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Multivariate Cumulative Sum Control Chart for Dispersion Matrix[1)]

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Abstract

Several different control statistics to simultaneously monitor dispersion matrix of several quality variables are presented since different control statistics can be used to describe variability. Multivariare cumulative sum(CUSUM) control charts are proposed and the performances of the proposed CUSUM charts are evaluated in terms of average run length(ARL). Multivariate Shewhart charts are also proposed to compare the properties of the proposed CUSUM charts. The numerical results show that multivariate CUSUM charts are more efficient than multivariate Shewhart charts for small or moderate shifts. And we also found that small reference value of the CUSUM chart is more efficient for small shift.

Keywords : control statistic, expected time to signal, false alarm

1. Introduction

Control chart is a repetitive statistical procedure for continuously monitoring paramters of the production process, and is also used to quickly detect when the process changes and eliminate assignable causes that may produce any deterioration in the quality of the manufactured products. To maintain a control chart, samples of size n are taken at regular sampling time interval of length d. The ability of a control chart is determined by the length of time required for the chart to signal when the process is out-of-control, and the rate of false alarm

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when the process is in-control. Therefore a good chart should quickly detect changes in the production process while producing few false alarms.

The basic Shewhart chart, although simple to understand and apply, uses only the information in the current samples and is thus relatively inefficient in detecting small changes on the process parameters. Modifications such as the use of supplementary runs rules may improve the efficiency of Shewhart chart somewhat, but other approaches which take full advantages of the information in past samples are needed.

The multivariate procedure to quality control was first introduced by Hotelling(1947) and became popular in recent years. Jackson(1959) and Ghare and Torgersen(1968) presented multivariate Shewhart chart based on Hotelling's T^2 statistic. Woodall and Ncube(1985) extended the univariate CUSUM procedure to the multivariate case for monitoring mean vector of quality variables. They operated p two-sided univariate CUSUM schemes simultaneously, and evaluated the performance of the collection of the scheme. Crosier(1988) and Pignatiello and Runger(1990) considered new multivariate CUSUM procedures that accumulate past sample information for each parameter and then form a univariate CUSUM statistic from the multivariate data for monitoring the mean vector.

Up to the present, multivariate control procedures have been widely used for monitoring process mean vector. But, relatively little attention has been given to the use of CUSUM charts for monitoring dispersion matrix. In this paper, we propose several different control statistics and CUSUM charts for monitoring the dispersion matrix Σ of correlated quality variables where the target process mean vector $\underline{\mu}$ remained known constant.

2. Evaluating Control Statistics

The quality of a product is often characterized by joint levels of several quality characteristics. Because it is inappropriate to use individual charts to detect any changes of each process parameters in this case, a multivariate quality control procedure for simultaneously monitoring correlated variables is needed. Since the assignable causes which produce random variation of the quality are treated as inherent to the process in its current state, these quality characteristics are random variables.

Assume that the production process of interest has $p\,(p \geq 2)$ correlated quality variables represented by the random vector $\underline{X} = (X_1, X_2, X_3, \cdots, X_p)'$ and we obtain an independent sequence of random vectors $\underline{X}_1, \underline{X}_2, \underline{X}_3, \cdots$, where the vector $\underline{X}_i = (\underline{X}'_{i1}, \cdots, \underline{X}'_{in})'$ is a sample of observations at each sampling time i and $\underline{X}_{ij} = (X_{ij1}, X_{ij2}, \cdots, X_{ijp})'$. The underlying probability distribution of the p quality variables is assumed to be multivariate normal distribution with mean vector $\underline{\mu}$ and dispersion matrix Σ.

