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Unified Estimations for Parameter Changes in a Generalized Uniform Distribution

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Abstract

We shall propose several estimators for the shape and scale parameters in a generalized uniform distribution when both parameters are polynomial of a known exposure level, and obtain expectations and variances for their proposed estimators. And we shall compare numerically efficiencies for the several proposed estimators for the shape and scale parameters in a generalized uniform distribution in the small sample sizes.

Keywords : Efficiency, Generalized uniform, Parameter Change.

1. Introduction

A random variable X is said to have a generalized uniform distribution (see, Tiwari et al(1996)) if its density function is of the form

$$f(x:\alpha,\beta)=\frac{\alpha+1}{\beta^{\alpha+1}}x^{\alpha},\qquad 0<x<\beta,\quad -1<\alpha, \tag{1.1}$$

where α and β are referred as the shape and the scale parameters, respectively. It is denoted by $X\sim GUNIF(\alpha,\beta)$.

Proctor(1987) introduced the four parameter generalized uniform distribution which is a counterpart to Burr type XII distribution. And Tiwari, Yang & Zalkikar(1996) studied Bayes estimation for parameters in the Pareto distribution

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using the generalized uniform distribution. Lee(2000) studied the estimations for parameters in the generalized uniform distribution.

The purpose of this work is to estimate the effects on the shape and scale parameters in a generalized uniform distribution when both parameters are polynomials of a known exposure level t.

Woo & Yoon(1990) considered unified estimations for parameter changes in a two parameters Pareto distribution. Woo & Ali(1994) considered the jackknife parametric estimations in the exponential distribution when its scale and location parameters change functions of environment dosage. And Woo & Lee(2000) considered applications of the Weibull distribution to the strength of materials when its shape and scale parameters are functions of a known exposure level.

In this paper, we shall propose several estimators for the shape and the scale parameters in a generalized uniform distribution when both parameters are polynomials of a known exposure level t, and obtain mean and variances for their proposed estimators. And we shall compare numerically efficiencies for the several proposed estimators for the shape and scale parameters in a generalized uniform distribution.

2. Estimates for Parameter Changes

Assume $X_{1j}, \cdots, X_{n_j j}$ be a simple random samples taken from $X_j \sim GUNIF(\alpha(t_j), \beta(t_j))$, $j = 1, \cdots, r+1$, $t_i \neq t_k$ for $i \neq k$ and $X_1, \cdots, X_{r+1}$ be independent. And Let $X_{(1)j}, \cdots, X_{(n_j)j}$ be corresponding the order statistics for $X_{1j}, \cdots, X_{n_j j}$.

We shall consider unified estimations for the parameter change of exposure levels or times in the generalized uniform distribution even when $\alpha(t)$ and $\beta(t)$ are polynomials of t ;

$$\alpha(t) = a_0 + a_1 t + \cdots + a_r t^r > -1, \text{ for all } t \text{ and}$$

$$\beta(t) = b_0 + b_1 t + \cdots + b_r t^r, \quad t > 0 \text{ and } b_i > 0, \text{ for all } i = 0, 1, \cdots, r.$$

Define the following notation :

$$\det[t_i^0, \cdots, t_i^r] = \begin{vmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^r \\ 1 & t_2 & t_2^2 & \cdots & t_2^r \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & t_{r+1} & t_{r+1}^2 & \cdots & t_{r+1}^r \end{vmatrix}.$$

By the maximum likelihood method, we can obtain MLE's for a_j and b_j, $j = 0, 1, \cdots, r$, as follows ;

$$\widehat{a}_j^{(1)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \left\{\frac{1}{n_i}\sum_{k=1}^{n_i}\ln(X_{(n_i)i}/X_{ki})\right\}^{-1} - 1, t_i^{j+1}, \cdots, t_i^r\right]}{\det[t_i^0, \cdots, t_i^r]},$$

$$\widehat{b}_j^{(1)} = \frac{\det[t_i^0, \cdots, t_i^{j-1}, X_{(n_i)i}, t_i^{j+1}, \cdots, t_i^r]}{\det[t_i^0, \cdots, t_i^r]}.$$

