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Noninformative Priors in Freund's Bivariate Exponential Distribution: Symmetry Case

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Abstract

In this paper, we develop noninformative priors that are used for estimating the ratio of failure rates under Freund's bivariate exponential distribution . A class of priors is found by matching the coverage probabilities of one-sided Baysian credible interval with the corresponding frequentist coverage probabilities. Also the propriety of posterior under the noninformative priors is proved and the frequentist coverage probabilities are investigated for small samples via simulation study.

Key Words : Frequentist coverage probability; Jeffreys prior; Matching prior; Reference prior .

1. INTRODUCTION

The quantification of the reliability of parallel systems is based on the assumption that, when a redundant component fails, the failure rate or the reliability of the surviving components is not affected by the failed component. In some situations, however, all the components share the load during the mission and the failure rate of the surviving components may increase due to increased load when a component fails. Systems such as a multi-processor computer and electric generators sharing an electrical load in plant can be described by a shared-load model. To correctly determine the reliability of such systems, the

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increase of failure rate of the surviving components has to be considered. Freund (1961) formulated a bivariate extension of the exponential model as a model for a system where the failure times of the two components may depend on each other. The Freund model applies, in particular, to two-component shared-load system that can function even if one component has failed. He also obtained the maximum likelihood estimators of the model parameters. Weier (1981) obtained Bayes estimators of parameters and reliability function for the Freund model. Cho (1999) obtained Bayesian tests of symmetry in Freund model.

Let the random variables (X, Y) be Freund's bivariate exponential model with parameters $\theta = (\alpha, \beta, \alpha', \beta')$. Then the joint probability density function(pdf) for (X, Y) by Freund (1961) is as follows:

$$f(x, y \mid \theta) = \begin{cases} \alpha \cdot \beta' \cdot \exp(-\beta' y - (\alpha + \beta - \beta')x), & y > x > 0 \\ \alpha' \cdot \beta \cdot \exp(-\alpha' x - (\alpha + \beta - \alpha')y), & x > y > 0 \end{cases}. \quad (1)$$

The Freund bivariate exponential model is symmetry case when the failure rates of two components are same, that is, $\alpha = \beta, \alpha' = \beta'$. In symmetry case, we let $r_1 = \alpha'/\alpha$, then the quantities of r_1 mean the degree of two-component shared-load. So, we consider Bayesian analysis for r_1 in this paper. In Bayesian analysis, inference problems are not simple because of problems associated with selection of priors as well as computational difficulties.

A commonly used noninformative prior is the Jeffreys prior (1961) utilizing a data translated likelihood. Berger and Bernardo (1989) argued that the Jeffreys prior has serious deficiencies in multiparameter case. Also Berger and Bernardo (1992) proposed a general algorithm to derive a reference prior. Datta and Ghosh (1995) introduced matching prior, that is founded by matching the coverage probabilities of one-sided Baysian credible interval with the corresponding frequentist coverage probabilities.

In this paper, we focus exclusively on developing noninformative priors for r_1. First, we derive the noninformative prior for r_1. Next, we obtain the propriety of posterior under the derived noninformative prior and we give the marginal density of r_1 under this prior. Finally, we provide simulated frequentist coverage probabilities under the derived noninformative prior for small sample sizes.

2. DEVELOPMENT OF THE NONINFORMATIVE PRIORS

Recently, development of noninformative prior has received a lot of attention. Notably among these are noninformative priors leading to Bayesian confidence regions with frequentist's desired confidence level.

When the parameter of interest is $t(\theta)$, an arbitrary function of p-dimensional parameter vector θ, Datta and Ghosh (1996) considered the following first-order asymptotic property:

$$P_\theta\left(\frac{\sqrt{n}(t(\theta)-t(\hat{\theta}))}{\sqrt{b}}\le z\right)=P_\pi\left(\frac{\sqrt{n}(t(\theta)-t(\hat{\theta}))}{\sqrt{b}}\le z|\boldsymbol{X}\right)+o(n^{-1/2}),$$

for all z, where $\hat{\theta}$ is the posterior mode or maximum likelihood estimator of theta and b is the asymptotic posterior variance of $\sqrt{n}(t(\theta)-t(\hat{\theta}))$up to $o_p(n^{-1/2})$. They proved that if $t(\theta)$ is a twice continuously differentiable function, $\pi(\theta)$ is a first-order matching prior if and only if $\pi(\theta)$ satisfies the following probability matching equation:

$$\sum_{i=1}^{p}\frac{\partial}{\partial\theta_i}\left(\frac{\rho_i^T I^{-1(\theta)}\nabla t(\theta)}{\sqrt{\nabla t(\theta)^T I^{-1(\theta)}\nabla t(\theta)}}\pi(\theta)\right)=0,$$

where $\nabla t(\theta)=\left(\frac{\partial}{\partial\theta_1}t(\theta),\frac{\partial}{\partial\theta_2}t(\theta),\ldots,\frac{\partial}{\partial\theta_p}t(\theta)\right)^T$ and ρ_i^T is the ith unit column p-vector and $I^{-1(\theta)}$ is the inverse of the per unit observation information matrix of θ.

In this section, the first-order matching priors are presented so that the asymptotic frequentist coverage probability of one sided posterior credible interval agree with frequentist's desired confidence level up to $o(n^{-1/2})$ when r_1 is the parameter of interest.

Lemma 1 A prior $\pi(\alpha,\alpha')$ is the first-order matching prior for r_1 if and only if:

$$\frac{\partial}{\partial\alpha}\left(-\frac{\alpha}{\sqrt{2}}\pi(\alpha,\alpha')\right)+\frac{\partial}{\partial\alpha'}\left(\frac{\alpha'}{\sqrt{2}}\pi(\alpha,\alpha')\right)=0. \quad (2)$$

Proof: Since the parameter of interest is $r_1=\alpha'/\alpha$, it follows

$$\nabla(r_1)=\left(-\frac{\alpha'}{\alpha^2},\frac{1}{\alpha}\right)^T,$$

$$I^{-1}(\alpha,\alpha')\nabla(r_1)=\left(-\alpha',-\frac{\alpha'^2}{\alpha}\right)^T,$$

$$\sqrt{\nabla(r_1)I^{-1}(\alpha,\alpha')\nabla(r_1)}=\frac{\sqrt{2\alpha'}}{\alpha}.$$

Hence, the probability matching equation is simplified to

$$\frac{\partial}{\partial \alpha}\left(-\frac{\alpha}{\sqrt{2}}\pi(\alpha,\alpha')\right)+\frac{\partial}{\partial \alpha'}\left(\frac{\alpha'}{\sqrt{2}}\pi(\alpha,\alpha')\right)=0. \tag{3}$$

Theorem 1. In the Freund's bivariate exponential distribution, the first-order matching prior is of the form:

$$\pi_M(\alpha,\alpha') \propto \alpha^{-1}\cdot \alpha'^{-1}. \tag{4}$$

Proof : By (3), it is trivial. □

By simple algebra, Fisher information matrix of (α,α') is given by

$$I(\theta)=\begin{pmatrix} 1/\alpha^2 & 0 \\ 0 & 1/\alpha'^2 \end{pmatrix}.$$

So $|I(\theta)|^{1/2} \propto \alpha^{-1}\alpha'^{-1}$. Then Jeffreys' prior is

$$\pi_J(\alpha,\alpha') \propto \alpha^{-1}\cdot \alpha'^{-1}. \tag{5}$$

By the transformation of variables, the joint prior for theta becomes $r_1^{-1}\cdot r_2^{-1}$,where r_1 is the parameter of interest and $r_2=\alpha$ is the nuisance parameter.

Other possible noninformative prior is the reference prior of Bernardo(1979). Choosing rectangle compacts for each one of r_1 and r_2, when r_1 is the parameter of interest, due to the orthogonality of r_1 with r_2, from Datta and Ghosh (1995b), the reference prior as well as the reverse prior is given by

$$\pi_R(r_1,r_2)=r_1^{-1}\cdot r_2^{-1}.$$

Back to (α,α') formulation the above reference prior transforms to

$$\pi_R(\alpha,\alpha')=\alpha^{-1}\cdot \alpha'^{-1}. \tag{6}$$

Thus it turns out that the Jeffreys' prior, the reference prior and the first order matching prior for (α,α') are the same in the Freund model. Therefore we denote

$$\pi(\alpha,\alpha') = \pi_J(\alpha,\alpha') = \pi_R(\alpha,\alpha') = \pi_M(\alpha,\alpha'). \tag{7}$$

Note that the same phenomenon was observed in Ghosh and Sun (1998) for the exponential case.