In practice, it may be necessary in many cases to estimate both $\underline{\mu}$ and Σ from the past data, but for simplicity we assume that $\underline{\mu}_0$ and Σ_0 are known. Let $\underline{\theta}_0 = (\underline{\mu}_0, \Sigma_0)$ be the known target values for the process parameters $\underline{\theta}$ of p quality characteristics and $\underline{\theta}_0$ is represented as

$$\underline{\mu}_0 = \begin{pmatrix} \mu_{10} \\ \mu_{20} \\ \vdots \\ \mu_{p0} \end{pmatrix} \text{ and } \Sigma_0 = \begin{pmatrix} \sigma_{10}^2 & \rho_{120}\sigma_{10}\sigma_{20} & \cdots & \rho_{1p0}\sigma_{10}\sigma_{p0} \\ \rho_{120}\sigma_{10}\sigma_{20} & \sigma_{20}^2 & \cdots & \rho_{2p0}\sigma_{20}\sigma_{p0} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1p0}\sigma_{10}\sigma_{p0} & \rho_{2p0}\sigma_{20}\sigma_{p0} & \cdots & \sigma_{p0}^2 \end{pmatrix},$$

where the target covariance component of X_r and X_s is $\sigma_{rs0} = \rho_{rs0}\sigma_{r0}\sigma_{s0}$ for $r, s = 1, 2, \cdots, p$.

In univariate case, the process dispersion can be monitored by S^2 chart or range chart, where S^2 denote an unbiased sample variance for a random sample of size n from a process. The S^2 chart signals for large values of S_i^2 or equivalently for large values of $T_i = (n-1)S_i^2/\sigma_0^2$ where σ_0^2 is target value of the process dispersion σ^2 and S_i^2 is obtained at sampling time i. When the process is in-control, the statistic T_i has a chi-squared distribution with $(n-1)$ degrees of freedom.

For multivariate case, one possible multivariate version of T_i is

$$V_i = \sum_{j=1}^{n}(\underline{X}_{ij} - \overline{\underline{X}}_i)'\Sigma_0^{-1}(\underline{X}_{ij} - \overline{\underline{X}}_i) = tr(A_i\Sigma_0^{-1}) \tag{2.1}$$

where $A_i = \sum_{j=1}^{n}(\underline{X}_{ij} - \overline{\underline{X}}_i)(\underline{X}_{ij} - \overline{\underline{X}}_i)'$ and the $p \times p$ sample dispersion matrix S_i is $A_i/(n-1)$. When the process is in-control, the dispersion matrix Σ is Σ_0 and the control statistic V_i has a chi-squared distribution with $(n-1)p$ degrees of freedom. Hotelling(1947) proposed that the statistic V_i can be used to monitor the process dispersion matrix of p quality variables.

The general multivariate statistical quality control chart can be considered as a repetitive tests of significance where each quality characteristic is defined by p quality variables $X_1, X_2, \cdots, X_p$. Therefore, we can obtain another chart statistic for momitoring Σ by using the likelihood ratio test(LRT) statistic for testing $H_0: \Sigma = \Sigma_0$ vs $H_1: \Sigma \neq \Sigma_0$ where target mean vector of the quality variables $\underline{\mu}_0$ is known. The regions above the UCL corresponds to the LRT rejection region. For the ith sample, likelihood ratio λ can be expressed as

$$\lambda = n^{-\frac{np}{2}} \cdot |A_i \Sigma_0^{-1}|^{\frac{n}{2}} \cdot \exp\left[-\frac{1}{2} tr(\Sigma_0^{-1} A_i) + \frac{1}{2} np\right].$$

Let TV_i be $-2\ln\lambda$. Then

$$TV_i = tr(A_i \Sigma_0^{-1}) - n \ln|A_i| + n \ln|\Sigma_0| + np \ln n - np. \tag{2.2}$$

Hence, the statistic TV_i can be used as the sample statistics for monitoring Σ. Alt(1982) proposed the use of sample generalized variance $|S_i|$ to monitor dispersion matrix. And Hui(1980) also used the sample generalized variances for monitoring the process dispersion by using the following statistic $\sqrt{n-1}\,(|S_i|/|\Sigma_0| - 1)$. This statistic is asymptotically normally distributed with mean 0 and variance $2p$.