Note that $-\ln(X_{ki}/\beta(t_i))$ has an exponential distribution with mean $(\alpha(t_i)+1)^{-1}$. Therefore, it is well known that $(\alpha(t_i)+1)\cdot\sum_{k=1}^{n_i}\ln(X_{(n_i)i}/X_{ki})$ has a Gamma distribution with a shape parameter $n_i - 1$ and a scale parameter 1 (see Johnson et al.(1995)). And note that

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{k1}A_{k1} + a_{k2}A_{k2} + \cdots + a_{kn}A_{kn}, \qquad (2.1)$$

where $A_{kj} = (-1)^{k+j}D_{kj}$ and D_{kj} is minor determinant for a_{kj} eliminated k-row and j-column in the determinant, $\det[t_i^0, \cdots, t_i^r]$..

Therefore, we can obtain the expectations and variances for $\widehat{a}_j^{(1)}$ and $\widehat{b}_j^{(1)}$, $j = 0, 1, \cdots, r$, as follows ;

$$E(\widehat{a}_j^{(1)}) = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \frac{n_i\cdot\alpha(t_i)+2}{n_i-2}, t_i^{j+1}, \cdots, t_i^r\right]}{\det[t_i^0, \cdots, t_i^r]},$$

$$E(\widehat{b}_j^{(1)}) = \sum_{k=0}^{r} b_k \cdot \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \frac{n_i(\alpha(t_i)+1)}{n_i(\alpha(t_i)+1)+1}\cdot t_i^k, t_i^{j+1}, \cdots, t_i^r\right]}{\det[t_i^0, \cdots, t_i^r]},$$

$$VAR(\widehat{a_j}^{(1)}) = \sum_{k=1}^{r+1} \frac{\det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i \neq k}}{\det^2[t_i^0, \cdots, t_i^r]} \cdot \frac{n_k^2(\alpha(t_k)+1)^2}{(n_k-2)^2(n_k-3)},$$

and

$$VAR(\widehat{b_j}^{(1)}) = \sum_{k=1}^{r+1} \frac{\det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i \neq k}}{\det^2[t_i^0, \cdots, t_i^r]}$$
$$\cdot \frac{n_k(\alpha(t_k)+1)}{(n_k(\alpha(t_k)+1)+1)^2(n_k(\alpha(t_k)+1)+2)} \cdot \beta^2(t_k), \quad (2.2)$$

where $\det[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i \neq k}$ is a minor determinant eliminated k-row and $(j+1)$-column in the determinant, $\det[t_i^0, \cdots, t_i^r]$.
Since the UMVUE's of the shape parameter $\alpha(t)$ and the scale parameter $\beta(t)$ in the generalized uniform distribution (see, Lee(2000)) are given by

$$\widehat{\alpha(t)}_U = \frac{n-2}{\sum_{i=1}^{n} \ln(X_{(n)}/X_i)} - 1,$$

$$\widehat{\beta(t)}_U = [1 + \frac{\sum_{i=1}^{n} \ln(X_{(n)}/X_i)}{n(n-1)}] \cdot X_{(n)},$$

we can propose the following estimators for a_j and b_j, $j = 0, 1, \cdots, r$, ;

$$\widehat{a_j}^{(2)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \left\{\frac{1}{n_i-2}\sum_{k=1}^{n_i} \ln(X_{(n_i)i}/X_{ki})\right\}^{-1} - 1, t_i^{j+1}, \cdots, t_i^r\right]}{\det[t_i^0, \cdots, t_i^r]},$$

$$\widehat{b_j}^{(2)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \{\frac{1}{n_i(n_i-1)}\sum_{k=1}^{n_i} \ln(X_{(n_i)i}/X_{ki}) + 1\} \cdot X_{(n_i)i}, t_i^{j+1}, \cdots, t_i^r\right]}{\det[t_i^0, \cdots, t_i^r]}.$$

