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Some Properties of Operations on Fuzzy Numbers[1)]

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Abstract

In this paper, we introduce a concept of (H)-property which generalize that of increasing(decreasing) property of binary operation. We also treat some works related to operations on fuzzy numbers and generalize earlier results of Kawaguchi and Da-te(1994).

Keywords : Fuzzy arithmetic ; Algebraic ; Sup-(t-norm) convolution.

1. Introduction

Out of the most basic concept of fuzzy set theory which can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. An important field of applications for the extension principle is given by algebraic operations such as addition and multiplication. Many researchers have studied the properties of the operations based on the so-called sup-min convolution [Dubois and Prade(1979a, 1980b, 1987f), Kaufmann and Gupta(1985), Kawaguchi and Da-te(1992a, 1994b)]. These operations generalize interval analysis and are computationally attractive. Although the set of real fuzzy numbers equipped with an extended addition or multiplication is no longer a group, many structural properties are preserved.

On the other hand, Dubois and Prade(1998a, 1984b, 1984c) studied the operations based on a sup- t-norm convolution. Fullér(1991a, 1991b) et al. illustrated the sum of triangular fuzzy numbers based on various t-norms. Hong and Hwang(1994) generalized these results to L - R fuzzy numbers based on Archimedean t-norms. Recently Kawaguchi and Da-te(1994)] treat the algebraic structure of fuzzy arithmetic, especially addition and multiplication, based

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on the sup- t-norm convolution.

In this paper, we introduce a concept of (H)-property on binary operation and generalize some result on fuzzy numbers and their operations in Kawaguchi and Da-te(1994).

2. Preliminary

In this section we recall the definition of triangular norm(t-norm) and the definition relating to fuzzy numbers(Dubois and Prade(1980b), Tanaka(1987)).

Definition 1. t-Norm $T : [0,1]^2 \rightarrow [0,1]$ is a function satisfying the following four conditions :

(T1) boundary condition : $T(a,1) = a$, $T(0,a) = 0$,

(T2) monotonicity: $a \leq b \rightarrow T(a,c) \leq T(b,c)$,

(T3) commutativity: $T(a,b) = T(b,a)$,

(T4) associativity: $T(T(a,b),c) = T(a,T(b,c))$.

Definition 2. By using commutativity (T3) and associativity (T4) of t-norm, a n-ary function

$$T^*(a_1, a_2, \cdots, a_n) \equiv T(T(\cdots T(T(a_1, a_2), a_3), \cdots, a_{n+1}), a_n).$$

Definition 3. A fuzzy number X is a fuzzy set on the real line R, whose membership function $\mu_X : R \rightarrow [0,1]$ satisfies the following properties:

(FN1) normality: there exists at least one point $x \in R$ such that

$$\mu_X(x) = 1,$$

(FN2) convexity: $\forall \lambda \in [0,1]$, $\forall x_1, x_2 \in R$,

$$\mu_X(\lambda x_1 + (1-\lambda) x_2) \leq \min(\mu_X(x_1), \mu_X(x_2)).$$

Definition 4. Given a binary operation $z = x*y$ on R and two fuzzy number X and Y, we define a fuzzy binary operation $Z = X*Y$ by the following extension principle, i.e., sup-(t-norm) convolution:

$$\mu_Z(z) = \sup_{z = x*y} T(\mu_X(x), \mu_Y(y)).$$

3. Algebraic operations with fuzzy numbers

Definition 5 [Dubois and Prade(1986b)]. A binary operation $*$ in R is called increasing (decreasing) if for $x_1 > y_1$ and $x_2 > y_2$

$$x_1 * x_2 > y_1 * y_2 \quad (x_1 * x_2 < y_1 * y_2).$$

For (x_0, y_0), $(x_1, y_1) \in R^2$, the box subset $B_{(x_0,y_0)}^{(x_1,y_1)}$ of R^2 is defined as

$$B_{(x_0,y_0)}^{(x_1,y_1)} = \{(x,y) \in R^2 \mid \min\{x_0, x_1\} \leq x \leq \max\{x_0, x_1\} \text{ and } \min\{y_0, y_1\} \leq y \leq \max\{y_0, y_1\}\}.$$

Definition 6. A binary operation $*$ in R is said to have (H_D)-property((H_I)-property) if for given (x_0, y_0), $(x_2, y_2) \in R^2$ and $z_1 \in R$ such that $x_0 * y_0 < z_1 < x_2 * y_2$ $(x_0 * y_0 > z_1 > x_2 * y_2)$ there exists a $(x_1, y_1) \in B_{(x_0,y_0)}^{(x_2,y_2)}$ such that $x_1 * y_1 = z_1$.