If the statistic V_i or TV_i plots above the upper control limits, the process dispersion matrix is deemed out-of-control state and assignable causes are sought. The statistic V_i has a chi-squared distribution with $(n-1)p$ degrees of freedom and the percentage point of the V_i can be obtained from chi-square distribution when the process is in-control. But, if the process shifts from Σ_0 then it is difficult to obtain the exact distribution of V_i. Thus in order to obtain the percentage points of V_i when the proess is out-of-control state, it is necessary to use computer simulations. And, it is difficult to obtain the exact distribution of TV_i when the process is in-control or out-of-control states. Thus, in order to evaluate the performances of the charts based on the statistics V_i or TV_i for Σ it is necessary to carry out computer simulations.

3. Multivariate Shewhart Chart

Shewhart chart is widely used to display sample data from a process for the purpose of determining whether a production process is in-control, for bringing an out-of-control process into in-control, and for monitoring a process to make sure that it stays in-control. A Shewhart chart has a good ability to detect large changes in monitored parameter quickly and is easy to implement the process. However, the Shewhart chart is slow to signal small or moderate changes in the process parameters.

The run length N is defined as the random number of samples required for the chart to signal and the time required to signal T is dN where d is the length of the sampling interval. The average run length(ARL) is $E(N)$ and the expected time to signal $E(T)$ is simply the product of the ARL and d. Therefore, the ARL can be thought of as the expected time to signal.

Let q be the probability that a chart statistic falls out-of-control limits, then N

is geometrically distributed with parameter q when the process is in-control. The expected time to signal $E(T)$ and the variance of the time to signal $V(T)$ can be represented as

$$E(T) = dE(N) = \frac{d}{q} \quad \text{and} \quad V(T) = \frac{d^2(1-q)}{q^2}.$$

Since the control limits for a multivariate Shewhart chart based on the sample statistic V_i would be set as $\{0, \chi^2_{1-\alpha}[(n-1)p]\}$, a Shewhart chart based on V_i signals whenever

$$V_i \geq \chi^2_{1-\alpha}[(n-1)p]. \tag{3.1}$$

And because the control limits for a multivariate Shewhart chart based on the statistic TV_i would be set by using percentage point of TV_i, a Shewhart chart based on TV_i signals whenever

$$TV_i \geq h_{TV(S)} \tag{3.2}$$

where $h_{TV(S)}$ can be obtained to satisfy a specified in-control ARL by simulation.

[Result 3.1] Let $\underline{X}_{ij} = (X_{ij1}, X_{ij2}, \cdots, X_{ijp})'$ be distributed as $N_p(\underline{\mu}_0, \Sigma_0)$ and $\underline{X}_{ij}'s$ be independent when the process is in-control. Assume that multivariate Shewhart chart based on the statistic V_i in (2.1) is used as stated above. If the process parameters of the distribution shifted as $N_p(\underline{\mu}_0, c\Sigma_0)$ where c is a constant, then

$$ARL = \frac{1}{1 - F(h_V / c)} \tag{3.3}$$

where $h_V = \chi^2_{1-\alpha}[(n-1)p]$ is the control limit of the chart in (3.1) and $F(\cdot)$ is a chi-squared distribution function with $(n-1)p$ degrees of freedom.

4. Multivariate CUSUM Chart

The CUSUM chart is a good alternative to the Shewhart chart and is often used instead of standard Shewhart chart when detection of small shifts in a production process is important. A CUSUM chart directly incorporates all of the information in the sequence of sample values by plotting the cumulative sum of the deviation of the sample values from the target value.

A multivariate CUSUM chart based on the statistic V_i in (2.1) is given by

$$Y_{V,i} = \max\{Y_{V,i-1}, 0\} + (V_i - k_V) \tag{4.1}$$

where $Y_{V,0} = \omega_V$ $(\omega_V \geq 0)$ and reference value $k_V \geq 0$. This chart for dispersion matrix signals whenever $Y_{V,i} \geq h_V$.

