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A Sequence of Improvements over the Lindley Type Estimator[1)]

Hoh Yoo Baek[2)]

ABSTRACT

In this paper, the problem of estimating a p-variate(p ≥ 4) normal mean vector in a decision-theoretic setup is considered. Using a technique of Guo and Pal (1992), a sequence of estimators dominating the Lindley type estimator is derived and each improved estimator is better than the previous one.

Key Words and Phrases : Improved estimator, Lindley type estimator, Normal mean vector.

1. INTRODUCTION

Let $\boldsymbol{X} = (X_1, \cdots, X_p)'$ be a p-variate random vector and $\boldsymbol{X} \sim N_p(\boldsymbol{\theta}, I_p)$, $\boldsymbol{\theta} \in R^p$. For any estimator $\boldsymbol{\delta}(\boldsymbol{X})$ of $\boldsymbol{\theta}$, the loss in estimating $\boldsymbol{\theta}$ by $\boldsymbol{\delta}(\boldsymbol{X})$ is

$$L(\boldsymbol{\delta}, \boldsymbol{\theta}) = ||\boldsymbol{\delta} - \boldsymbol{\theta}||^2 = (\boldsymbol{\delta} - \boldsymbol{\theta})'(\boldsymbol{\delta} - \boldsymbol{\theta}). \tag{1.1}$$

The standard estimator(MLE as well as the best location estimator) of $\boldsymbol{\theta}$ is

$$\boldsymbol{\delta}^0 = \boldsymbol{X} \tag{1.2}$$

which is admissible for $p \leq 2$. Stein(1956) and James and Stein(1961) showed that $\boldsymbol{\delta}^o$ is inadmissible for $p \geq 3$ and it is dominated by

1) This work was supported by Wonkwang University Research Fund, 2000.
2) Associate Professor, Division of Mathematics and Informational Statistics, Wonkwang University, Iksan, 570-549, Korea.
E-mail : hybaek@wonkwang.ac.kr

$$\boldsymbol{\delta}^{JS} = \left(1 - \frac{(p-2)}{\|\boldsymbol{X}\|^2}\right)\boldsymbol{X}, \qquad p \geq 3, \tag{1.3}$$

which shrinks $\boldsymbol{X}$ toward the origin. Subsequently a number of authors provided classes of Stein-type estimators dominating $\boldsymbol{X}$ (see, for example, Efron and Morris (1976), Ghosh, Hwang, and Tsui(1984) where other references are cited). One common feature of the above classes of estimators dominating $\boldsymbol{X}$ is that they are all spherically symmetric shrinking $\boldsymbol{X}$ toward some particular point, not necessarily the origin.

Guo and Pal(1992) considered a sequence of improved estimators providing successive improvement over $\boldsymbol{\delta}^{JS}$. The Lindley(1962) type estimator is

$$\boldsymbol{\delta}^1 = \overline{X}\,\mathbf{1} + \left(1 - \frac{p-3}{\|\boldsymbol{X} - \overline{X}\,\mathbf{1}\|^2}\right)(\boldsymbol{X} - \overline{X}\,\mathbf{1}), \quad p \geq 4 \tag{1.4}$$

where $\overline{X} = (X_1 + \cdots + X_p)/p$ and $\mathbf{1} = (1, \cdots, 1)'$. His estimator possesses better risk properties than the ordinary James-Stein estimator over a large region of the parameter space, suggesting that from a sampling theoretic viewpoint the shrinkage should be taken toward $\overline{X}\,\mathbf{1}$ as opposed to the origin.

In this paper, a sequence of improved estimators providing successive improvements over $\boldsymbol{\delta}^1$ is constructed. In Section 2, such improved estimators are derived and in Section 3, the above results is generalized when $\boldsymbol{X}$ $N_p(\boldsymbol{\theta}, \sigma^2 I_p)$, where covariance matrix is $\sigma^2 I_p$ for some unknown scalar $\sigma^2 > 0$.