Definition 7. A binary operation $\ast$ in R is said to have (H)-property iff it has both (H_D)-property and (H_I)-property.

The following lemma shows that (H)-property is a generalized concept of increasing and decreasing property.

Lemma 1. Let $*$ be a continuous increasing binary operation. Then $*$ has (H)-property.

Proof. First it is noted that for (x, y), $(x', y') \in R^2$ such that $x \leq x'$ and $y \leq y'$, we have $x * y \leq x' * y'$. Let (x_0, y_0), $(x_2, y_2) \in R^2$ and $z_1 \in R$ be given such that $x_0 * y_0 < z_1 < x_2 * y_2$. By the increasing property, there are three possible cases where

i) $x_0 \leq x_2$ and $y_0 \leq y_2$,
ii) $x_0 \leq x_2$ and $y_0 \geq y_2$,
iii) $x_0 \geq x_2$ and $y_0 \leq y_2$.

In the case of i) we can be proved easily by the continuity and increasing property of $*$. We prove for the case of ii) since in the case of iii) it is proved similarly. We see that $x_0 * y_0 < x_2 * y_0$. If not, then $x_0 * y_0 = x_2 * y_0 < x_2 * y_2$. Let $x_2 * y_2 - x_2 * y_0 = \alpha > 0$. By the continuity of $*$, there exist $\varepsilon > 0$ such that $|(x_2 + \varepsilon) * y_0 - x_2 * y_0| < \alpha/2$. Then $x_2 * y_2 - (x_2 + \varepsilon) * y_0 = (x_2 * y_2 - x_2 * y_0) - ((x_2 + \varepsilon) * y_0 - x_2 * y_0) > \alpha - \alpha/2 = \alpha/2 > 0$. This is contradict that $*$ is increasing. Now the continuity of $*$ on $\{(x, y) | x_0 \leq x \leq x_2, \ y = y_0\}$, the ordering of $x_0 * y_0 < x_2 * y_2 < x_2 * y_0$ and the intermediate value theorem [Rudin (1964), p93] proves that $*$ has (H_D)-property. The case of (H_I)-property is similar.

Similarly we can prove that decreasing property implies (H)-property.

Example 1. Let $f(x)$, $g(x)$ and $h(x)$ be non-decreasing functions on R. Then $x * y = \pm h(f(x) + g(x))$, $x * y = \pm h(f(x) \cdot g(x))$, have (H)-property. Special cases are $x * y = \pm(x + y)$, $x * y = \pm(x \cdot y)$, $x * y = \pm e^{x+y}$.

The proof of Example 1 is easy to check so we leave the proof to the leader. It is noted that $x * y = x \cdot y$ is not increasing operation on R but it has (H)-property.

Lemma 2. Let M and N be two fuzzy numbers such that μ_M is non-decreasing on $(-\infty, m]$ and non-increasing on $[m, +\infty)$ and μ_N is non-decreasing on $(-\infty, n)$ and non-increasing on $[n, +\infty)$. Let $*$ has (H)-property. Then μ_{M*N} is non-increasing on $(-\infty, m*n]$ and non-increasing on $[m*n, \infty)$.

Proof. Let $t \geq m*n$. For given (x, y), t' and t with $x*y = t$ and $m*n < t' < t$, there exists a $(x', y') \in B^{(x,y)}_{(m,n)}$ such that $x' * y' = t'$ by the definition of (H)-property. Then, using non-decreasing and non-decreasing property on some region and the monotonicity of T, we can easily check that $T(\mu_M(x), \mu_N(y)) \leq T(\mu_M(x'), \mu_N(y'))$.

Then it follows that

$$\begin{aligned} \mu_{M*N}(t') &= \sup_{x*y=t'} T(\mu_M(x), \mu_N(y)) \\ &\geq \sup_{x*y=t} T(\mu_M(x), \mu_N(y)) = \mu_{M*N}(t). \end{aligned}$$

When $t \leq m*n$, a similar proof holds.

Now we can write the following theorem which is a generalization of Theorem 1(Kawaguchi and Da-te(1994)) by Lemma 1 and 2.

Theorem 1. If M and N are fuzzy numbers and $*$ is a continuous binary operation which has (H)-property then $M*N$ based on the sup-t-norm convolution are also normal and convex.

Proof. The convexity is immediate from Lemma 2, so we prove the normality. Since M and N are normal, there exist m, $n \in R$ such that $\mu_M(m) = \mu_{N(m)} = 1$

and $\mu_{M*N}(m*n) = 1$, which proves the normality of $M*N$.

We also consider some other properties of $M*N$.