When the process parameters are on-target, decision interval h_V can be evaluated by the Markov chain or integral equation approach to satisfy a specified in-control ARL. And when the process parameters in Σ have changed, the performances and properties of this chart can be evaluated by simulation.

The CUSUM procedure can be considered as a sequence of independent tests where each test is actually equivalent to a sequential probability ratio test(SPRT) for testing $H_0 : \Sigma = \Sigma_0$ vs $H_1 : \Sigma \neq \Sigma_0$. This sequence of SPRT's is equivalent to using the CUSUM statistic

$$Y_i = \max\{Y_{i-1}, 0\} + (L_i - k) \tag{4.2}$$

where the likelihood ratio $L_i = \max\{L(\underline{\mu}, \Sigma)\} / \max\{L(\underline{\mu}, \Sigma_0)\}$ and $L(\underline{\mu}, \Sigma)$ is likelihood function of random vector $\underline{X}_i = (\underline{X}'_{i1}, \underline{X}'_{i1}, \cdots, \underline{X}'_{in})'$ and $k \geq 0$. This chart signals whenever $Y_i > c$. Therefore, a CUSUM procedure based on the statistic TV_i in (2.2) can also be constructed as

$$Y_{TV,i} = \max\{Y_{TV,i-1}, 0\} + (TV_i - k_{TV}) \tag{4.3}$$

where $Y_{TV,0} = \omega_{TV}$ $(\omega_{TV} \geq 0)$ and $k_{TV} \geq 0$. This chart signals whenever $Y_{TV,i} \geq h_{TV(C)}$.

Since it is difficult to obtain the exact performances of multivariate CUSUM scheme based on TV_i, the percentage point and properties of this chart can be evaluated by simulation under the process parameters of the process are on-target or changed.

5. Numerical Performances and Concluding Remarks

The design of a CUSUM chart requires the specification of the sample size n, the sampling interval d, and the chart parameters h and k. A good choice for the chart parameters depends on the number of quality variables in the proposed control scheme and the size of shift on interesting.

In order to evaluate the performances and compare the proposed multivariate CUSUM and Shewhart charts fairly, it is necessary to calibrate each schemes so that on-target ARL $E(N \mid \underline{\mu}_0, \Sigma_0)$ be the same for all the proposed schemes. In our computation, each scheme was calibrated so that the on-target ARL was approximately equal to 370.4 and the sample size for each characteristic was five for $p = 3$ and $p = 4$. For convenience, we let that the sampling interval of unit time $d = 1$ and known target mean vector $\underline{\mu}_0 = \underline{0}$. The performance of the charts for monitoring a dispersion matrix depends on the components of Σ. For computational simplicity in our computation, we assume that $\sigma_{r0}^2 = 1$, $\rho_{rs0} = 0.3$ for $r, s = 1, 2, \cdots, p$.

Since it is not possible to investigate all of the different ways in which Σ could change, we consider the following typical types of shifts for comparison in the process parameters :

(1) V_i : σ_{10} of Σ_0 is increased to $[1 + (4i - 3)/10]$.

(2) C_i : ρ_{120} and ρ_{210} of Σ_0 are changed to $[0.3 + (2i - 1)/10]$

(3) (V_i, C_i) for $i = 1, 2, 3$.

(4) S_i : Σ_0 is changed to $c_i \Sigma_0$ where $c_i = [1 + (3i - 2)/10]^2$.

After the reference value of the proposed CUSUM chart based on the control statistic V_i, decision interval h_V was calculated by Markov chain method with the number of transient states $r = 100$. And the parameters $h_{TV(C)}$ based on TV_i for CUSUM scheme, and the ARL values for all the proposed types of shifts for Shewhart and CUSUM charts based on V_i or TV_i were obtained by simulation with 10,000 iterations.

