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Estimation of the Number of Change-Points with Local Linear Fit[1)]

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Abstract

The aim of this paper is to consider of detecting the location, the jump size and the number of change-points in regression functions by using the local linear fit which is one of nonparametric regression techniques. It is obtained the asymptotic properties of the change points and the jump sizes. and the corresponding rates of convergence for change-point estimators.

Keywords : Local linear regression fit; Change-point; Kernel smoother.

1. 서론

Nonparametric regression techniques are generally used in order to obtain a smooth fit of regression function whenever there is no suitable parametric model available. Sometimes a generally smooth function might contain some isolated discontinuity or multiple change points in the function or in a derivative, and in many cases interest focuses on the occurrence of such change points. The analysis of change points usually occurring in economics, engineering medicine and biological sciences has recently found increasing interest.

The purpose of this paper is to obtain asymptotic distributions and corresponding rates of convergence for change-point estimators. This results is very important to suggest the methods for testing and estimation to detect the location and size of change-points in regression function by using the local linear fit

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In the literature, testing and estimation about changes in the nonparametric regression function has been studied by many authors. Muller(1992) gave estimators for location and size of change-points in nonparametric regression based on a comparison of a left and right one-sided kernel smoothers. Loard(1996) proposed an estimate of the location of discontinuity based on one-side nonparametric regression estimates of the mean function. Chen and Gupta(1997) studied testing and locating variance change points with application to stock price using the Schwarz information criterion.

2. Estimates of location and jump size of change points

The nonparametric regression model considered in this paper is given by

$$Y_i = m(x_i) + \varepsilon_i, \quad x_i \in [0,1], \ i = 1,2,\ldots,n, \tag{2.1}$$

where x_i are fixed design points and ε_i are *iid* errors with mean 0 and variance $\sigma^2 < \infty$. The design points x_i are assumed to be equidistant, i. e. $x_i = i/n$. m is the unknown regression function defined on the interval [0, 1].

The regression estimators we consider are based on the local least squares fitting of kernel weighted polynomial regression function. The locally weighted polynomial regression estimator of m is $\widehat{\beta_0}$, the solution for β_0 to minimize the kernel weighted local-likelihood function

$$\sum_{i=1}^{n} \left\{ Y_i - \sum_{l=0}^{p} \beta_l (x_i - x) \right\}^2 K \left(\frac{x_i - x}{h} \right) \tag{2.2}$$

Assume that some change-points exist for m in the following sense. There exists $g \in C^2([0,1])$ such that

$$m(\cdot) = g(\cdot) + \sum_{i=1}^{nc} \triangle_i 1_{(\cdot \geq c_i)}, \tag{2.3}$$

where c_i, nc are unknown and $g(\cdot)$ has a bounded second derivative.

Let K_+ and K_- be one-sided kernel functions with support $(K_+) \in [-1,0]$ and support $(K_-) \in [0,1]$. For simplicity, let $\mu_l(F) = \int_{-\infty}^{\infty} y^l F(y)dy$. The kernel $K_-(\cdot)$ is $(\mu - 1)$ times differentiable on R and $K_-^{(\mu-1)}$ is absolutely continuous density function satisfying $\mu_1(K) = \mu_3(K) = 0$, $\mu_2(K) > 0$, $K_-^{(j)}(0) = K_-^{(j)}(1) = 0$, $0 \leq j < \mu$, and $K_+(x) = K_-(-x)$. Define $m_+^{(j)}(x_0) = \lim_{x \downarrow x_0} m^{(j)}(x)$, $m_-^{(j)}(x_0) = \lim_{x \uparrow x_0} m^{(j)}(x)$, $j = 0,1,2$.

$$\triangle(x_0) = m_+(x_0) - m_-(x_0). \tag{2.4}$$

Then the following holds, for any $x_0 \in [0, 1]$,

$m_+(x_0) = m(x_0)$, $m_{\pm}^{(j)} = g^{(j)}(x_0)$, $j = 1, 2$.

