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Bivariate EWMA Control Charts for Autocorrelated Processes

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Abstract

In this paper we establish bivariate exponentially weighted moving average (EWMA) control charts for autocorrelated processes using residual vectors. We first derive the residual vectors, their expectation, variance-covariance matrix, then evaluate the control chart based on the average run length (ARL).

Keywords : Bivariate EWMA Control Chart, Residual Vectors, ARL

1. Introduction

In statistical process control, it is usually assumed on the process output at different times are IID. However, for many processes the observations are correlated and control charts for monitoring these processes have recently received much attention. Many authors suggested fitting on appropriate time series model to the process and using residuals as control statistic on a control chart(Abraham and Kartha(1979), Alwan and Roberts(1988), Yourstone and Montgomery(1989), Harris and Ross(1991), Box and Ramirez(1992), Longnecker and Ryan(1992), Yaschin(1993), Wardell et al.(1993), and Runger et al.(1995)).

There are many situations in which the simultaneous control of two or more related quality characteristics is necessary so, many authors suggested multivariate control charts (Alt(1985), Jackson(1985), Lowry and Montgomery(1995), Mason et al(1997)). Lowry et al(1992) studied multivariate EWMA chart under process

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output at different times are IID.

Autocorrelation is quite common in in chemical and process industries and often several process variables are measured and stored simultaneously. Runger(1996), Mason et al(1996), Bakshi(1998), Negiz and Cinar(1998) studied multivariate control charts for autocorrelated processes.

Our objective is to evaluate the properties of multivariate EWMA chart when the residual vectors are used to monitor an autocorrelated process. A simple vector time series model is used here to represent the observations from an autocorrelated process.

2. Modeling and Controlling the Process

2.1 Vector Time Series Process Measured with Random Error Vector

In this section, a vector time series process with random error vector will be developed. Let

$$\underline{x}_t = \underline{\mu}_t + \underline{\varepsilon}_t , \tag{2.1}$$

where $\underline{\chi}_t$ is the observation vector at time t, $\underline{\mu}_t$ is the process mean vector at time t, and $\underline{\varepsilon}_t$ is the measurement error vector at time t.

It is also assumed that $\underline{\mu}_t$ can be described as a vector first order autoregressive (vector AR(1)) process with process level $\underline{\xi}$, that is (Box and Jenkins(1976)),

$$\underline{\mu}_t = (I - \Phi)\underline{\xi} + \Phi\,\underline{\mu}_{t-1} + \underline{\delta}_t , \tag{2.2}$$

where $\underline{\delta}_t$,, the random shock vector at time t, is assumed to be multivariate normally distributed mean vector $\underline{0}$ and variance-covanance matrix $\sum\underline{\delta}$, independent of measurement error vector, and independent of the random shock vector at any other time.

The vector AR(1) process observed with measurement error vector is equivalent to a vector first-order antoregressive first-order moving average (vector ARMA(1,1)) process (Box and Jenkins(1976)). The ARMA(1, 1) process can be written as

$$(I - \Phi B)\,\underline{\chi}_t = (I - \Phi)\underline{\xi} + (I - \Theta B)\,\underline{a}_t \, . \tag{2.3}$$

where B is a back shift operator such that $B\,\underline{x}_t = \underline{x}_{t-1}$, and $\underline{a}_t$ is uncorrelated and multivariate normally distributed with mean vector $\underline{0}$ and variance - covariance matrix $\sum \underline{a}$ and Θ is the moving -average parameter matrix and Φ is the autoregressive parameter matrix.

2.2 Step Change in the Process Level

When the process is in-control ($\underline{\xi} = \underline{\xi}_0$), minimum MSE forecast vector made at time t for $t+1$ is

$$\underline{\hat{x}}_t = \underline{\xi}_0 + \Phi(\underline{x}_t - \underline{\xi}_0) - \Theta \underline{e}_t,$$

where $\underline{e}_t = \underline{x}_t - \underline{\hat{x}}_{t-1}$

$$= \underline{x}_t - \underline{\xi}_0 - \Phi(\underline{x}_{t-1} - \underline{\xi}_0) + \Theta \underline{e}_{t-1}$$

and $\underline{e}_t$ is the residual vector at time t. Suppose that there is a step change from $\underline{\xi}_0$ to $\underline{\xi}_1$ in the process level between time $t=k-1$ and k. Then the process can be written as