3. IMPLEMENTATION OF THE BAYESIAN PROCEDURE

Let $D_1 = \{i \mid \delta_i = 1\}$, $|D_1|$ is number of elements for $D_1$and D_0 , respectively. We now prove that the posterior is proper under the noninformative prior given in (7).

Theorem 2. The posterior distribution of (α,α') under the prior π is proper.

Proof : Note that

$$\int_0^\infty \int_0^\infty L(\theta)\alpha^{-1}\beta^{-1} d\alpha d\alpha' = \frac{\Gamma(|D_1| + |D_0|)}{(2\sum_{i\in D_1} x_i + 2\sum_{j\in D_0} y_j)^{|D_1| + |D_0|}}$$

$$\times \frac{\Gamma(|D_1| + |D_0|)}{(\sum_{i\in D_0} x_i + \sum_{j\in D_1} y_j - \sum_{i\in D_1} x_i - \sum_{j\in D_0} y_j)^{|D_1| + |D_0|}} < \infty .$$

This completes the proof. □

Next, we provide the marginal posterior density of r_1 under the above noninformative prior.

Theorem 3. Under the prior r_1, the marginal posterior density of $r_1 = \alpha'/\alpha$, is given by

$$p(r_1|X, Y) \propto \frac{r_1^{|D_1| + |D_0| - 1}}{(r_1(\sum_{i\in D_0} x_i + \sum_{j\in D_1} y_j) - (2 - r_1)(\sum_{j\in D_0} y_j + \sum_{i\in D_1} x_i))^{2|D_0| + + 2|D_1|}} .$$

Proof : It is trivial. □

The normalizing constant for the marginal density of r_1 requires one dimensional integration. Therefore we have the marginal posterior density of r_1, and so it is easy to compute the marginal moment of r_1.

4. SIMULATION STUDY

We evaluate the frequentist coverage probability by investigating the credible interval of the marginal posterior density of r_1 under our prior π for small sample sizes (n=3,5,10,20). The computation of these numerical values is based on the following algorithm for any fixed true (α, α') and any prespecified probability value γ. Here γ is 0.05 (0.95). Let $r_1^{\pi}(\gamma|\boldsymbol{X}, \boldsymbol{Y})$ be the posterior γ-quantile of r_1 given $(\boldsymbol{X}, \boldsymbol{Y})$. That is to say, $F(r_1^{\pi}(\gamma|\boldsymbol{X}, \boldsymbol{Y})|\boldsymbol{X}, \boldsymbol{Y}) = \gamma$, where $F(\cdot|\boldsymbol{X}, \boldsymbol{Y})$ is the marginal posterior distribution of r_1.

Then the frequentist coverage probability of this one sided credible interval of r_1 is

$$P_{(\alpha,\alpha')}(\gamma; r_1) = P_{(\alpha,\alpha')}(0 < r_1 \le r_1^{\pi}(\gamma|\boldsymbol{X}, \boldsymbol{Y})). \tag{8}$$

Table1: Frequentist Coverage Probability of 0.05 and 0.95, Posterior Quantiles of r_1

r_1	n	γ	
		0.05	0.95
1	3	0.052	0.949
	5	0.049	0.950
	10	0.053	0.950
	20	0.046	0.950
2	3	0.050	0.952
	5	0.049	0.948
	10	0.052	0.948
	20	0.050	0.952
3	3	0.049	0.954
	5	0.053	0.951
	10	0.050	0.948
	20	0.052	0.949
5	3	0.050	0.952
	5	0.050	0.949
	10	0.045	0.950
	20	0.053	0.953

The estimated $P_{(\alpha,\alpha')}(\gamma;r_1)$ when γ = 0.05 (0.95) is shown in Table 1. In particular, for fixed (r_1, n), we take 10,000 independent random samples from $\boldsymbol{X} = (X_1, \cdots, X_n)$ and $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$, respectively. Note that under $r_1 \le r_1^{\pi}(\gamma|\boldsymbol{X}, \boldsymbol{Y})$ if and only if $F(r_1|\boldsymbol{X}, \boldsymbol{Y}) \le \gamma$. For the cases presented in Table 1, we see that the noninformative prior π meets very well the target coverage probability. Also note that the results in Table 1 are not much sensitive to the change of the values of (α, α').

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