2. IMPROVED ESTIMATORS DOMINATING $\boldsymbol{\delta}^1$

Consider a sequence of estimators of the form

$$\boldsymbol{\delta}^n = \overline{X}\,\mathbf{1} + K_n(\boldsymbol{X} - \overline{X}\,\mathbf{1}), \quad n = 1, 2, 3, \cdots, \tag{2.1}$$

where $K_n = K_n(\boldsymbol{X})$ is a suitable function of $\boldsymbol{X}$. We choose $K_1 = \left(1 - \frac{p-3}{\|\boldsymbol{X} - \overline{X}\,\mathbf{1}\|^2}\right)$ to make the first element $\boldsymbol{\delta}^1$ of the sequence $\{\boldsymbol{\delta}^n\}$. Our goal is to construct $\boldsymbol{\delta}^n$, $n \geq 2$, such that for any integer $n \geq 1$ and $p \geq 4$,

$$R(\boldsymbol{\delta}^{n+1}, \boldsymbol{\theta}) \leq R(\boldsymbol{\delta}^n, \boldsymbol{\theta}), \ \forall\ \boldsymbol{\theta} \in R^p. \tag{2.2}$$

To dominate the estimator $\boldsymbol{\delta}^n$ for any $n \geq 1$, define $\boldsymbol{\delta}^{n+1}$ as

$$\boldsymbol{\delta}^{n+1} = \boldsymbol{\delta}^n + r_n^*(\boldsymbol{X} - \overline{X}\,\mathbf{1}), \ i.e., K_{n+1} = K_n + r_n^*, \tag{2.3}$$

where $r_n^* = r_n^*(\boldsymbol{X})$ is a suitable real valued function. Let $\boldsymbol{r}_n = r_n^* \cdot (\boldsymbol{X} - \overline{X}\,\mathbf{1})$. Define the risk difference (RD) between $R(\boldsymbol{\delta}^{n+1}, \boldsymbol{\theta})$ and $R(\boldsymbol{\delta}^n, \boldsymbol{\theta})$ as

$$RD(n+1,n) = R(\boldsymbol{\delta}^{n+1}, \boldsymbol{\theta}) - R(\boldsymbol{\delta}^{n}, \boldsymbol{\theta})$$
$$= E\{\sum_{i=1}^{p} r_{ni}^2 + 2\sum_{i=1}^{p} r_{ni}(\delta_i^n - \theta_i)\}, \tag{2.4}$$

where δ_i^n, θ_i, and r_{ni} denote the i^{th} elements of $\boldsymbol{\delta}^n$, $\boldsymbol{\theta}$, and $\boldsymbol{r}_n$, respectively. The second term of (2.4) can be simplified as

$$E\left\{\sum_{i=1}^{p} r_{ni}(\delta_i^n - \theta_i)\right\} = \sum_{i=1}^{p} E[r_{ni}\{\overline{X} + K_n(X_i - \overline{X}) - \theta_i\}]$$
$$= \sum_{i=1}^{p} E[(K_n - 1)r_{ni}(X_i - \overline{X}) + r_{ni}(X_i - \theta_i)]$$
$$= \sum_{i=1}^{p} E[(K_n - 1)r_{ni}(X_i - \overline{X}) + \frac{\partial}{\partial X_i} r_{ni}]. \tag{2.5}$$

The expression (2.5) is obtained by using Stein's normal identity assuming that r_{ni}'s $(i = 1,2,\cdots,p)$ satisfy all the regularity conditions of the identity. Combining (2.4) and (2.5) we get

$$RD(n+1,n) = E\left[\sum_{i=1}^{p}\left\{r_{ni}^2 + 2(K_n - 1)r_{ni}(X_i - \overline{X}) + 2\sigma^2\frac{\partial}{\partial X_i} r_{ni}\right\}\right]. \tag{2.6}$$

We now look for suitable $\boldsymbol{r}_n = r_n^* \cdot (\boldsymbol{X} - \overline{X}\ \mathbf{1})$ such that $RD(n+1,n) \le 0$, $\forall n \ge 1$.

Before we derive the general result, let us look at some special cases.