$$\triangle(x_0) = \begin{cases} \triangle_i & \text{if } x_0 = c_i, \\ 0 & \text{otherwis}. \end{cases}$$

Define one-sided regression estimates of $m(x)$,

$$\widehat{m}_{\pm}(x) = \frac{\sum W_j^{\pm} Y_j}{\sum W_j^{\pm}}, \tag{2.5}$$

where

$$W_j^{\pm} = \left(\frac{x - x_j}{h}\right)\left(S_{n,2}^{\pm} - (x - x_j)\, S_{n,1}^{\pm}\right),$$

$$S_{nl}^{\pm} = \sum K_{\pm}\left(\frac{x - x_j}{h}\right)(x - x_j)^l, \quad l = 0, 1, 2.$$

Inference for change points will be based on the following estimates

$$\widehat{\triangle}(x) = \widehat{m}_+(x) - \widehat{m}_-(x). \tag{2.6}$$

where x_j is called to be a change point if $|\widehat{\triangle}(x_j)| > C_\alpha$, $j = 1, 2, \ldots, n$, for some constant C_α at a given significant level $1 - \alpha$ where C_α will be given below.

Observe the following before investigating bias and variance of $\widehat{\triangle}(x)$.

Proposition 2.1 Define $S_{nl}^{\pm} = \sum K_{\pm}\left(\frac{x - x_j}{h}\right)(x - x_j)^l$, $l = 0, 1, 2$..

(1) $S_{nl}^{\pm} = nh^{l+1}\left(\int K_{\pm}(u) u^l dy + O(1/n)\right) = nh^{l+1}[\mu_l(K_-) + O(1/n)]$.

(2) $\sum W_j^{\pm} = S_{n,0}^{\pm} S_{n,2}^{\pm} - (S_{n,1}^{\pm})^2 = n^2 h^4 [\mu_2(K_-) + O(1/n)]$.

(3) $$\sum (W_j^{\pm})^2 = (S_{n,2}^{\pm})^2 \sum K_{\pm}^2\left(\frac{x - x_j}{h}\right) + (S_{n,1}^{\pm})^2 \sum K_{\pm}^2\left(\frac{x - x_j}{h}\right)(x - x_j)^2$$
$$- 2 S_{n,1}^{\pm} S_{n,2}^{\pm} \sum K_{\pm}^2\left(\frac{x - x_j}{h}\right)(x - x_j)$$
$$= n^3 h^7 [\mu_0^2(K_{\pm}^2) + O(1/n)].$$

Now, we consider the bias and variance of $\widehat{\triangle}(x_0)$, $x_0 \in [h, 1 - h]$ in the following.

Proposition 2.2 Assume that the jump size is $\triangle(x_0)$ at point x_0.

(1) Bias : $E\left(\widehat{\triangle}(x_0) - \triangle(x_0)\right) = o(h^2)$.

(2) Variance : $\mathrm{Var}(\widehat{\triangle}(x_0)) = \dfrac{2\sigma^2}{nh}(\mu_0(K_-^2) + o(1))$.

***Proof* (1).**

$$\begin{aligned}
E&\left[\widehat{\triangle}(x_0) - \triangle(x_0)\right] \\
&= E\left[\widehat{m}_+(x_0) - \widehat{m}_-(x_0)\right] - E\left[m_+(x_0) - m_-(x_0)\right] \\
&= E\left[\widehat{m}_+(x_0) - m_+(x_0)\right] - E\left[\widehat{m}_-(x_0) - m_-(x_0)\right] \\
&= E\frac{\sum W_j^+(Y_j - m_+(x_0))}{\sum W_j^+} - E\frac{\sum W_j^-(Y_j - m_-(x_0))}{\sum W_j^-} \\
&= \frac{1}{n^2h^4[\mu_2(K_-) + O(1/n)]}\left[\sum\{m(x_j) - m_+(x_0)\}W_j^+ - \sum\{m(x_j) - m_-(x_0)\}W_j^-\right]
\end{aligned}$$

by Proposition 2.1-(2). Let now be $R_\pm(x_j) = m(x_j) - m_\pm(x_0) - (x_j - x_0)m'_\pm(x_0)$.

