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Noninformative Priors for the Ratio of the Failure Rates in Exponential Model[1]

Jang Sik Cho[2] · Sung Uk Baek[3]

Abstract

In this paper, we derive noninformative priors for the ratio of failure rates in exponential model. A class of priors is found by matching the coverage probabilities of one-sided Baysian credible interval with the corresponding frequentist coverage probabilities. And we prove that the noninformative prior matches the alternative coverage probabilities and is a HPD matching prior up to the second order. Finally, we provide simulated freqentist coverage probabilities under the derived noninformative prior for small samples.

Key Words : Alternative coverage probability; HPD maching prior; Matching prior; Reference prior.

1. INTRODUCTION

In lifetime studies, exponential model has been widely used as a model in areas ranging from studies on the lifetimes of manufactured items to research involving survival or remission times in chronic diseases.

Now let X be a random variable having exponential model with parameter θ. Then probability density function(pdf) for random variable is as follows:

$$f(x \mid \theta) = \theta \exp^{-\theta x} \quad x > 0, \quad \theta > 0. \tag{1}$$

And let Y be another random variable having exponential model with parameter

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1. This Research was supported by Kyungsung University Research Grants in 2002

2. Assist Professor, Department of Statistical Information Science, Kyungsung University, Pusan, 608-736, Korea
E-mail : jscho@star.kyungsung.ac.kr

3. Researcher, Research Institute for Basic Science, Kyungsung University, Pusan, 608-736, Korea

ϕ. Here X and Y are independent. We focus on the ratio of these failure rates, $r_1 = \phi/\theta$. Then we can see that the ratio of failure rates means the comparativereliability of Y for X. That is, if $r_1 = 1$ then X and Y are same failure rate. If $r_1 = 1$ (or $r_1 < 1$) then the failure rate of Y is greater(or less) than that of X. So we focus exclusively on developing noninformative priors for r_1.

In Bayesian analysis, inference problems are not simple because of problems associated with selection of priors as well as computational difficulties. A commonly used noninformative prior is the Jeffreys prior (1961) utilizing a data translated likelihood. Although its successful commitment, Berger and Bernardo (1989) argue that the Jeffreys prior has serious deficiencies in multiparameter case. To overcome these deficiencies, Berger and Bernardo (1992) and Ghosh and Mukerjee (1992) introduced the reference prior. Recently, Mukerjee and Ghosh (1997) developed a second order matching prior.

In this paper, we derive the second order matching prior for r_1 and obtain the propriety of posterior under the derived noninformative prior. We prove that the second order matching prior matches the altenative coverage probabilities up to the same order and is a HPD matching prior up to the same order. Finally, we provide simulated freqentist coverage probabilities under the derived noninformative prior for small samples.

2. NONINFORMATIVE PRIORS

2.1 The Probability Matching Priors

Suppose that $\boldsymbol{X} = (X_1, \cdots, X_m)$are random samples from exponential (θ), and $\boldsymbol{Y} = (Y_1, \cdots, Y_m)$ are random samples from exponential (ϕ). Here, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent. The log-likelihood function of (θ, ϕ) is

$$l(\theta, \phi) \propto m \ln(\theta) + n \ln(\phi) - \theta \sum_{i=1}^{m} x_i - \phi \sum_{j=1}^{n} y_j .$$

By a simple algebra, the Fisher information matrix of (θ, ϕ) is given by

$$I(\theta, \phi) = \begin{pmatrix} \frac{m}{\theta^2} & 0 \\ 0 & \frac{n}{\phi^2} \end{pmatrix} .$$

Then Jeffreys' prior is $|I(\theta, \phi)|^{\frac{1}{2}} \propto (\theta\phi)^{-1}$. That is,

$$\pi_J(\theta,\phi)\propto\frac{1}{\theta\phi}\ . \qquad (2)$$

In this paper, since the parameter of interest is $r_1=\phi/\theta$, our interest is to find the probability matching prior for r_1.