Special cases

(1) When $n = 1$, i.e., we are trying to dominate $\boldsymbol{\delta}^1$(Lindley), take $r_1^* = c_1\|\boldsymbol{X} - \overline{X}\ \mathbf{1}\|^{-(2+\alpha_1)}$ where $\alpha_1 > 0$ and c_1 is a suitable constant. Then

$$2\sum_{i=1}^{p}\frac{\partial}{\partial X_i} r_{1i} = \frac{2c_1\{p - (3+\alpha_1)\}}{\|\boldsymbol{X} - \overline{X}\ \mathbf{1}\|^{2+\alpha_1}}, \quad \sum_{i=1}^{p} r_{1i}^2 = \frac{c_1^2}{\|\boldsymbol{X} - \overline{X}\ \mathbf{1}\|^{2+2\alpha_1}}$$

and $2(K_1 - 1)\sum_{i=1}^{n} r_{1i}(X_i - \overline{X}) = -\dfrac{2c_1(p-3)}{\|\boldsymbol{X} - \overline{X}\ \mathbf{1}\|^{2+\alpha_1}}$.

Therefore, from (2.6) one can get

$$RD(2,1) = E\left[\frac{c_1^2}{T^{1+\alpha_1}} - \frac{2c_1\alpha_1}{T^{1+\frac{\alpha_1}{2}}}\right], \tag{2.7}$$

where $T = \|\boldsymbol{X} - \overline{X}\ \mathbf{1}\|^2 \sim noncentral\ \chi_{p-1}^2(\lambda)$ with $\lambda = \|\boldsymbol{\theta} - \overline{\theta}\ \mathbf{1}\|^2$ and $\overline{\theta} = (\theta_1 + \cdots + \theta_p)/p$. It is well known that T can be treated as a mixture of central χ_{p-1+2U}^2 and $U \sim Poisson(\frac{\lambda}{2})$. Let $\beta_U = U + \frac{p-1}{2}$, then

$$RD(2,1) = E_U\left[c_1^2\, 2^{-(1+\alpha_1)}\frac{\Gamma(\beta_U-(1+\alpha_1))}{\Gamma(\beta_U)} - 2c_1\alpha_1 2^{-(1+\frac{\alpha_1}{2})}\frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U)}\right]. \tag{2.8}$$

To make $RD(2,1)\le 0$, it is sufficient to $\left[c_1^2\, 2^{-(1+\alpha_1)}\frac{\Gamma(\beta_U-(1+\alpha_1))}{\Gamma(\beta_U)}\right.$ $\left.\le 2c_1\alpha_1 2^{-(1+\frac{\alpha_1}{2})}\frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U)}\right]$ for all U, $U=0,1,2,\cdots$. Hence, the condition c_1 is

$$0<c_1<\alpha_1 2^{(1+\frac{\alpha_1}{2})}\frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U-(1+\alpha_1))} \quad \text{for } U=0,1,2,\cdots.$$

Let

$$\varepsilon_1(p,\alpha_1) = \min_U \frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U-(1+\alpha_1))}. \tag{2.9}$$

Then a sufficient condition on c_1 is

$$0<c_1<\alpha_1 2^{1+\frac{\alpha_1}{2}}\varepsilon_1(p,\alpha_1), \tag{2.10}$$

provided $p-1>2(1+\alpha_1)$. In fact, the optimal value of c_1 which minimizes

$$\left[c_1^2\, 2^{-(1+\alpha_1)}\frac{\Gamma(\beta_U-(1+\alpha_1))}{\Gamma(\beta_U)} - 2c_1\alpha_1 2^{-(1+\frac{\alpha_1}{2})}\frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U)}\right] \quad \text{for all } U$$

is

$$c_1^0 = \alpha_1 2^{\frac{\alpha_1}{2}}\varepsilon_1(p,\alpha_1). \tag{2.11}$$

It can be proved as a part of a more general result that the minimum in (2.9) is attained at $U=0$, *i.e.*, $\beta_U=\frac{(p-1)}{2}$ (see Guo and Pal (1992)). The condition that $p-1>2(1+\alpha_1)$ is necessary to ensure that all the expectation exist. The following result is immediate from the above derivation.