<Table 1> ARL values for dispersion matrix ($p = 3$)

types of shifts	Shewhart		CUSUM					
	V_i	TV_i	based on V_i			based on TV_i		
no shift	370.4	370.4	370.4	370.4	370.4	370.3	370.3	370.3
V_1	177.7	340.6	86.1	91.5	99.5	302.2	311.9	318.0
V_2	15.8	53.9	12.6	10.7	9.8	24.5	25.2	26.4
V_3	4.5	8.8	6.2	5.0	4.5	7.2	6.6	6.3
C_1	419.7	354.4	623.3	564.2	530.6	321.6	329.1	334.2
C_2	403.8	244.7	1663.7	1121.5	880.8	103.5	119.9	136.4
C_3	309.0	101.3	3019.5	1502.3	996.9	18.5	19.1	20.9
(V_1, C_1)	203.8	328.2	116.3	123.9	133.6	274.5	288.6	296.8
(V_2, C_2)	19.2	46.3	15.6	13.3	12.4	20.3	20.4	21.3
(V_3, C_3)	5.6	6.9	7.5	6.2	5.5	5.7	5.2	4.9
S_1	65.3	299.6	26.6	24.5	24.7	213.8	229.0	242.0
S_2	4.5	27.2	5.8	4.7	4.1	12.3	11.9	11.9
S_3	1.7	4.3	3.1	2.6	2.0	4.2	3.8	3.5
S_4	1.2	1.8	2.2	1.8	1.6	2.4	2.1	2.0
			$k_V = 12.5$	$k_V = 13$	$k_V = 13.5$	$k_{TV} = 9$	$k_{TV} = 9.5$	$k_V = 10$

The performances of the proposed multivariate Shewhart and CUSUM chart are given in Table 1 and 2. By various numerical computation, we found the following properties. When a shift in variance components have occured, multivariate scheme based on the control statistic V_i is efficient. And a shift in correlation coefficients have occured, control procedure based on the TV_i will be recommended. A shift for both variances and correlation coefficients in Σ has occurred, the multivariate CUSUM procedure based on TV_i will be recommended.

When small or moderate changes in the production process have occurred, CUSUM procedures are more efficient than Shewhart charts in terms of ARL. Numerical results for various reference values of CUSUM schemes show that small reference values are more efficient in detecting small shifts and large reference values are more efficient for large shifts.

Hence, we recommend CUSUM chart based on the control statistic TV_i to simultaneously monitor both variances and correlation coefficients of Σ in the multivariate normal process when small or moderate shift has occured in the production process.

<Table 2> ARL values for dispersion matrix ($p = 4$)

types	Shewhart		CUSUM					
of shifts	V_i	TV_i	based on V_i			based on TV_i		
no shift	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
V_1	199.9	357.3	97.9	103.6	111.9	316.3	322.8	325.8
V_2	19.3	118.7	14.9	12.6	11.6	43.5	41.8	42.8
V_3	5.3	19.3	7.2	5.9	5.2	13.8	12.0	11.0
C_1	397.6	359.7	540.7	506.5	488.8	331.8	334.7	339.9
C_2	374.3	291.6	1107.9	859.3	721.2	130.8	140.9	153.6
C_3	291.8	166.4	1795.4	1106.6	815.8	33.8	32.1	32.8
(V_1, C_1)	2191.1	349.6	123.6	130.5	139.9	290.7	298.0	305.9
(V_2, C_2)	22.5	100.7	17.7	15.1	14.0	35.7	33.6	33.9
(V_3, C_3)	6.2	14.2	8.5	7.0	6.2	10.7	9.2	8.4
S_1	55.3	326.6	23.2	20.7	20.0	215.1	226.6	237.2
S_2	3.4	47.9	5.2	4.2	3.7	17.9	15.7	14.7
S_3	1.4	5.7	2.9	2.4	2.1	6.3	5.3	4.8
S_4	1.1	2.0	2.0	1.7	1.5	3.5	2.9	2.6
			$k_V = 16.5$	$k_V = 17$	$k_V = 17.5$	$k_{TV} = 16$	$k_{TV} = 16.5$	$k_V = 17$

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