

On Jackknife Reliability Estimation in the Weibull Case¹⁾

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Abstract

We compare MISE of the MLE, UMVUE, invariantly optimal estimator and Jackknife estimator for the reliability function of the Weibull distribution when the sample size is small.

Key Words and Phrases : Weibull distribution, Reliability function, Jackknife estimation, mean integrated squared error.

1. Introduction

Recent tragedies such as the Space Shuttle Challenger accident, the Chernobyl and Three Mile Island nuclear power plant accidents, and aircraft catastrophes highlight reliability problems in engineering design. Reliability engineers, actuaries, and biostatisticians are all interested in lifetimes. A reliability engineer may study the lifetimes of products used in the marketplace. An actuary might be interested in the distribution of the lifetime of a person in order to determine the appropriate premium for a life insurance policy. A biostatistician might analyze the survival times of cancer patients in order to compare the effectiveness of treatment techniques, such as radiation and chemotherapy.

Tanaka(1998) compared the asymptotic mean integrated squared errors(MISE) of the ML estimator, the uniformly minimum variance unbiased(UMVU) estimator and the invariantly optimal(IO) estimator in the Weibull case. He showed that these estimators have the same asymptotic mean integrated squared error up to the order $o(n^{-2})$ using the concept of the asymptotic deficiency.

On the other hand, Lee and Kwon(2000) compared the mean square error(MSE) and bias of the maximum likelihood(ML) estimator, and the Jackknife

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estimator(JML) of the reliability function in the Weibull distribution for small sample. They showed that bias of JML estimator is smaller than the bias of ML estimator without increasing MSE in the small sample size. However they showed that MSE of the UMVU estimator is small than MSE of the JML when the sample size is small. Hence we shall compare the MISEs of the ML estimator, the UMVU estimator, the IO estimator and the JML estimator of the reliability function for the Weibull distribution when the sample size n is small.

2. Mean integrated squared errors(MISE) of the estimators

Suppose that X_1, X_2, \dots, X_n is a sequence of random variables independently and identically distributed according to the Weibull distribution with a probability density function (p.d.f.)

$$f(x; \theta) = \frac{c}{\theta} \left(\frac{x}{\theta} \right)^{c-1} \exp \left\{ - \left(\frac{x}{\theta} \right)^c \right\} (x > 0),$$

where $c > 0$ is a known constance and $\theta \in \Theta = (0, \infty)$ is an unknown parameter. Then $S_n = \sum_{i=1}^n X_i^c$ is distribution according to the Gamma distribution $\Gamma(n, \theta^c)$ with a p.d.f.

$$g(s; \theta) = \frac{1}{\Gamma(n)} \frac{s^{n-1}}{\theta^{cn}} \exp \left\{ - \frac{s}{\theta^c} \right\} (s > 0),$$

where $\Gamma(\cdot)$ is a gamma function.

And the reliability function of the Weibull Distribution is

$$R_\theta(t) = P_\theta(X_1 \geq t) = \exp \left\{ - \left(\frac{t}{\theta} \right)^c \right\} (t > 0).$$

Tanaka(1998) obtained the maximum likelihood estimator $\hat{R}_{ML}(t)$, the uniformly minimum variance unbiased estimator $\hat{R}_{UMVU}(t)$ and the invariantly optimal estimator $\hat{R}_{IO}(t)$ of $R_\theta(t)$ which are given by, respectively,

$$\begin{aligned} \hat{R}_{ML}(t) &= \exp \left\{ - \frac{n t^c}{S_n} \right\} (t > 0), \\ \hat{R}_{UMVU}(t) &= \left(\frac{S_n - t^c}{S_n} \right)^{n-1} (0 < t < S_n^{1/c}), \\ \hat{R}_{IO}(t) &= \left(\frac{S_n}{S_n + t^c} \right) (t > 0), \end{aligned}$$

and Lee and Kwon(2000) obtained the Jackknife estimator(JML) of the reliability function $\hat{R}_{JML}(t)$ of $R_\theta(t)$ that are given by

$$\widehat{R}_{JML}(t) = n \exp \left\{ -\frac{n t^c}{S_n} \right\} \frac{n-1}{n} \sum_{i=1}^n \exp \left\{ -\frac{(n-1) t^c}{S_n - X_i^c} \right\} (t > 0).$$