Since $(\alpha(t_i)+1)\sum_{k=1}^{n_i} \ln(X_{(n_i)i}/X_{ki})$ and $X_{(n_i)i}$ are independent (see Johnson et al.(1995)), we can obtain expectations and variances for $\widehat{a_j}^{(2)}$ and $\widehat{b_j}^{(2)}$, $j = 0, 1, \cdots, r$, as follows ;

$$E(\widehat{a_j}^{(2)}) = a_j, \qquad E(\widehat{b_j}^{(2)}) = b_j,$$

$$VAR(\widehat{a_j}^{(2)}) = \sum_{k=1}^{r+1} \frac{\det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i\neq k}}{\det^2[t_i^0, \cdots, t_i^r]} \cdot \frac{(\alpha(t_k)+1)^2}{(n_k-3)}, \qquad (2.3)$$

and

$$VAR(\widehat{b_j}^{(2)}) = \sum_{k=1}^{r+1} \frac{\det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i\neq k}}{\det^2[t_i^0, \cdots, t_i^r]}$$

$$\cdot \frac{1}{(n_k-1)(\alpha(t_k)+1)(n_k(\alpha(t_k)+1)+2)} \cdot \beta^2(t_k).$$

Next, the minimum risk estimator for the shape parameter $\alpha(t)$ among $c \cdot (\sum_{k=1}^{n} \ln(X_{(n)}/X_k))^{-1} - 1$, and the modified MLE for the scale parameter $\beta(t)$ in the generalized uniform distribution are given by ;

$$\widehat{\alpha(t)}_R = \frac{n-3}{\sum_{k=1}^{n} \ln(X_{(n)}/X_k)} - 1, \qquad \widehat{\beta(t)}_{MM} = [1 + \frac{\sum_{k=1}^{n} \ln(X_{(n)}/X_k)}{n^2}]X_{(n)}.$$

we can propose the following estimators for a_j and b_j, $j = 0, 1, \cdots, r$, ;

$$\widehat{a_j}^{(3)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \left\{\frac{1}{n_i-3}\sum_{k=1}^{n_i} \ln(X_{(n_i)i}/X_{ki})\right\}^{-1} - 1, t_i^{j+1}, \cdots, t_i^r\right]}{\det[t_i^0, \cdots, t_i^r]},$$

$$\widehat{b_j}^{(3)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \left\{1 - \frac{1}{n_i^2}\sum_{k=1}^{n_i} \ln(X_{(n_i)i}/X_{ki})\right\} \cdot X_{(n_i)i}, t_i^{j+1}, \cdots, t_i^r\right]}{\det[t_i^0, \cdots, t_i^r]}.$$

Since $(\alpha+1)\sum_{i=1}^{n} \ln(X_{(n)}/X_i)$ has a Gamma distribution with a shape parameter *n-1* and a scale parameter *1*, and $(\alpha+1)\sum_{i=1}^{n} \ln(X_{(n)}/X_i)$ and $X_{(n)}$ are independent, we can obtain the expectations and variances for $\widehat{a_j}^{(3)}$ and $\widehat{b_j}^{(3)}$ as follows ;

$$E(\widehat{a}_j^{(3)}) = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \left\{\frac{(n_i-3)\cdot(\alpha(t_i)+1)}{n_i-2} - 1\right\}, t_i^{j+1}, \cdots, t_i^r\right]}{\det[t_i^0, \cdots, t_i^r]},$$

$$E(\widehat{b}_j^{(3)}) = \sum_{k=0}^{r} b_k \cdot \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \frac{n_i[n_i(\alpha(t_i)+1)+1]-1}{n_i[n_i(\alpha(t_i)+1)+1]}\cdot t_i^k, t_i^{j+1}, \cdots, t_i^r\right]}{\det[t_i^0, \cdots, t_i^r]},$$

$$VAR(\widehat{a}_j^{(3)}) = \sum_{k=1}^{r+1} \frac{\det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i\neq k}}{\det^2[t_i^0, \cdots, t_i^r]} \cdot \frac{(n_k-3)(\alpha(t_k)+1)^2}{(n_k-2)^2}, \tag{2.4}$$

and

$$VAR(\widehat{b}_j^{(3)}) = \sum_{k=1}^{r+1} \frac{\det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i\neq k}}{\det^2[t_i^0, \cdots, t_i^r]} \cdot \left[\frac{1}{n_k[n_k(\alpha(t_k)+1)+1]}\left\{2 - \frac{1}{n_k[n_k(\alpha(t_k)+1)+1]}\right\} - \frac{2n_k(\alpha(t_k)+1) - n_k + 1}{n_k^2(\alpha(t_k)+1)[n_k(\alpha(t_k)+1)+2]}\right]\beta^2(t_k).$$

From the results (2.2) through (2.4), we can obtain the followings.