Proposition 2.1. The estimator $\boldsymbol{\delta}^2 = \boldsymbol{\delta}^1 + (\frac{c_1^0}{\|\boldsymbol{X}-\overline{X}\mathbf{1}\|^{2+\alpha_1}})(\boldsymbol{X}-\overline{X}\mathbf{1})$

with $\alpha_1 > 0$ dominates δ^1(Lindley) uniformly under the quadratic loss (1.1) provided $p - 1 > 2(1+\alpha_1)$.

Remark 2.1. It is interesting to look at various choice of $\alpha_1 > 0$ in the above proposition.

(a) If $0 < \alpha_1 < 0.5$, then δ^2 dominates δ^1 for $p \geq 4$.

(b) If $\alpha_1 = 1$, then $\varepsilon_1(p, \alpha_1) = \dfrac{\Gamma\left(\frac{p-4}{2}\right)}{\Gamma\left(\frac{p-5}{2}\right)}$. Hence, $\delta^2 = \delta^1 + \dfrac{\sqrt{2}}{\|\boldsymbol{X} - \overline{X}\,\mathbf{1}\|^3}$ $\varepsilon_1(p, \alpha_1)\,(\boldsymbol{X} - \overline{X}\,\mathbf{1})$ dominates δ^1 whenever $p > 5$.

(c) If $\alpha_1 = 2$ then $\varepsilon_1(p, \alpha_1) = \dfrac{p-7}{2}$. In this case, δ^1 is uniformly dominated by $\delta^2 = \overline{X}\,\mathbf{1} + \left(1 - \dfrac{p-3}{\|\boldsymbol{X} - \overline{X}\,\mathbf{1}\|^2} + \dfrac{2(p-7)}{\|\boldsymbol{X} - \overline{X}\,\mathbf{1}\|^4}\right)(\boldsymbol{X} - \overline{X}\,\mathbf{1})$ for $p > 7$.

(2) When $n = 2$, and we want to dominate $\delta^2 = \overline{X}\,\mathbf{1} + K_2(\boldsymbol{X} - \overline{X}\,\mathbf{1})$, where $K_2 = \left(1 - \dfrac{(p-3)}{\|\boldsymbol{X} - \overline{X}\,\mathbf{1}\|^2} + \dfrac{c_1}{\|\boldsymbol{X} - \overline{X}\,\mathbf{1}\|^{2+\alpha_1}}\right)$

choose $r_2^* = \dfrac{c_2}{\|\boldsymbol{X} - \overline{X}\,\mathbf{1}\|^{2+\alpha_2}}$, where $\alpha_2 > \alpha_1 > 0$ and c_2 is suitable constant. Similar to the case $n = 1$, $RD(3,2)$ can be derived form (2.6) as

$$RD(3,2) = E\left[\frac{c_2^2}{T^{1+\alpha_2}} + \frac{2c_1c_2}{T^{1+(\alpha_1+\alpha_2)}} - \frac{2c\alpha_2}{T^{1+\frac{\alpha_2}{2}}}\right].$$

Following the earlier approach, a sufficient condition for $RD(3,2) \leq 0$ is

$$0 < c_2 < \alpha_2\, 2^{(1+\frac{\alpha_2}{2})}\varepsilon_2(p, \alpha_1, \alpha_2), \tag{2.12}$$

where

$$\varepsilon_2(p, \alpha_1, \alpha_2) = \min_U \left\{\frac{\Gamma(\beta_U - (1+\frac{\alpha_2}{2}))}{\Gamma(\beta_U - (1+\alpha_2))} \times \left|1 - c_1\alpha_2^{-1} 2^{-\frac{\alpha_1}{2}} \frac{\Gamma(\beta_U - (1+\frac{\alpha_1+\alpha_2}{2}))}{\Gamma(\beta_U - (1+\frac{\alpha_1+\alpha_2}{2}))}\right|\right\}$$

provided that $p-1>2(1+\alpha_2)$. Again, the optimal value of c_2 is $c_2^0=\alpha_2 2^{\frac{\alpha_2}{2}}\varepsilon_2(p,\alpha_1,\alpha_2)$.