Now we denote the MISE for $\widehat{R}(t)$ of $R_\theta(t)$ by

$$MISE_\theta(\widehat{R}) := \int_{-\infty}^{\infty} E_\theta [\{\widehat{R}(t) - R_\theta(t)\}^2] dt.$$

Theorem. The MISE of the Jackknife reliability estimator $\widehat{R}_{JML}(t)$ is obtained as

$$MISE_\theta(\widehat{R}_{JML}(t)) = -\frac{\Gamma(2+1/c)}{2^{2+1/c}} \left\{ \frac{1}{n} + \frac{1}{48n^2} \left(2 - \frac{105}{c} - \frac{10}{c^2} \right) \right\}$$

$$\begin{aligned} \text{Proof : } MISE_\theta(\widehat{R}_{JML}(t)) &= \int_0^\infty E [n \widehat{R}_{ML}(t) - (n-1) \overline{\widehat{R}_{ML}^{(i)}(t)} - R_\theta(t)]^2 dt \\ &= \int_0^\infty \{n^2 E[\widehat{R}_{ML}(t)]^2 + (n-1)^2 E[\overline{\widehat{R}_{ML}^{(i)}(t)}]^2 \\ &\quad + R^2(t) - 2n(n-1)E[\widehat{R}_{ML}(t) \cdot \overline{\widehat{R}_{ML}^{(i)}(t)}] \\ &\quad - 2(n-1)R_\theta(t)E[\widehat{R}_{ML}^{(i)}(t)] \\ &\quad - 2nR_\theta(t)E[\widehat{R}_{ML}(t)]\} dt \\ &= n^2 \int_0^\infty E[\widehat{R}_{ML}(t)]^2 dt + (n-1)^2 \int_0^\infty E[\overline{\widehat{R}_{ML}^{(i)}(t)}]^2 dt \\ &\quad + \int_0^\infty R_\theta^2(t) dt - 2n(n-1) \int_0^\infty E[\widehat{R}_{ML}(t) \cdot \overline{\widehat{R}_{ML}^{(i)}(t)}] dt \\ &\quad + 2(n-1) \int_0^\infty R_\theta(t)E[\widehat{R}_{ML}^{(i)}(t)] dt \\ &\quad - 2n \int_0^\infty R_\theta(t)E[\widehat{R}_{ML}(t)] dt, \end{aligned}$$

where $\overline{\widehat{R}_{ML}^{(i)}(t)} = \frac{1}{n} \sum_{i=1}^n \widehat{R}_{ML}^{(i)}(t)$ and $\widehat{R}_{ML}^{(i)}(t)$ is the MLE of the $R(t)$ for the sample that deleted i^{th} sample from original sample (X_1, X_2, \dots, X_n) . After tedious calculation we can obtain

$$\begin{aligned} I_1 &= E[\overline{\widehat{R}_{ML}^{(i)}(t)}]^2 \\ &= \frac{(1/c)}{2^{1/c}} \frac{(n-1+1/c)}{(n-1)^{1/c}} \left(\frac{1}{c} \right). \quad (3.1) \end{aligned}$$

Since

$$E [\widehat{R}_{ML}(t) \cdot \overline{\widehat{R}_{ML}^{(i)}(t)}] = E [\widehat{R}_{ML}(t)] \cdot E [\overline{\widehat{R}_{ML}^{(i)}(t)}],$$

we have

$$\begin{aligned} I_2 &= \int_0^t E [\widehat{R}_{ML}(t) \cdot \overline{\widehat{R}_{ML}^{(i)}(t)}] dt \\ &= \int_0^t \left[\left\{ R_{ML}(t) + \frac{1}{2n} (\log^2 R_{ML}(t) + 2\log R_{ML}(t)R_{ML}(t)) \right\} \right. \\ &\quad \times \left. \left\{ R_{ML}(t) + \frac{1}{2(n-1)} (\log^2 R_{ML}(t) + 2\log R_{ML}(t)R_{ML}(t)) \right\} \right] dt. \quad (3.2) \end{aligned}$$

Hence we obtain $MISE_\theta(\widehat{R}_{JML}(t))$ by Tanaka(1998), (3.1) and (3.2).