For a prior π, let $r_1^{1-\alpha}(\pi;\ \boldsymbol{X},\ \boldsymbol{Y})$ denote the $100(1-\alpha)th$ percentile of the posterior distribution of r_1, that is,

$$P^{\pi}[r_1\le r_1^{1-\alpha}(\pi;\ \boldsymbol{X},\ \boldsymbol{Y})\mid \boldsymbol{X},\ \boldsymbol{Y}]=1-\alpha. \qquad (3)$$

We want to find priors π for which

$$P[r_1\le r_1^{1-\alpha}(\pi;\ \boldsymbol{X},\ \boldsymbol{Y})|\theta,\phi]=1-\alpha+o(n^{-u}), \qquad (4)$$

for some $u>0$, as n goes to infinity. Priors π satisfying (4) are called matching priors. If $u=1/2$, then π is referred to as a first order matching prior, while if $u=1$, π is referred to as a second order matching prior.

In order to find such matching priors π, it is convenient to introduce orthogonal parametrization (Cox and Reid, 1987 ; Tibshirani,1989). Now we let

$$r_1=\frac{\phi}{\theta}\ ,\ r_2=\theta\phi. \qquad (5)$$

With this parametrization, the log-likelihood has the alternate representation.

$$l(r_1,r_2)=\ln r_2-r_1^{-\frac{1}{2}}r_2^{\frac{1}{2}}x-r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}y. \qquad (6)$$

Based on (6), the Fisher information matrix is given by

$$I(r_1,r_2)=\begin{pmatrix}\frac{1}{2r_1^2} & 0\\ 0 & \frac{1}{2r_2^2}\end{pmatrix}.$$

Thus r_1 is orthogonal to r_2 in the sense of Cox and Reid (1987). Following Tibshirani (1989), the class of first order matching prior is characterized by

$$\pi_M^{(1)}(r_1,r_2)\propto\frac{1}{r_1}\,d(r_2), \qquad (7)$$

where $d(\cdot)$ is an arbitrary function differentiable in its argument.

Clearly the class of prior of given (7) is quite large and it is important to narrow down this class of prior. To do this, we consider the class of second order probability matching priors as given in Mukerjee and Ghosh (1997). A second order matching prior is also of the form (7), but the function d must satisfy an additional differential equation, namely

$$\frac{1}{6} d(r_2) \frac{\partial}{\partial r_1} (I_{11}^{-\frac{3}{2}} L_{1,1,1}) + \frac{\partial}{\partial r_2} \{I_{11}^{-\frac{1}{2}} L_{112} I^{22} d(r_2)\} = 0, \quad (8)$$

where

$$L_{1,1,1} = E[(\frac{\partial l}{\partial r_1})^3] = 0, \quad L_{112} = E[\frac{\partial^3 l}{\partial^2 r_1 \partial r_2}] = -\frac{1}{4} r_1^{-2} r_2^{-1},$$

and

$$\begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}^{-1} = \begin{pmatrix} 2r_1^2 & 0 \\ 0 & 2r_2^2 \end{pmatrix}.$$

Then (8) simplifies to

$$\frac{\partial}{\partial r_2} \{2^{-\frac{1}{2}} r_1^{-1} r_2 d(r_2)\} = 0. \quad (9)$$

Hence the set of (9) is of the form $d(r_2) = 1/r_2$. That is, the unique second order matching prior is given by

$$\pi_M^{(2)}(r_1, r_2) \propto 1/(r_1 r_2).$$

Back to (θ, ϕ) formulation the above second order matching prior transforms to

$$\pi_M^{(2)}(\theta, \phi) \propto 1/(\theta\phi). \quad (10)$$

Other possible noninformative prior is the reference prior of Bernardo (1979). Due to the orthogonality of r_1 with r_2 from Datta and Ghosh (1995b), the reference prior as well as the reverse reference prior is given by

$$\pi_R(r_1, r_2) \propto 1/(r_1 r_2).$$

This prior is clearly a second order probability matching prior. Thus it turns out that the Jeffreys' prior, the reference priors and second order matching prior for (θ, ϕ) are the same in the exponential model. Therefore we denote

$$\pi(\theta, \phi) = \pi_J(\theta, \phi) = \pi_R(\theta, \phi) = \pi_M^{(2)}(\theta, \phi). \quad (11)$$