In general, consider the estimator δ^n (in (2.1)) with

$$K_n = 1-\frac{p-3}{\|X-\bar{X}\mathbf{1}\|^2}+\sum_{j=1}^{n-1}\frac{c_j}{\|X-\bar{X}\mathbf{1}\|^{2+\alpha_j}}, \tag{2.13}$$

where $\alpha_{n-1}>\alpha_{n-2}>\cdots>\alpha_1>0$ and $0<c_j<\alpha_j 2^{1+\frac{\alpha_j}{2}}\varepsilon_j(p,\alpha_1,\cdots,\alpha_j)$, $j=1,2,\cdots,n-1$. Take $r_n^*=\frac{c_n}{\|X-\bar{X}\mathbf{1}\|^{2+\alpha_n}}$, where $\alpha_n>\alpha_{n-1}$ and c_n is a suitable constant. Similar to the special cases $n=1,2$, one can get

$$\begin{aligned} RD(n+1,n) = E_U \Bigg\{\frac{1}{\Gamma(\beta_u)}\Bigg[& c_n^2 2^{-(1+\alpha)}\Gamma(\beta_u-(1+\alpha_n)) \\ & + 2c_n\sum_{j=1}^{n-1}c_j 2^{-(1+\frac{(\alpha_j+\alpha_n)}{2})}\Gamma\left(\beta_u-\left(1+\frac{\alpha_j+\alpha_n}{2}\right)\right) \\ & - 2c_n\alpha_n 2^{-(1+\frac{\alpha_n}{2})}\Gamma\left(\beta_u-\left(1+\frac{\alpha_n}{2}\right)\right)\Bigg]\Bigg\} \end{aligned} \tag{2.14}$$

All the expectations in (2.14) exist provided $p-1>2(1+\alpha_n)$. Define $\varepsilon_n(p,\alpha_1,\cdots,\alpha_n)$ as

$$\varepsilon_n(p,\alpha_1,\cdots,\alpha_n)=\min_U\left[\frac{\Gamma(\beta_U-1+\frac{\alpha_n}{2})}{\Gamma(\beta_U-1+\alpha_n)}\left(1-\sum_{j=1}^{n-1}c_j\alpha_n^{-1}2^{\frac{-\alpha_j}{2}} \times\frac{\Gamma(\beta_U-(1+\frac{\alpha_j+\alpha_n}{2}))}{\Gamma(\beta_U-(1+\frac{\alpha_n}{2}))}\right)\right]. \tag{2.15}$$

Then a sufficient condition for δ^{n+1} dominating over δ^n is

$$0<c_n<\alpha_n 2^{1+\frac{\alpha_n}{2}}\varepsilon_n(p,\alpha_1,\cdots,\alpha_n) \tag{2.16}$$

and the optimal value of c_n is $c_n^0=\alpha_n 2^{\frac{\alpha_n}{2}}\varepsilon_n(p,\alpha_1,\cdots,\alpha_n)$. The minimum in (2.15) is attained $U=0$ (see Guo and Pal(1992)). We now state the main theorem of this section.

Theorem 2.1. An estimator δ^n with K_n given by (2.13) is uniformly dominated by

$$\boldsymbol{\delta}^{n+1} = \boldsymbol{\delta}^n + \frac{c_n}{\|\boldsymbol{X} - \overline{X}\,\boldsymbol{1}\|^{2+\alpha_n}}(\boldsymbol{X} - \overline{X}\,\boldsymbol{1})$$

provided $p - 1 > 2(1+\alpha_n)$ and c_n satisfies the condition (2.16).

Remark 2.2. Note that the functions r_{ni}, $i = 1, 2, \cdots, p$ satisfy the regularity condition of Stein's normal identity which enables us to derive (2.14).

If we choose $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} < \cdots < 0.5$, then we get $\{\boldsymbol{\delta}^n\}$, a sequence of improved estimators giving successive improvements for $p \geq 4$ (since, $2(1+\alpha_n) < 3 \ \forall n \geq 1$).

Remark 2.3. The limiting value of $\{c_n^0\}$ is hard to find analytically due to the complicated structure of c_n^0(see(2.15)). Hence, the problem of finding the close form of the limiting estimator of the sequence $\{\boldsymbol{\delta}^n\}$ still remains open.