3. Simulation and conclusion

Design parameters in the simulation experiments include sample size n , shape parameter c and scale parameter $1/\theta$. We generate Weibull random variables generating IMSL and run 1000 replicates in each fixed parameter values. We estimate each reliability estimators and their MISEs. In Table , we observe that MISE of the \widehat{R}_{JML} estimator is smaller than \widehat{R}_{ML} , \widehat{R}_{UMVU} , and \widehat{R}_{IO} in the small size

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Table : Mean Integrated Squared Error of Estimators

c	θ	n = 10			
		MISE of ML	MISE of UMVU	MISE of IO	MISE of JML
3.0	6.0	4.732873E-02	4.736611E-02	4.990840E-02	4.724767E-02
	6.2	4.890635E-02	4.894498E-02	5.157201E-02	4.882259E-02
	6.4	5.048398E-02	5.052385E-02	5.323562E-02	5.039751E-02
	6.6	5.206160E-02	5.210272E-02	5.489923E-02	5.197243E-02
	6.8	5.363922E-02	5.368159E-02	5.656285E-02	5.354736E-02
3.2	6.0	4.443927E-02	4.447947E-02	4.600405E-02	4.436908E-02
	6.2	4.592058E-02	4.596212E-02	4.753752E-02	4.584804E-02
	6.4	4.740189E-02	4.744477E-02	4.907099E-02	4.732701E-02
	6.6	4.888320E-02	4.892742E-02	5.060446E-02	4.880598E-02
	6.8	5.036451E-02	5.041007E-02	5.213793E-02	5.028495E-02
3.4	6.0	4.189106E-02	4.193250E-02	4.301264E-02	4.182910E-02
	6.2	4.328743E-02	4.333025E-02	4.444639E-02	4.322340E-02
	6.4	4.468380E-02	4.472801E-02	4.588015E-02	4.461771E-02
	6.6	4.608016E-02	4.612575E-02	4.731390E-02	4.601201E-02
	6.8	4.747653E-02	4.752351E-02	4.874766E-02	4.740631E-02
3.6	6.0	3.962567E-02	3.966746E-02	4.049955E-02	3.957018E-02
	6.2	4.094653E-02	4.098971E-02	4.184953E-02	4.088918E-02
	6.4	4.226739E-02	4.231196E-02	4.319952E-02	4.220819E-02
	6.6	4.358824E-02	4.363421E-02	4.454951E-02	4.352719E-02
	6.8	4.490910E-02	4.495646E-02	4.589949E-02	4.484620E-02
3.8	6.0	3.759760E-02	3.763917E-02	3.831353E-02	3.754729E-02
	6.0	3.885085E-02	3.889380E-02	3.959065E-02	3.879887E-02
	6.0	4.010411E-02	4.014844E-02	4.086777E-02	4.005045E-02
	6.0	4.135736E-02	4.140308E-02	4.214489E-02	4.130203E-02
	6.0	4.261061E-02	4.265773E-02	4.342201E-02	4.255360E-02

Table : (continue)

