha|\boldsymbol{y})|\boldsymbol{y}) \leq \alpha$. Under the prior π, $P_{\sigma^2}(\alpha:\sigma^2)$ can be estimated by the relative frequency of $F((\sigma^2)^{\pi}|\boldsymbol{y}) \leq \alpha$. From the Table 5.1, π_2 is the most appealing reference prior distribution in the sense of the asymptotic frequentist coverage probability.

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