2.2 Matching the Alternative Coverage Probability

Mukerjee and Reid (1999) studied that a prior satisfying (10) matches $P[\theta_1 + \beta(I^{11})^{1/2} \le \theta_1^{1-\alpha}(\pi; \mathbf{Z}) \mid \boldsymbol{\theta}]$ with the corresponding posterior probability, up to the same order and for each β and α, where the scalar β is free from n, $\boldsymbol{\theta}$ and $\boldsymbol{Z}$. If a matching prior matches the alternative coverage probabilities then there is a stronger justification for calling it noninformative in so far as agreement with a frequentist is concerned. In general a second order matching prior may or may not match the alternative coverage probabilities up to the same order of approximation.

Under orthogonal parametrization, Mukerjee and Reid (1999) gives the simple differential equations that a second order probability matching prior matches alternative coverage probabilities up to the second order. The differential equations are given by

$$\sum_{i=2}^{t}\sum_{j=2}^{t}\frac{\partial}{\partial\theta_i}\{L_{11j}I^{ij}I_{11}^{-1/2}d(\theta_2,\cdots,\theta_t)\}=0, \tag{12}$$

$$\sum_{i=2}^{t}\sum_{j=2}^{t}\frac{\partial}{\partial\theta_i}\{L_{j,11}I^{ij}I_{11}^{-1/2}d(\theta_2,\cdots,\theta_t)\}=0, \tag{13}$$

$$\frac{\partial}{\partial\theta_1}\{I_{11}^{-3/2}L_{111}\}=0,\quad \frac{\partial}{\partial\theta_1}\{I_{11}^{-3/2}L_{1,11}\}=0, \tag{14}$$

where $\boldsymbol{\theta}=(\theta_1,\cdots,\theta_t)^T$, θ_1 is the parameter of interest.

Theorem 1. The second order probability matching prior for r_1,

$$\pi_M^{(2)}(r_1,r_2)=(r_1\cdot r_2)^{-1}, \tag{15}$$

matches the alternative coverage probabilities up to the second order.

Proof : Since r_1 is orthogonal to r_2, the differential equations for the case of r_1 are given as

$$\frac{\partial\log L}{\partial r_2}\{L_{112}I^{22}I_{11}^{-1/2}d(r_2)\}=0,$$

$$\frac{\partial\log L}{\partial r_2}\{L_{2,11}I^{22}I_{11}^{-1/2}d(r_2)\}=0,$$

$$\frac{\partial\log L}{\partial r_1}\{I_{11}^{-3/2}L_{111}\}=0,\ \frac{\partial\log L}{\partial r_1}\{I_{11}^{-3/2}L_{1,11}\}=0.$$

Since

$$d(r_2)=(r_2)^{-1},L_{111}=E\left[\frac{\partial^3\log L}{\partial r_1^3}\right]=\frac{3}{2}r_1^{-3},$$

$$L_{112} = E\left[\frac{\partial^3 \log L}{\partial r_1^2 \partial r_2}\right] = -\frac{1}{4} r_1^{-2} \cdot r_2^{-1}, L_{1,11} = E\left[\frac{\partial \log L}{\partial r_1} \frac{\partial^2 \log L}{\partial r_1^2}\right] = -\frac{1}{12} r_1^{-3},$$

$$L_{2,11} = E\left[\frac{\partial \log L}{\partial r_2} \frac{\partial^2 \log L}{\partial r_1^2}\right] = \frac{1}{4} r_1^{-2} \cdot r_2^{-1}, I_{11} = \frac{1}{2r_1^2}, I^{22} = 2r_2^2,$$

we can see that

$$\frac{\partial \log L}{\partial r_2}\{L_{112} I^{22} I_{11}^{-1/2} d(r_2)\} = 0,$$

$$\frac{\partial \log L}{\partial r_2}\{L_{2,11} I^{22} I_{11}^{-1/2} d(r_2)\} = 0,$$

$$\frac{\partial \log L}{\partial r_1}\{I_{11}^{-3/2} L_{111}\} = 0, \frac{\partial \log L}{\partial r_1}\{I_{11}^{-3/2} L_{1,11}\} = 0.$$