Since

$$\begin{aligned}
\sum(x_j - x_0)W_j^\pm &= \sum(x_j - x_0)K_\pm\left(\frac{x - x_j}{h}\right)(S_{n,2}^\pm - (x - x_j)S_{n,1}^\pm) \\
&= -S_{n,2}^\pm S_{n,1}^\pm + S_{n,2}^\pm S_{n,1}^\pm = 0,
\end{aligned}$$

$$\sum\{m(x_j) - m_\pm(x_0)\}w_j^\pm = \sum\left\{m(x_j) - m_\pm(x_0) - (x_j - x_0)m'_\pm(x_0)\right\}w_j^\pm$$

and hence we have

$$\begin{aligned}
\frac{1}{n}&\sum R_+(x_j)K_+\left(\frac{x_0 - x_j}{h}\right) \\
&= \int\left\{m(x) - m_+(x_0) - (x - x_0)m'_+(x)\right\}K_+\left(\frac{x_0 - x}{h}\right)dx + O(1/n) \\
&= h\int_{-1}^{0}\left\{m(x_0 - hu) - m_+(x_0) + hu\,m'_+(x_0)\right\}K_+(u)\,du + O(1/n) \\
&= h\int_{-1}^{0}\Big(m_+(x_0) - hu\,m'_+(x_0) + \frac{h^2u^2}{2}m''_+(x_0) - m_+(x_0) + hu\,m'_+(x_0) \\
&\qquad + o(h^2u^2)\Big)K_+(u)\,du + O(1/n) \\
&= \frac{h^3}{2}m''_+(x_0)\mu_2(K_-) + o(h^3) + O(1/n).
\end{aligned}$$

$$\frac{1}{n} \sum R_{+}(x_j)(x_0 - x_j) K_{+}\left(\frac{x_0 - x_j}{h}\right)$$

$$= \int \{m(x) - m_{+}(x_0) - (x - x_0) m_{+}'(x)\}(x_0 - x_j) K_{+}\left(\frac{x_0 - x}{h}\right) dx + O(1/n)$$

$$= h \int_{-1}^{0} \{m(x_0 - hu) - m_{+}(x_0) + hu\, m_{+}'(x_0)\} hu\, K_{+}(u)\, du + O(1/n)$$

$$= h^2 \int_{-1}^{0} (m_{+}(x_0) - hum_{+}'(x_0) + \frac{h^2u^2}{2} m_{+}''(x_0) - m_{+}(x_0) + hum_{+}'(x_0)$$

$$+ o(h^2u^2))\, u\, K_{+}(u)\, du + O(1/n)$$

$$= \frac{h^4}{2} m_{+}''(x_0)\mu_3(K_{+}) + o(h^4) + O(1/n).$$

Thus it follows from Proposition 2.1-(1), (2) and (3) that

$$\sum (m(x_j) - m_{+}(x_0))\, W_j^{+} = \sum R_{+}(x_j) W_j^{+} = \frac{n^2h^6}{2} [m_{+}''(x_0)\mu_2^2(K_{-}) + o(1)].$$

Similarly, $\sum (m(x_j) - m_{-}(x_0))\, W_j^{-} = \frac{n^2h^6}{2} [m_{+}''(x_0)\, \mu_2^2(K_{-}) + o(1)].$

