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A note on a triangular norm hierarchy[1)]

Dug Hun Hong[2)]

Abstract

In Cretu(2001), triangular norms and their hierarchy are investigated. In this paper, we give new proofs which are significantly shorter than those given in Cretu, applying a known result which involves only one argument of one-place rather than two-place arguments by Klement et al.(1997).

Keywords : t-norm; Fuzzy logic; Comparison of t-norms

1. Introduction

Triangular norms (t-norms) and the corresponding t-conorms are used in several branches of mathematics in different manners, e.g., in probabilistic metric spaces, many-valued logic, fuzzy sets, decomposable measures and their applications[Butnariu and Klement(1993), Klement(1982), Mesiar(1993), Zimmermann(1991)]. A t-norm T is a two-place function from the unit square into the unit interval which is associative, commutative, non-decreasing, and fulfills, for all x in $[0,1]$, the boundary condition $T(1,x) = x$.

Its dual function S defined via $S(x,y) = 1 - T(1-x, 1-y)$ is called a t-conorm (see Schweizer and Sklar(1961)).

We are now interested in the question whether, given two t-norms T_1 and T_2, T_1 is weaker than T_2 or, equivalently, T_2 is stronger than T_1 (in symbols $T_1 \leq$ and T_2, i.e., $T_1(x,y) \leq$ and $T_2(x,y)$ for all points (x,y)) in the unit square.

Cretu(2001), recently, showed the monotonicity of some well-known classes of t

1. This research was supported by the Catholic University of Daegu, 2002.

2. Associate Professor, School of Mechanical and Automotive Engineering, Catholic University of Daegu, Kyungbuk 712 - 702, South Korea
E-mail : dhhong@cuth.cataegu.ac.kr

-norms. In this paper, we simplify this result, applying a known result which involves only one argument of one-place rather than two place arguments by klement et al.(1997).

2. Some known results about t-norm

The followings are the most important t-norms, together with their corresponding t-conorms:

$T_M(x,y) = \min(x,y)$, $S_M(x,y) = \max(x,y)$,
$T_P(x,y) = x \cdot y$, $S_P(x,y) = x + y - x \cdot y$,
$T_L(x,y) = \max(x + y - 1, 0)$,
$S_L(x,y) = \min(x + y, 1)$,

The following lemma is obvious from the monotonicity and boundary conditions.

Lemma 2.1. Let T be a t-norm. Then the following statement holds

$$T(x,y) \leq T_M(x,y), \qquad \text{for } all \quad x,y \in [0,1].$$

Proof. $T(x,y) \leq T(x,1) = x$, $T(x,y) = T(1,y) = y$
which means that

$$T(x,y) \leq \min(x,y) = T_M(x,y).$$

Continuous t-norms (t-conorms) were studied extensively by Ling(1965), among others. A continuous t-norm T is called Archimedean if $T(x,x) < x$ for all $x \in (0,1)$. A continuous t-norm T is strict if $T(x,y) < T(x,z)$ whenever $x \in (0,1)$ and $y < z$. Each strict t-norm T is Archimedean. Non-strict continuous Archimedean t-norms are called nilpotent. Aczél(1969), Mostert and Shields(1997) and Ling(1965) have proved the following result:

Theorem 2.1. T is a continuous Archimedean t-norm if and only if there is a continuous strictly decreasing function: $f : [0,1] \to [0,\infty]$ such that $f(1) = 0$ and $T(a,b) = f^*(f(a) + f(b))$ where f^* is the pseudoinverse of f, i.e., for all $x \in [0,\infty]$, $f^*(x) = f^{-1}(\min(x, f(0)))$.

T is strict if and only if $f(0) = +\infty$, i.e., f is bijective and $f^* = f^{-1}$.

The function f is called an additive generator of T and it is unique up to a positive multiplicative constant.

Now, let T_1, T_2 be two continuous Archimedean t-norms with additive generators f_1 and f_2, espectively. The full information about T_i is contained in f_i and, as a consequence, it should be possible to decide whether T_1 is weaker than T_2 only by means of f_1 and f_2.