Therefore the second order matching prior matches the alternative coverage probabilities up to the second order. □

2.3 HPD Matching Priors

There are alternative ways through which matching can be accomplished. One such approach(Ghosh and Mukerjee, 1995) is matching through the HPD region. Specifically, if $\tilde{\pi}$ denotes the posterior distribution of θ_1 under a prior π, and $k_\alpha \equiv k_\alpha(\pi; \mathbf{Z})$ is such that

$$P^\pi[\tilde{\pi}(\theta_1 \mid \mathbf{Z}) \geq k_\alpha \mid \mathbf{Z}] = 1 - \alpha + o(n^{-u}), \tag{16}$$

then the HPD region for θ_1 with posterior coverage probability $1-\alpha+o(n^{-u})$ is given by

$$H_\alpha(\pi; \mathbf{Z}) = \{\theta_1 : \tilde{\pi}(\theta_1 \mid \mathbf{Z}) \geq k_\alpha\}. \tag{17}$$

Ghosh and Murkerjee (1995) characterized priors π for which

$$P[\theta_1 \in H_\alpha(\pi; \mathbf{Z}) \mid \boldsymbol{\theta}] = 1 - \alpha + o(n^{-u}), \tag{18}$$

for all $\boldsymbol{\theta}$ and all $\alpha \in (0,1)$. They found necessary and sufficient conditions which π satisfies (18). Due to the orthogonality of r_1 with r_2, a prior π is a HPD matching prior if and only if it satisties

$$\frac{\partial^2}{\partial r_1^2}\{I^{11}\pi\} - \frac{\partial}{\partial r_1}\{L_{111}(I^{11})^2\pi\} - \frac{\partial}{\partial r_2}\{L_{112} I^{22} I^{11}\pi\} = 0. \tag{19}$$

Theorem 2. The second order probability matching prior for r_1,

$$\pi_M^{(2)}(r_1, r_2) = 1/(r_1 r_2), \tag{20}$$

is a HPD matching prior up to the same order.

Proof : The second order probability matching prior must satisfies the differential equations (19). Since

$$L_{111} = E[-\frac{\partial^3 \log L}{\partial r_1^3}] = \frac{3}{2} r_1^{-3}, L_{112} = E[-\frac{\partial^3 \log L}{\partial r_1^2 \partial r_2}] = -\frac{1}{4} r_1^{-2} \cdot r_2^{-1},$$
$$I^{11} = \frac{1}{2} r_1^2, \quad I^{22} = 2 r_2^2,$$

thus

$$\frac{\partial^2}{\partial r_1^2}\{I^{11}\pi_M^{(2)}\} - \frac{\partial}{\partial r_1}\{L_{111}(I^{11})^2\pi_M^{(2)}\} - \frac{\partial}{\partial r_2}\{L_{112}I^{22}I^{11}\pi_M^{(2)}\} = 0.$$

Therefore the second order matching prior is a HPD matching prior up to the second order. This completes the proof. □

3. IMPLEMENTATION OF THE BAYESIAN PROCEDURE

We now prove that the posterior is proper under the noninformative prior given in (10).

Theorem 3. The posterior distribution of (θ, ϕ) under the prior π, (10), is proper.