Fact. (a) $\widehat{a}_j^{(2)}$ and $\widehat{b}_j^{(2)}$ are unbiased and MSE consistent estimators for a_j and b_j, respectively.

(b) $\widehat{a}_j^{(i)}$ and $\widehat{b}_j^{(i)}$, $i = 1, 3$, are asymptotically unbiased and MSE consistent estimators for a_j and b_j, respectively.

Tables show the numerical values of MSE's for $\widehat{a}_j^{(i)}$ and $\widehat{b}_j^{(i)}$, $j = 0, 1$ and $i = 1, 2, 3$, for in an assumed generalized uniform distribution for sample sizes n_1=10(10)30, n_2=10(10)30, $a_0 = 0$, $a_1 = 1$, $b_0 = 0$, $b_1 = 1$, and $t_1 = 1$, $t_2 = 2$ when $r = 1$. From Table 1, $\widehat{a}_j^{(3)}$ proposed by minimum risk estimator is more efficient than other proposed estimators for a_j, $j = 1, 2$. From Table 2, the MLE

$\widehat{b_0}^{(1)}$ is more efficient than other proposed estimators for b_0 when $n_1 > n_2$ but $\widehat{b_0}^{(3)}$ is more efficient than other proposed estimators for b_0 when $n_1 < n_2$. And $\widehat{b_1}^{(1)}$ is more efficient than other proposed estimators for b_1 when $n_1 = n_2$.

Table 1. Mean squared errors of proposed estimators for a_j in the generalized uniform distribution.

size		parameter	MSE		
n_1	n_2		$\widehat{a_j}^{(1)}$	$\widehat{a_j}^{(2)}$	$\widehat{a_j}^{(3)}$
10	10	a_0	5.64283	3.57143	2.75000
		a_1	2.96429	1.85714	1.43750
	15	a_0	4.85989	3.03571	2.46154
		a_1	1.89286	1.32143	1.07692
	20	a_0	4.66947	2.81513	2.33333
		a_1	1.57423	1.10084	0.91667
	25	a_0	4.60107	2.69481	2.26087
		a_1	1.43337	0.98052	0.82609
	30	a_0	4.57143	2.61905	2.21429
		a_1	1.35714	0.90476	0.76786
15	10	a_0	3.80220	2.61905	2.12500
		a_1	2.64835	1.61905	1.31731
	15	a_0	2.79734	2.08333	1.78107
		a_1	1.46598	1.08333	0.92899
	20	a_0	2.50830	1.86275	1.62821
		a_1	1.09804	0.86275	0.75641
	25	a_0	2.38416	1.74242	1.54181
		a_1	0.92931	0.74242	0.65886
	30	a_0	2.31868	1.66667	1.48626
		a_1	0.83516	0.66667	0.59615