The first step into this direction was done by Schweizer and Sklar(1961), who proved that if both T_1 and T_2 are strict, then $T_1 \leq T_2$ if and only if the composite $h = f_1 \circ f_2^{-1}$ is a subadditive function, i.e., if for all $s, t \geq 0$
$h(s+t) \leq h(s) + h(t)$.

Klement et al.(1997) showed the following result as a corollary of this result.

Theorem 2.2. Let T_1, T_2 be two continuous Archimedean t-norms with differentiable additive generators f_1 and f_2, respectively. If $g = f_1' / f_2'$ is a non-decreasing function on $(0,1)$, then we have $T_1 \leq T_2$.

3. *t*-norm hierarchy

In this section, we reconsider families of t-norms which are investigated by Cretu(2001). The reasonings are significantly shorten the the proofs given in Klement, Mesiar and Pap(1997). Many applications deal with the Frank(1979) family of t-norms, where for $s \in [0, \infty]$

$$T_s^F(x,y) = \begin{cases} T_M(x,y) & \text{if } s = 0, \\ T_P(x,y) & \text{if } s = 1, \\ T_L(x,y) & \text{if } s = \infty, \\ \log_s\left(1 + \dfrac{(s^x - 1)(s^y - 1)}{s - 1}\right) & \text{otherwise.} \end{cases}$$

Cretu(2001) showed at Proposition 2.1, 2.2, and 2.3 in his paper that, for $0 < r < 1 < s < \infty$,

$$T_\infty^F \leq T_s^F \leq T_1^F \leq T_r^F .$$

But Klement et al.(1997) already gave a proof using Theorem 2.2 which is significantly shorter than that given in Butnariu and Klement(1993) and Cretu(2001). Here we summarize them.

Frank showed that this family is continuous with respect to the parameter s. Note that trivially $T_0^F = T_M \geq T_s^F$ for all $s \in (0, \infty)$. For each $s \in (0, \infty)$, T_s^F is a strict t-norm whose generator is given by

$$f_s(x) = \begin{cases} \log x & \text{if } s = 1, \\ \log \dfrac{s-1}{s^x - 1} & \text{if } s \neq 1. \end{cases}$$

T_{∞}^{F} is a nilpotent t-norm and its generator is given by $f_{\infty}(x) = 1 - x$. Then

$$\begin{cases} \dfrac{f_{\infty}'(\mu)}{f_s'(\mu)} = \dfrac{1}{\log} s(1 - s^{-\mu}) & \text{for } s \in (0, \infty) \setminus \{1\}, \\ \dfrac{f_{\infty}'(\mu)}{f_1'(\mu)} = \mu & \text{for } s \in (0, \infty), \\ \dfrac{f_t'(\mu)}{f_s'(\mu)} = \dfrac{\log s}{\log t} \dfrac{1 - b^{\mu}}{1 - a^{\mu}} & \text{for } 1 < s < t < \infty, \\ & \text{(the case } 0 < s < t < 1 \text{ is completely analogous)} \end{cases}$$

are non-decreasing on $(0, 1)$. Hence, Theorem 2.2 implies the following three results which are Proposition 2.1, 2.2, and 2.3 in Cretu (2001).

Proposition 1. Let $s \in (0, 1)$, $T_s(x, y) = \log_s(1 + (s^x - 1)(s^y - 1)/(s - 1))$ and $T_1(x, y) = xy$, for all $x, y \in [0, 1]$.
Then
$T_s(x, y) \geq T_1(x, y)$, for all $x, y \in [0, 1]$.

Proposition 2. Let $s \in (1, \infty)$, $T_s(x, y) = \log_s(1 + (s^x - 1)(s^y - 1)/(s - 1))$ and $T_1(x, y) = xy$, for all $x, y \in [0, 1]$.
Then
$T_s(x, y) \leq T_1(x, y)$, for all $x, y \in [0, 1]$.

Proposition 3. Let $s \in (1, \infty)$, $T_s(x, y) = \log_s(1 + (s^x - 1)(s^y - 1)/(s - 1))$ and $T_{\infty}(x, y) \leq T_s(x, y)$, for all $x, y \in [0, 1]$.
Then
$T_{\infty}(x, y) \leq T_s(x, y)$, for all $x, y \in [0, 1]$.