Proof. Note that

$$\int_0^\infty \int_0^\infty L(\theta,\phi)\frac{1}{\theta\phi}\, d\theta d\phi = \frac{\Gamma(m)}{(\sum_{i=1}^m x_i)^m} \times \frac{\Gamma(n)}{(\sum_{j=1}^n y_j)^n}$$

$$\times \int_0^\infty \int_0^\infty \frac{\theta^{m-1}\exp(-\theta\sum_{i=1}^m x_i)}{\Gamma(m)/(\sum_{i=1}^m x_i)^m} \times \frac{\phi^{n-1}\exp(-\phi\sum_{j=1}^n y_j)}{\Gamma(n)/(\sum_{j=1}^n y_j)^n}\, d\theta d\phi$$

$$= \frac{\Gamma(m)}{(\sum_{i=1}^m x_i)^m} \cdot \frac{\Gamma(n)}{(\sum_{j=1}^n y_j)^n} < \infty. \tag{21}$$

This completes the proof. □

Next, we provide the marginal density of r_1 under the second order matching prior.

Theorem 4. Under the prior r_1, the marginal posterior density of $r_1 = \phi/\theta$, is given by

$$p(r_1 | \boldsymbol{X}, \boldsymbol{Y}) \propto \frac{r_1^{(n-m-2)/2}}{(r_1^{-1/2} \sum_{i=1}^{m} x_i + r_1^{1/2} \sum_{j=1}^{n} y_j)^{m+n}}. \tag{22}$$

Proof : It is trivial. □

The normalizing constant for the marginal density of r_1 requires a one dimensional integration. Therefore we have the marginal posterior density of r_1, and so it is easy to compute the marginal moment of r_1.

5. SIMULATION STUDY

We evaluate the frequentist coverage probability by investigating the credible interval of the marginal posterior density of r_1 when m and n are small and moderate size. To do this, we need the frequentist coverage probability for 0 to 0.05 (0 to 0.95) posterior quantile for the our prior. The computation of these numerical values is based on the following algorithm. For any fixed true (θ, ϕ) and any prespecified probability value $\alpha = 0.05(0.95)$, let $r_1^{\pi}(\alpha | \boldsymbol{X}, \boldsymbol{Y})$ be the posterior α-quantile of r_1 under prior π. Then the frequentist coverage probability of this one sided credible interval of r_1 is

$$P_{(\theta,\phi)}(\alpha; r_1) = P_{(\theta,\phi)}(0 < r_1 \le r_1^{\pi}(\alpha | \boldsymbol{X}, \boldsymbol{Y})). \tag{23}$$

The estimated $P_{(\theta,\phi)}(\gamma\, ;\, r_1)$ is shown in Table 1. In particular, for fixed $(\theta.\phi, m, n)$, we take 10,000 independent random samples of size m exponential(θ) and of size n exponential(ϕ). Note that for fixed $\boldsymbol{X}$ and $\boldsymbol{Y}$, $r_1 \le r_1^{\pi}(\alpha | \boldsymbol{X}, \boldsymbol{Y})$ if and only if $F(r_1 | \boldsymbol{X}, \boldsymbol{Y}) \le \alpha$. Under the prior π, $P_{(\theta,\phi)}(\gamma\, ;\, r_1)$ can be estimate by the relative frequency of

$F(r_1^{\pi}(\gamma|\boldsymbol{X}, \boldsymbol{Y})) \le \alpha$. For the cases presented in Table 1, we can see that the first order matching prior π meets very well the target coverage probability. Also we note that the results in Table 1 are not much sensitive to the change of the values of (θ, ϕ).

Table1: Frequentist Coverage Probability of 0.05 and 0.95, Posterior Quantiles of r_1

r_1	$m = n$	α	
		0.05	0.95
1	3	0.046	0.947
	5	0.049	0.951
	10	0.051	0.947
	20	0.050	0.949
2	3	0.050	0.955
	5	0.048	0.949
	10	0.055	0.953
	20	0.051	0.951
3	3	0.048	0.950
	5	0.049	0.950
	10	0.053	0.952
	20	0.054	0.951
5	3	0.052	0.950
	5	0.053	0.950
	10	0.053	0.948
	20	0.048	0.949

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