Therefore

$$E\left(\widehat{\triangle}(x_0) - \triangle(x_0) \right) = \frac{1}{n^2h^4 \{\mu_2(K_{-}) + O(1/n)\}} \left(\sum R_{+}(x_j) W_j^{+} - \sum R_{-}(x_j) W_j^{-}\right)$$

$$= o(h^2).$$

Proof **(2).** First, note that $\mathrm{Cov}(\widehat{m}_{+}(x_0), \widehat{m}_{-}(x_0)) = 0$ since ε_j's are independent and $(j : -1 \le (x_0 - x_j)/h < 0)$ and $(j : 0 \le (x_0 - x_j)/h < 1)$ are disjoint. Thus, $\mathrm{Var}(\widehat{\triangle}(x_0)) = \mathrm{Var}(\widehat{m}_{+}(x_0)) + \mathrm{Var}(\widehat{m}_{-}(x_0))$. Since it now follows from Proposition 2.1-(2) and (3) that

$$\mathrm{Var}(\widehat{m}_{\pm}(x_0)) = \frac{\sigma^2 \sum (W_j^{\pm})^2}{(\sum W_j^{\pm})^2} = \frac{\sigma^2}{nh} [\mu_0(K_{\pm}^2) + o(1)],$$

$$\mathrm{Var}(\widehat{\triangle}(x_0)) = \frac{2\sigma^2}{nh} (\mu_0(K_{-}^2) + o(1)).$$

3. Main Results

In this section, we consider the asymptotic distributions and corresponding rates of convergence for change-point estimators.

Lemma 3.1. For fixed $x_0 \in (0,1)$ as $n \to \infty$, $h \to 0$ in such a way that $nh \to \infty$

$$CK_n(x_0) \equiv \frac{\widehat{\triangle}(x_0) - \triangle(x_0)}{\hat{\sigma}_n \sqrt{V_n}} \rightarrow N(0,1), \tag{3.1}$$

where $\widehat{\sigma_n}$ is a consistent estimate of σ and

$$V_n = \sum (W_j^-)^2 / \sum (W_j^-)^2 + \sum (W_j^-)^2 / (\sum W_j^-)^2.$$

In addition,

$$\frac{\sqrt{nh}\,(\widehat{\triangle}(x_0) - \triangle(x_0))}{\hat{\sigma}_n \sqrt{2\,\mu_0(K_-^2)}} \xrightarrow{d} N(0,1). \tag{3.2}$$

Proof. For simplicity, let a_j denote $a_j = W_j^+ / \sum W_j^+ - W_j^- / \sum W_j^-$. Then $\sum_j a_j^2 = V_n$ and hence CK_n can be rewritten as $\sum_j a_j \varepsilon_j / \sqrt{\widehat{\sigma_n}^2 \sum_j a_j^2}$. Since it is shown that $\widehat{\sigma_n}^2 \rightarrow \sigma^2$, it is sufficient to prove (3.1) with respect to CK_n with σ in place of $\hat{\sigma}_n$. It follows, in Proposition 2.1-(2) and (3), that $\max_{1 \le j \le n} \frac{a_j^2}{\sum a_j^2} = O(\frac{1}{nh})$ and hence Lindberg's condition is satisfied. Therefore asymptotic normality of CK_n follows. Since $\sum a_j^2 = 2\mu_0(K_-^2)/(nh) + o(1/(nh))$, the equation (3.2) follows.

Let τ denote a location of a change point of $m(x)$. Assume that we know A is a closed neighborhood of τ in which τ is only one change point of $m(x)$. Define the estimator

$$\hat{\tau} = \inf\{\rho \in A : \widehat{\triangle}(\rho) = \sup_{x \in A} \widehat{\triangle}(x)\}$$

for the location of a change point τ. Now, we investigate limiting properties of $\hat{\tau}$. In order to do that define

$$\hat{\delta}(y) = \widehat{\triangle}(\tau + yh) = \widehat{m}_+(\tau + yh) - \widehat{m}_-(\tau + yh)$$

and define for some $0 < M < \infty$, $-M \le z \le M$, the sequence of stochastic processes

$$\zeta_n(z) = (nh)^{(\mu+1)/(2\mu)} \left(\hat{\delta}\left(\frac{z}{(nh)^{1/(2\mu)}}\right) - \hat{\delta}(0)\right).$$