3. THE CASE OF COVARIANCE MATRIX $\sigma^2 I_p$ ($\sigma^2 > 0$ unknown)

In this section, we extend the results derived in section 2 to the case where covariance matrix is $\sigma^2 I_p$, $\sigma^2 > 0$ unknown. Let $\boldsymbol{X}$ and S be independent observations with $\boldsymbol{X} \sim N_p(\boldsymbol{\theta}, \sigma^2 I_p)$ and $S \sim \sigma^2\chi_k^2$. Here we want to estimate $\boldsymbol{\theta}$ under the loss function

$$L(\boldsymbol{\delta}, \boldsymbol{\theta}) = \frac{\|\boldsymbol{\delta} - \boldsymbol{\theta}\|^2}{\sigma^2}. \tag{3.1}$$

Again the usual estimator is $\boldsymbol{\delta}^0 = \boldsymbol{X}$ and the Lindley type estimator dominating $\boldsymbol{\delta}^0$ is

$$\boldsymbol{\delta}^1 = \overline{X}\,\boldsymbol{1} + \left(1 - \frac{(p-3)S}{(k+2)\|\boldsymbol{X} - \overline{X}\,\boldsymbol{1}\|^2}\right)(\boldsymbol{X} - \overline{X}\,\boldsymbol{1}), \quad p \geq 4. \tag{3.2}$$

We construct the sequence $\{\boldsymbol{\delta}^n\}$ of improved estimators as follows. Let $\boldsymbol{\delta}^n = \overline{X}\,\boldsymbol{1} + K_n(\boldsymbol{X} - \overline{X}\,\boldsymbol{1})$, where

$$K_n = 1 - \frac{(p-3)S}{(k+2)\|\boldsymbol{X} - \overline{X}\,\boldsymbol{1}\|^2} + \sum_{j=1}^{n-1} \frac{c_j S^{1+\frac{\alpha_j}{2}}}{\|\boldsymbol{X} - \overline{X}\,\boldsymbol{1}\|^{2+\alpha_j}},$$
$$\alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_1 > 0 \tag{3.3}$$

and

$$\boldsymbol{\delta}^{n+1} = \boldsymbol{\delta}^n + \boldsymbol{r}_n = \boldsymbol{\delta}^n + r_n^*(\boldsymbol{X} - \overline{X}\,\boldsymbol{1}).$$

Then,

$$RD(n+1,n) = R(\boldsymbol{\delta}^{n+1}, \boldsymbol{\theta}) - R(\boldsymbol{\delta}^{n}, \boldsymbol{\theta})$$

$$= \frac{1}{\sigma^2} E\left[\sum_{i=1}^{p} r_{ni}^2 + 2\sum_{i=1}^{p} r_{ni}(\delta_i^n - \theta_i)\right]$$

$$= \frac{1}{\sigma^2} E\left[\sum_{i=1}^{p} r_{ni}^2 + 2(K_n - 1) r_{ni}(X_i - \overline{X}) + 2\sigma^2 \frac{\partial}{\partial X_i} r_{ni}\right] \quad (3.4)$$

The last expression follows from Stein's normal identity assuming that all the expectations exist. By taking $r_n^* = \dfrac{c_n S^{1+\frac{\alpha_n}{2}}}{\|\boldsymbol{X} - \overline{X}\mathbf{1}\|^{2+\alpha_n}}$, where c_n is a suitable constant and $\alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_1 > 0$, one can get

$$RD(n+1,n) = \frac{1}{\sigma^2} E\left[c_n^2 \frac{S^{2+\alpha_n}}{\|\boldsymbol{X} - \overline{X}\mathbf{1}\|^{2+2\alpha_n}} + 2\sigma^2 c_n(p-(3+\alpha_n)) \right.$$

$$\times \frac{S^{1+\frac{\alpha_n}{2}}}{\|\boldsymbol{X} - \overline{X}\mathbf{1}\|^{2+\alpha_n}} - 2c_n \frac{(p-3)S^{2+\frac{\alpha_n}{2}}}{(k+2)\|\boldsymbol{X} - \overline{X}\mathbf{1}\|^{2+\alpha_n}}$$