Theorem 2. Let M and N be two fuzzy numbers such that μ_M is non-decreasing on $(-\infty, m]$ and non-increasing on $[m, \infty)$ and μ_N is non-decreasing on $(-\infty, n]$ and non-increasing on $[n, +\infty)$. Let $*$ be a continuous binary operation. Assume $\lim_{|x|\to\infty} \mu_M(x) = \lim_{|x|\to\infty} \mu_N(x) = 0$ and $\lim_{|x|, |y|\to\infty} |x*y| = \infty$. hen we have

$$\lim_{|x|\to\infty} \mu_{M*N}(x) = 0.$$

Proof. For given $\varepsilon>0$, let $\inf\{x \ |\mu_M(x) \geq \varepsilon\} = m_\alpha$, $\sup\{x \ |\mu_M(x) \geq \varepsilon\} = m_\beta$, $\inf\{x \ |\mu_N(x) \geq \varepsilon\} = n_\alpha$ and $\sup\{x \ |\mu_N(x) \geq \varepsilon\} = n_\beta$. Let $\max\{|m_\alpha|, |m_\beta|, |n_\alpha|, |n_\beta|\} = K$. Since $*$ is continuous and $[-K, K] \times [-K, K]$ is compact,

$$\sup\{|x*y| \mid |x| \leq K, \ |y| \leq K\} = M < \infty.$$

Then for $|z|>M$, we have

$$\begin{aligned} \sup_{x*y=z} T(\mu_M(x), \mu_N(Y)) &\leq \sup_{x*y=z} \min\{\mu_M(x), \mu_N(y)\} \\ &\leq \sup\{\min\{\mu_M(x), \mu_N(y)\} | (x,y) \\ &\in ([-K \times K] \times [-K, K])^c\} \\ &\leq \varepsilon, \end{aligned}$$

which completes the lemma.

Theorem 3. Let T be continuous and $*$ be a continuous binary operation which has (H)-property. If M and N are upper semicontinuous then $M*N$ is upper semicontinuous.

Proof. Let m, n be real numbers such that $\mu_M(m) = 1$ and $\mu_N(n) = 1$. Then by Lemma 2, μ_{M*N} is non-increasing on $[m*n, \infty)$. Let $z_0 > m*n$, then it is enough to show the left continuity by Lemma 2. Since $*$ is continuous, there exist (x_0, y_0) such that $x_0 * y_0 = z_0$, $T(\mu_M(x_0), \mu_N(y_0)) = \mu_{M*N}(z_0)$ and $T(\mu_M(x), \ \mu_N(y)) \neq \mu_{M*N}(z_0)$ for any (x,y) satisfying $x*y = z_0$ and

$(x,y)<(x_{0},y_0)$. Now let $z_n \uparrow z_0$, $z_n > m*n$, $n=1,2,\cdots$, then by (H)-property and Lemma 2, there exist $(x_1,y_1)\in B^{(x_0,y_0)}_{(m,n)}$ and $(x_n,y_n)\in B^{(x_0,y_0)}_{(x_{n-1},y_{n-1})}$, $n=1,2,\cdots$ such that $\mu_{M*N}(z_n) = T(\mu_M(x_n),\mu_N(y_n))$ and $x_n * y_n = z_n$. Since $x_n\uparrow$, $y_n\uparrow$ and $z_n \uparrow z$, by the construction of (x_{0},y_0) and continuity of $*$, we have $x_n\uparrow x_0$ and $y_n \uparrow y_0$. Then by upper semicontinuity of μ_M and μ_N, and continuity of T,

$$\begin{aligned}\lim_{z_n\uparrow z_0} \mu_{M*N}(z_n) &= \lim_{n\to\infty} T(\mu_M(x_n),\mu_N(y_n)) \\ &= T(\mu_M(x_0),\mu_N(y_0)) = \mu_{M*N}(z_0)\end{aligned}$$

The case for $z<m*n$ is similar.

We consider the following example for above result.

Example 2. Let $T=\min$ and let

$$\mu_M(x)=\mu_N(x)=\begin{cases}1-|x| & \text{if } |x|\le 1,\\ 0 & \text{otherwise}.\end{cases}$$

We define $*$ as follows :

$$x*y=f(x+y)=\begin{cases}0 & |x+y|\le 1,\\ (x+y)-1 & x+y>1,\\ (x+y)+1 & x+y<-1.\end{cases}$$

Then, T, $*$ and M, N satisfy conditions in Theorem 3. It is easy to have that

$$\mu_{M*N}(x)=\begin{cases}1 & x=0,\\ \frac{1}{2}(1-|x|) & 0<x\le 1,\\ 0 & \text{otherwise}.\end{cases}$$

We note that even though M and N are continuous, $M*N$ is not continuous but upper semicontinuous.

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