Note that

$$\sum K_\pm\left(\frac{x - x_j}{h}\right) m(x_j)\, S_{n,2}^{\pm} = n^2 h^4 \mu_2(K_-) \int K_\pm(u)\, m(x - hu)\, du + O(nh^4),$$

$$\sum K_\pm\left(\frac{x - x_j}{h}\right)(x - x_j)\, m(x_j)\, S_{n,1}^{\pm} = O(nh^4).$$

Thus we have $E\{\widehat{m}_{\pm}(\tau+yh)\} = \int K_{\pm}(u)\, m(\tau+yh-uh)\,du + O(1/n)$ and hence

$$E\{\hat{\delta}(y)\} = \int_{-1}^{1}(K_{+}(u)-K_{-}(u))\,g(\tau+yh-uh)\,du + O(1/n),$$

which is the same as the equation (6.2) of Muller(1992). Therefore following the same techniques used below (6.2) of Lemma 6.1 in Muller gives

$$E\zeta_n(z) = -\triangle z^{\mu+1}K_{-}^{\mu}(0)\,/\,(\mu+1)! + o(1).$$

Now, we consider covariance of $\zeta_n(z)$. First, let $\beta=(\mu+1)/(2\mu)$, $\gamma=1/(2\mu)$,

$$S_{n,l}^{\pm}(z) = \sum K_{\pm}\left(\frac{\tau+(zh)/(nh)^{\gamma}-x_j}{h}\right)(\tau+(zh)/(nh)^{\gamma}-x_j)^{l}, \quad l=0,1,2.$$

and

$$W_{j,z}^{\pm} = K_{\pm}\left(\frac{\tau+(zh)/(nh)^{\gamma}-x_j}{h}\right)\{S_{n,2}^{\pm}(z) - (\tau+(zh)/(nh)^{\gamma}-x_j)\,S_{n,1}^{\pm}(z)\}.$$

Then

$$\zeta_n(z) - E\{\zeta_n(z)\} = (nh)^{\beta}\left[\sum_j\left(\frac{W_{j,z}^{+}}{\sum_j W_{j,z}^{+}} - \frac{W_{j,0}^{+}}{\sum_j W_{j,0}^{+}}\right)\varepsilon_j - \sum_j\left(\frac{W_{j,z}^{-}}{\sum_j W_{j,z}^{-}} - \frac{W_{j,0}^{-}}{\sum_j W_{j,0}^{-}}\right)\varepsilon_j\right].$$

Observe now for any $z\in[-M, M]$

$$S_{n,l}^{\pm}(z) = nh^{l+1}[\mu_l(K_{-}) + O(1/n)], \quad \sum w_{j,z}^{\pm} = n^2h^4[\mu_2(K_{-}) + O(1/n)].$$

Thus we have

$$\begin{aligned}
&\mathrm{Cov}(\zeta_n(z_1), \zeta_n(z_2)) \\
&= \frac{(nh)^{2\beta}\sigma^2}{n^4h^8\mu_2(K_{-})^2}\sum_j\Big[(W_{j,z_1}^{+}-W_{j,0}^{+})(W_{j,z_2}^{+}-W_{j,0}^{+}) - (W_{j,z_1}^{+}-W_{j,0}^{+})(W_{j,z_2}^{-}-W_{j,0}^{-}) \\
&\qquad - (W_{j,z_1}^{-}-W_{j,0}^{-})(W_{j,z_2}^{+}-W_{j,0}^{+}) + (W_{j,z_1}^{-}-W_{j,0}^{-})(W_{j,z_2}^{-}-W_{j,0}^{-})\Big] \\
&= \frac{(nh)^{2\beta}}{nh^2}\sigma^2\Big\{\!\int\!\Big\{\Big[K_{+}\Big(\frac{\tau+(z_{1h})/(nh)^{\gamma}-x}{h}\Big) - K_{+}\Big(\frac{\tau-x}{h}\Big)\Big] \\
&\qquad \times\Big[K_{+}\Big(\frac{\tau+(z_{2h})/(nh)^{\gamma}-x}{h}\Big) - K_{+}\Big(\frac{\tau-x}{h}\Big)\Big] \\
&\qquad - \Big[K_{+}\Big(\frac{\tau+(z_1h)/(nh)^{\gamma}-x}{h}\Big) - K_{+}\Big(\frac{\tau-x}{h}\Big)\Big] \\
&\qquad \times\Big[K_{-}\Big(\frac{\tau+(z_2h)/(nh)^{\gamma}-x}{h}\Big) - K_{-}\Big(\frac{\tau-x}{h}\Big)\Big]
\end{aligned}$$