size		parameter	MSE		
n_1	n_2		$\widehat{a_j}^{(1)}$	$\widehat{a_j}^{(2)}$	$\widehat{a_j}^{(3)}$
20	10	a_0	3.26424	2.22689	1.84722
		a_1	2.57796	1.52101	1.26389
	15	a_0	2.16076	1.69118	1.47863
		a_1	1.34628	0.98529	0.86325
	20	a_0	1.82789	1.47059	1.31481
		a_1	0.95643	0.76471	0.68519
	25	a_0	1.67898	1.35027	1.22222
		a_1	0.77531	0.64439	0.58454
	30	a_0	1.59757	1.27451	1.16270
		a_1	0.67320	0.56863	0.51984
25	10	a_0	3.02993	2.01299	1.69022
		a_1	2.55562	1.46753	1.23370
	15	a_0	1.87071	1.47727	1.30769
		a_1	1.29606	0.93182	0.82609
	20	a_0	1.51306	1.25668	1.13768
		a_1	0.89382	0.71123	0.64493
	25	a_0	1.35015	1.13636	1.04159
		a_1	0.70571	0.59091	0.54253
	30	a_0	1.25974	1.06061	0.97981
		a_1	0.59910	0.51515	0.47671
30	10	a_0	2.90476	1.87831	1.58929
		a_1	2.54762	1.43386	1.21429
	15	a_0	1.70971	1.34259	1.19780
		a_1	1.27015	0.89815	0.80220
	20	a_0	1.33613	1.12200	1.02381
		a_1	0.85994	0.67756	0.61905
	25	a_0	1.16422	1.00168	0.92547
		a_1	0.66733	0.55724	0.51553
	30	a_0	1.06803	0.92593	0.86224
		a_1	0.55782	0.48148	0.44898

Table 2. Mean squared errors of proposed estimators for b_j in the generalized uniform distribution.

size		parameter	MSE		
n_1	n_2		$\widehat{b_j}^{(1)}$	$\widehat{b_j}^{(2)}$	$\widehat{b_j}^{(3)}$
10	10	b_0	0.01309	0.01473	0.01434
		b_1	0.00624	0.00715	0.00697
	15	b_0	0.01273	0.01212	0.01187
		b_1	0.00388	0.00455	0.00446
	20	b_0	0.01318	0.01123	0.00110
		b_1	0.00332	0.00365	0.00359
	25	b_0	0.01367	0.01082	0.01061
		b_1	0.00318	0.00324	0.00318
	30	b_0	0.01408	0.01060	0.01039
		b_1	0.00319	0.00302	0.00297
15	10	b_0	0.00780	0.00909	0.00897
		b_1	0.00591	0.00574	0.00563
	15	b_0	0.00615	0.00649	0.00641
		b_1	0.00291	0.00314	0.00310
	20	b_0	0.00594	0.00559	0.00554
		b_1	0.00201	0.00224	0.00222
	25	b_0	0.00603	0.00518	0.00513
		b_1	0.00168	0.00183	0.00182
	30	b_0	0.00618	0.00496	0.00491
		b_1	0.00155	0.00161	0.00160

size		parameter	MSE		
n_1	n_2		$\widehat{b_j}^{(1)}$	$\widehat{b_j}^{(2)}$	$\widehat{b_j}^{(3)}$
20	10	b_0	0.00641	0.00713	0.00709
		b_1	0.00607	0.00525	0.00516
	15	b_0	0.00410	0.00453	0.00449
		b_1	0.00274	0.00265	0.00262
	20	b_0	0.00356	0.00363	0.00361
		b_1	0.00167	0.00175	0.00174
	25	b_0	0.00344	0.00322	0.00320
		b_1	0.00124	0.00135	0.00134
	30	b_0	0.00345	0.00300	0.00298
		b_1	0.00104	0.00113	0.00112
25	10	b_0	0.00602	0.00623	0.00613
		b_1	0.00628	0.00503	0.00494
	15	b_0	0.00330	0.00362	0.00360
		b_1	0.00274	0.00242	0.00240
	20	b_0	0.00256	0.00273	0.00272
		b_1	0.00158	0.00153	0.00152
	25	b_0	0.00232	0.00232	0.00231
		b_1	0.00108	0.00112	0.00111
	30	b_0	0.00224	0.00210	0.00209
		b_1	0.00084	0.00090	0.00089
30	10	b_0	0.00594	0.00572	0.00565
		b_1	0.00647	0.00490	0.00482
	15	b_0	0.00296	0.00313	0.00311
		b_1	0.00280	0.00230	0.00228
	20	b_0	0.00208	0.00224	0.00223
		b_1	0.00156	0.00141	0.00140
	25	b_0	0.00175	0.00183	0.00182
		b_1	0.00103	0.00100	0.00099
	30	b_0	0.00163	0.00161	0.00160
		b_1	0.00076	0.00077	0.00077

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