c	θ	n = 13			
		MISE of ML	MISE of UMVU	MISE of IO	MISE of JML
3.0	6.0	3.639283E-02	3.641496E-02	3.791927E-02	3.634487E-02
	6.2	3.760593E-02	3.762879E-02	3.918324E-02	3.755637E-02
	6.4	3.881902E-02	3.884262E-02	4.044722E-02	3.876786E-02
	6.6	4.003212E-02	4.005645E-02	4.171119E-02	3.997936E-02
	6.8	4.124521E-02	4.127099E-02	4.297517E-02	4.119086E-02
3.2	6.0	3.417211E-02	3.419590E-02	3.509802E-02	3.413057E-02
	6.2	3.531118E-02	3.533576E-02	3.626795E-02	3.526826E-02
	6.4	3.645026E-02	3.647562E-02	3.743789E-02	3.640594E-02
	6.6	3.758932E-02	3.761549E-02	3.860782E-02	3.754363E-02
	6.8	3.872839E-02	3.875535E-02	3.977776E-02	3.868132E-02
3.4	6.0	3.221341E-02	3.223794E-02	3.287707E-02	3.217675E-02
	6.2	3.328719E-02	3.331253E-02	3.397297E-02	3.324931E-02
	6.4	3.436098E-02	3.438713E-02	3.506887E-02	3.432187E-02
	6.6	3.543475E-02	3.546173E-02	3.616478E-02	3.539442E-02
	6.8	3.650853E-02	3.653633E-02	3.726068E-02	3.646698E-02
3.6	6.0	3.047196E-02	3.049669E-02	3.098905E-02	3.043912E-02
	6.2	3.148769E-02	3.151324E-02	3.202011E-02	3.145376E-02
	6.4	3.250343E-02	3.252980E-02	3.305499E-02	3.246840E-02
	6.6	3.351916E-02	3.354635E-02	3.408795E-02	3.348303E-02
	6.8	3.453489E-02	3.456291E-02	3.512092E-02	3.449767E-02
3.8	6.0	2.891283E-02	2.893743E-02	2.933646E-02	2.888306E-02
	6.0	2.987659E-02	2.990200E-02	3.031434E-02	2.984583E-02
	6.0	3.084035E-02	3.086659E-02	3.129222E-02	3.080860E-02
	6.0	3.180411E-02	3.183117E-02	3.227010E-02	3.177137E-02
	6.0	3.276787E-02	3.279575E-02	3.324799E-02	3.273414E-02

Table : (continue)

c	θ	$n = 16$			
		MISE of ML	MISE of UMVU	MISE of IO	MISE of JML
3.0	6.0	2.956213E-02	2.957674E-02	3.056982E-02	2.953047E-02
	6.2	3.054754E-02	3.056263E-02	3.158881E-02	3.051482E-02
	6.4	3.153294E-02	3.154852E-02	3.260780E-02	3.149917E-02
	6.6	3.251835E-02	3.253441E-02	3.362680E-02	3.248351E-02
	6.8	3.350375E-02	3.352030E-02	3.464579E-02	3.346787E-02
3.2	6.0	2.775878E-02	2.777448E-02	2.837002E-02	2.773135E-02
	6.2	2.868407E-02	2.870029E-02	2.931568E-02	2.865573E-02
	6.4	2.960936E-02	2.962611E-02	3.026135E-02	2.958011E-02
	6.6	3.053465E-02	3.055192E-02	3.120702E-02	3.050449E-02
	6.8	3.145995E-02	3.147744E-02	3.215269E-02	3.142887E-02
3.4	6.0	2.616808E-02	2.618427E-02	2.660619E-02	2.614387E-02
	6.2	2.704034E-02	2.705707E-02	2.749307E-02	2.701533E-02
	6.4	2.791261E-02	2.792989E-02	2.837994E-02	2.788680E-02
	6.6	2.878488E-02	2.880269E-02	2.926681E-02	2.875826E-02
	6.8	2.965715E-02	2.967550E-02	3.015369E-02	2.962972E-02
3.6	6.0	2.475373E-02	2.477005E-02	2.509509E-02	2.473205E-02
	6.2	2.557885E-02	2.559572E-02	2.593159E-02	2.555645E-02
	6.4	2.640398E-02	2.642139E-02	2.676810E-02	2.638086E-02
	6.6	2.722910E-02	2.724706E-02	2.760460E-02	2.720526E-02
	6.8	2.805423E-02	2.807273E-02	2.844110E-02	2.802966E-02
3.8	6.0	2.348741E-02	2.350364E-02	2.376707E-02	2.346775E-02
	6.0	2.427032E-02	2.428710E-02	2.455930E-02	2.425001E-02
	6.0	2.505323E-02	2.507055E-02	2.535154E-02	2.503227E-02
	6.0	2.583615E-02	2.585401E-02	2.614377E-02	2.581453E-02
	6.0	2.661906E-02	2.663746E-02	2.693601E-02	2.659679E-02

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