$$+ \left[K_- \left(\frac{\tau + (z_1 h)/(nh)^\gamma - x}{h} \right) - K_+ \left(\frac{\tau - x}{h} \right) \right]$$
$$\times \left[K_- \left(\frac{\tau + (z_2 h)/(nh)^\gamma - x}{h} \right) - K_- \left(\frac{\tau - x}{h} \right) \right] \Bigg] dx + O(1/n) \Bigg\}$$

By the assumptions for $K_\pm$, then

$$K_\pm \left(\frac{\tau + (zh)/(nh)^\gamma - x}{h} \right) - K_\pm \left(\frac{\tau - x}{h} \right)$$
$$= K'_\pm \left(\frac{\tau - x}{h} \right) \frac{z}{(nh)^\gamma} + O\left(\frac{1}{(nh)^{2\gamma}} \right) \times \mathbf{1}_{\left\{ K'_\pm\left(\frac{\tau - x}{h}\right) \neq 0 \right\} \cup \left\{ K'_\pm\left(\frac{\tau + (zh)/(nh)^\gamma - x}{h}\right) \neq 0 \right\}}$$

Since $\int 1_{\left\{ K'_\pm\left(\frac{\tau - x}{h}\right) \neq 0 \right\} \cup K'_\pm\left(\frac{\tau + (zh)/(nh)^\gamma - x}{h}\right) \neq 0 \}} dx = O(h)$. $\int K'^2_\pm \left(\frac{\tau - x}{h} \right) dx = h \int K'^2_-(u)\, du$, and $\int K'_+ \left(\frac{\tau - x}{h} \right) K'_- \left(\frac{\tau - x}{h} \right) dx = 0$, then

$$\mathrm{Cov}\,(\zeta_n(z_1), \zeta_n(z_2)) = 2 z_1 z_2 \sigma^2 \int K'^2_-(u)\, du + O(1/(nh)^\gamma). \tag{3.3}$$

Let

$$a_j = (nh)^\beta \left[\left(\frac{W^+_{j,z}}{\sum_j W^+_{j,z}} - \frac{W^+_{j,0}}{\sum_j W^+_{j,0}} \right) - \left(\frac{W^-_{j,z}}{\sum_j W^-_{j,z}} - \frac{W^-_{j,0}}{\sum_j W^-_{j,0}} \right) \right].$$

Then $\zeta_n(z) - E\,\zeta_n(z) = \sum a_j \varepsilon_j$. Since $\max_{1 \le j \le n} a_j^2 / \sum a_j^2 \to 0$,

$\zeta_n(z) - E\,\zeta_n(z)$ satisfies Lindberg's condition. Thus we have asymptotic normality of $\zeta_n(z)$ as follows.

Lemma 3.2. For fixed $z \in [-M, M]$,

$$\zeta_n(z) - E\,\zeta_n(z) \xrightarrow{d} N\left(0,\ 2z^2\sigma^2 \int K'^2_-(u)\, du\right).$$

For fixed $z_1, z_2, \ldots, z_l \in [-M, M]$,

$$(\zeta_n(z_1) - E\,\zeta_n(z_1), \ \ldots,\ \zeta_n(z_l) - E\,\zeta_n(z_l)) \xrightarrow{d} N(0, \Sigma),$$

where $\Sigma = (\sigma_{i,j})$ and $\sigma_{i,j} = 2 z_i z_j \sigma^2 \int K'^2_-(u)\, du$.