$$\left. + 2c_n S^{1+\frac{\alpha_n}{2}} \sum_{j=1}^{n-1} c_j \frac{S^{1+\frac{\alpha_j}{2}}}{\|\boldsymbol{X} - \overline{X}\mathbf{1}\|^{2+\alpha_j+\alpha_n}}\right]$$

$$= E_{T,S}\left[c_n^2 \frac{1}{T^{1+\alpha_n}} \left(\frac{S}{\sigma^2}\right)^{2+\alpha_n} \right.$$

$$+ 2c_n(p-(3+\alpha_n)) \frac{1}{T^{1+\frac{\alpha}{2}}} \left(\frac{S}{\sigma^2}\right)^{1+\frac{\alpha_n}{2}}$$

$$- 2c_n \frac{p-3}{(k+2)T^{1+\frac{\alpha_n}{2}}} \left(\frac{S}{\sigma^2}\right)^{2+\frac{\alpha_n}{2}}$$

$$\left. + 2c_n \sum_{j=1}^{n-1} c_j \frac{1}{T^{1+\frac{(\alpha_j+\alpha_2)}{2}}} \left(\frac{S}{\sigma^2}\right)^{2+\frac{(\alpha_j+\alpha_n)}{2}}\right],$$

where $T = \dfrac{\|\boldsymbol{X} - \overline{X}\mathbf{1}\|}{\sigma^2}, (\dfrac{S}{\sigma^2})\ \chi_k^2(\lambda)$ with $\lambda = \dfrac{\|\boldsymbol{\theta} - \overline{\theta}\mathbf{1}\|}{\sigma^2}, (\dfrac{S}{\sigma^2})\ \chi_k^2$, and they are independent. Similar to the previous section a sufficient condition for $RD(n+1,n) \le 0$ is

$$0 < c_n < \alpha_n \frac{k+p-1}{k+2} \varepsilon_n(p, \alpha_1, \cdots, \alpha_n), \quad (3.5)$$

where

$$\varepsilon_n(p,\alpha_1,\cdots,\alpha_n)=\frac{\Gamma(\frac{(p-3-\alpha_n)}{2})\ \Gamma(\frac{k}{2}+1+\frac{\alpha_n}{2})}{\Gamma(\frac{(p-3)}{2}-\alpha_n)\ \Gamma(\frac{k}{2}+1+\alpha_n)}\left\{1-\frac{2(k+1)}{\alpha_n(k+p-1)}\right.$$

$$\left.\times\sum_{j=1}^{n-1}c_j\frac{\Gamma(\frac{k}{2}+2+\frac{(\alpha_j+\alpha_n)}{2})\Gamma(\frac{(p-3)}{2}-\frac{(\alpha_j+\alpha_n)}{2})}{\Gamma(\frac{k}{2}+1+\frac{\alpha_n}{2})\Gamma(\frac{p-3-\alpha_n}{2})}\right\}. \quad (3.6)$$

The optimal value c_n is $c_n^0=\alpha_n\frac{k+p-1}{2(k+2)}\varepsilon_n(p,\alpha_1,\cdots,\alpha_n)$.

Theorem 3.1. An estimator $\boldsymbol{\delta}^n$ of the form (3.3) is uniformly dominated by $\boldsymbol{\delta}^{n+1}=\boldsymbol{\delta}^n+(\frac{c_n S^{1+\frac{\alpha_n}{2}}}{||\boldsymbol{X}-\overline{X}\ \boldsymbol{1}||^{2+\alpha_n}})(\boldsymbol{X}-\overline{X}\ \boldsymbol{1})$ under the loss (3.1) provided $p-1>2(1+\alpha_n)$ and c_n satisfies the condition (3.5).

Remark 3.1. Note that the function r_{ni}, $i=1,2,\cdots,p$, satisfies the regularity condition of Stein's normal identity which enables us to derive (3.4). Here also the question of convergence of $\{\boldsymbol{\delta}^n\}$ remains open.

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