The following two propositions, which are Proposition 2.4 and 2.5 in Cretu (2001), can be easily obtained by Lemma 2.1.

Proposition 4. Let $s \in (0, 1)$, $T_s(x, y) = \log_s(1 + (s^x - 1)(s^y - 1)/(s - 1))$ and $T_0(x, y) = \min(x, y)$, for all $x, y \in [0, 1]$.
Then
$T_s(x, y) \leq T_0(x, y)$, forall $x, y \in [0, 1]$.

Proposition 5. Let $a \in [0,1]$ and $T^a(x,y) = \dfrac{xy}{\max(x,y,a)}$ (Dubois and Prade intersection), $T_0(x,y) = \min(x,y)$.
Then
$T^a(x,y) \le T_0(x,y)$, for all $x,y \in [0,1]$.

Next, we give a simple proof of Proposition 2.12 in Cretu(2001).

Proposition 6. Let $T_{Yager(p)}:[0,1]\times[0,1]\to[0,1]$, such that

$$T_{Yager(p)}(x,y) = 1 - \min(1, ((1-x)^p + (1-y)^p)^{\frac{1}{p}}, \quad p \ge 1.$$

Then
$T_\infty(x,y) \le T_{Yager(p)}(x,y) \le T_0(x,y)$.

Proof. By Lemma 2.1 $T_{Yager(p)} \le T_0$ is trivial. So we prove that $T_\infty \le T_{Yager(p)}$, $p \ge 1$. We know that
$f_\infty(\mu) = 1-\mu$ and $f_{Yager(p)}(\mu) = (1-\mu)^p$.
Then

$$\frac{f_\infty{}'(\mu)}{f_{Yager}{}'(\mu)} = \frac{1}{p(1-\mu)^{p-1}}$$

is clearly non-decreasing on $(0,1)$ for $p \ge 1$, which completes the proof by Theorem 2.2.

Finally, we prove the following two propositions which are Proposition 3.1 and 3.2 in Cretu(2001).

Proposition 7. Let $T_{Einstein}:[0,1]\times[0,1]\to[0,1]$, such that

$$T_{Einstein}(x,y) = \frac{xy}{1+(1-x)(1-y)}.$$

Then
$T_\infty(x,y) \le T_{Einstein}(x,y) \le T_1(x,y)$, for all $x,y \in [0,1]$.

Proposition 8. Let $T_{Hamacher(\lambda)}:[0,1]\times[0,1]\to[0,1]$, such that

$$T_{Hamacher(\lambda)}(x,y) = \frac{xy}{\lambda + (1-\lambda)(x+y-xy)}, \quad \lambda \in [1,2].$$

Then

$$T_{\infty}(x,y) \leq T_{Hamacher(\lambda)}(x,y) \leq T_1(x,y).$$

Since $T_{Hamacher(2)} = T_{Einstein}$, it is enough to prove Proposition 8.

Proof of Proposition 8. It is known that

$$f_{Hamacher(\lambda)}(\mu) = \log \frac{\lambda + (1-\lambda)\mu}{\mu}.$$

Then

$$f_{Hamacher(\lambda)}{}'(\mu) = -\frac{\lambda}{\mu[\lambda + (1-\lambda)\mu]}$$

and hence

$$\frac{f_{\infty}{}'(\mu)}{f_{Hamacher(\lambda)}{}'(\mu)} = \frac{\mu[\lambda + (1-\lambda)\mu]}{\lambda}$$

is increasing for $\lambda \in [1,2]$ and similarly $\frac{f_{Hamacher(\lambda)}(\mu)}{f_1'(\mu)} = \frac{\lambda}{\lambda + (1-\lambda)\mu}$ is non-decreasing for $\lambda \in [1,2]$, which completes the proof.

4. Conclusion

We reconsidered most results about t-norm hierarchy given by Cretu(2001) and gave significantly simple proofs using a known result which involves only one argument of one-place rather than two-place arguments by Klement et al.(1997).

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