By following the similar lines to the proof of Lemma 6.5 in Muller(1992), it can be shown that the sequence $\overline{\zeta_n} = \zeta_n(z) - E\,\zeta_n(z)$ is tight. This and Lemma 3.2 together imply the following.

Theorem 3.1 For fixed $z \in [-M, M]$,

$$\zeta_n \xrightarrow{w} \zeta \quad \text{on } C([-M, M]), \tag{3.4}$$

where ζ is a continuous Gaussian process with

$$E\,\zeta(z) = -\Delta z^{\mu+1} K_-^{\mu}(0)/(\mu+1).$$

$$\mathrm{Cov}(\zeta(z_1), \zeta(z_2)) = 2 z_1 z_2 \sigma^2 \int K_-^{'2}(u)\,du.$$

Note that $\zeta(z)$ in above theorem can be written equivalently as

$$\zeta(z) = -\Delta z^{\mu+1} K_-^{\mu}(0)/(\mu+1)! + Xz,$$

where $X \sim N(0, \sigma^2 \int K_-^{'2}(u)\,du)$ and ζ has a unique maximum at

$$Z^* = [x\mu!/(\Delta(\tau)\, K_-^{\mu}(0))]^{1/\mu}.$$

Let Z_n be the location of the maximum of ζ_n. Then

$$\hat{\tau} = \tau + Z_{nh}/(nh)^{1/(2\mu)}.$$

Since $Z_n \xrightarrow{d} Z^*$ by Theorem 3.1 the asymptotic normality of $\hat{\tau}$ follows.

Corollary 3.1

$$\sqrt{nh}\left(\frac{\hat{\tau}-\tau}{h}\right)^{\mu} \xrightarrow{d} N\left(0,\ 2\sigma^2\left(\frac{\mu!}{\Delta(\tau) K_-^{(\mu)}(0)}\right)^2 \int K_-^{'2}(u)\,du\right) \tag{3.6}$$

Applying the functional mapping theorem gives that $\zeta_n(Z_n) \xrightarrow{d} \zeta(Z^*)$ and hence $(nh)^{1/2}\zeta_n(Z_n)/(nh)^{(\mu+1)/(2\mu)} \to 0$. This implies $(nh)^{1/2}\{\hat{\Delta}(\hat{\tau}) - \hat{\Delta}(\tau)\} \xrightarrow{p} 0$ by the definitions of $\zeta_n(\cdot)$, Z_n, $\hat{\Delta}(\cdot)$ and $\hat{\tau}$. Moreover since $\sqrt{nh}\,\hat{\sigma}_n\sqrt{V_n} \xrightarrow{p} K$, for some finite constant K, $\{\hat{\Delta}(\hat{\tau}) - \hat{\Delta}(\tau)\}/\hat{\sigma}_n\sqrt{V_n} \xrightarrow{p} 0$. Therefore combining this with Lemma 3.1, we have the asymptotic normality of $\hat{\Delta}(\hat{\tau})$.

Corollary 3.2

$$CK_n^*(\tau) = \frac{\hat{\Delta}(\hat{\tau}) - \Delta(\tau)}{\hat{\sigma}_n\sqrt{V_n}} \xrightarrow{d} N(0,1). \tag{3.7}$$

Asymptotic $100(1-\alpha)\%$ confidence intervals of τ and $\Delta(\tau)$ are

$$\hat{\tau} \pm h[\Phi^{-1}(1-\alpha/2)\,\mu!\,\hat{\sigma}_n/(\widehat{\triangle}(\hat{\tau})K_{-}(0))]^{1/\mu} \times [2\int K_{-}'^2(u)\,du/(nh)]^{1/(2\mu)}, \quad (3.8)$$

$$\widehat{\triangle}(\hat{\tau}) \pm \Phi^{-1}(1-\alpha/2)\,\hat{\sigma}_n\sqrt{V_n}.\,. \quad (3.9)$$

By Corollary 3.2. the number of change points is equivalent to the number of the clusters of the closed neighborhood satisfied as

$$CK_n^*(\tau) > \Phi^{-1}(1-\alpha/2).$$

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