$$\underline{x}_t = \begin{cases} \underline{\xi}_0 - \Phi\,\underline{\xi}_0 + \Phi\,\underline{x}_{t-1} + \underline{a}_t - \Theta\,\underline{a}_{t-1}, & t=k-1, k-2, \cdots \\ \underline{\xi}_1 - \Phi\,\underline{\xi}_0 + \Phi\,\underline{x}_{t-1} + \underline{a}_t - \Theta\,\underline{a}_{t-1}, & t=k \\ \underline{\xi}_1 - \Phi\,\underline{\xi}_1 + \Phi\,\underline{x}_{t-1} + \underline{a}_t - \Theta\,\underline{a}_{t-1}, & t=k+1, k+2, \cdots \end{cases}$$

and the expectation of the observation vector $\underline{x}_t$ is

$$E(\underline{x}_t) = \begin{cases} \underline{\xi}_0, & t=k-1, k-2, \cdots \\ \underline{\xi}_1, & t=k, k+1, k+2, \cdots \end{cases}$$

Therefore the residual vectors are

$$\underline{e}_t = \begin{cases} \underline{a}_t, & t<k \\ (\underline{\xi}_1 - \underline{\xi}_0) + \underline{a}_t, & t=k \\ (I+\Theta+\Theta^{2+\cdots}+\Theta^{l-1})(I-\Phi)+\Theta^l](\underline{\xi}_1 - \underline{\xi}_0) + \underline{a}_t, & t=k+l,\ l=1,2,3,\cdots \end{cases}$$

Since $\underline{a}_t$ is white noise, we have

$$E(\underline{e}_t) = \begin{cases} \underline{0}, & t=k-1, k-2, \cdots \\ \underline{\xi}_1 - \underline{\xi}_0, & t=k \\ (I+\Theta+\Theta^2+\cdots+\Theta^{l-1})(I-\Phi)+\Theta^l](\underline{\xi}_1 - \underline{\xi}_0), & t=k+1, k+2, \cdots \end{cases}$$

3. Multivariate EWMA Chart of Residual Vectors

The Multivariate EWMA control statistic using residual vectors is

$$Y_t = (1-\lambda) Y_{t-1} + \lambda W_t, \qquad 0 < \lambda \le 1$$

where $W_t = [\underline{e}_t - E(\underline{e}_t)]' \sum_a^{-1} [\underline{e}_t - E(\underline{e}_t)]$.

The continuous-state Markov chain is discretized by dividing the interval between 0 and the control limit L into n subintervals of width $w = L/n$. Let S_j be the midpoint of the jth interval, j = 1, 2, · · · , n. So, Y_t is in transient state j at time t if $S_j - \frac{w}{2} < Y_t < S_j + \frac{w}{2}$. Brook and Evans (1972) has shown that the ARL vector $\underline{N}$ when the process is in control is given by

$$\underline{N} = (I - R)^{-1} \underline{1},$$

where the jth element of $\underline{N}$ represents the ARL for the process starts from state j. Transition probability matrix, represented in partitioned matrix form, for this Markov chain can be written as

$$P = (p_{jk}) = P \quad (\text{going to state } k \mid \text{in state } j)$$

$$= \begin{pmatrix} R & (I-R)\underline{1} \\ 0^T & 1 \end{pmatrix},$$

where the submatrix R is a $(n \times n)$ matrix which contains the transient probability of going from one transient state to another, I is a $(n \times n)$ identity matrix, 0 is a column vector of zeros and 1 is the column vector of ones.

4. Numerical Results and Conclusions

The parameters $\sum_{\underline{\varepsilon}}$, ρ, Φ and Θ in the time series model can be estimated from the preliminary data. For simplicity, suppose that those parameters are known. The following control procedures will be compared on the basis of their ARL performances.

(1) $\sum_{\underline{\varepsilon}} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, ρ = 0.2, 0.5, 0.8

(2) $\Phi = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}$, $\Phi = \begin{pmatrix} 0.3 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}$, $\Phi = \begin{pmatrix} 0.8 & 0.5 \\ 0.5 & 0.8 \end{pmatrix}$,

$\Phi = \begin{pmatrix} 0.9 & 0.8 \\ 0.8 & 0.9 \end{pmatrix}$

(3) $\underset{\sim}{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underset{\sim}{\xi}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\underset{\sim}{\xi}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\underset{\sim}{\xi}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

When comparing control charts, some kinds of standard for comparison is necessary. The charts are matched for ARL when the process is in control. This enables the performance to be evaluated when the process has shifted away from its target value. In our computation, the ARL in control was fixed to be 200.

Tables 1-4 show that ARL of the bivariate EWMA control charts decrease as λ increases, and decreases as ρ and $\underset{\sim}{\xi}$ increase.

In conclusion, we drive residual vector, variance-covariance matrix and then construct multivariate EWMA control statistic for autocorrelated process. Also we evaluate bivariate EWMA control charts based on the ARL and identify the relationship among λ, ρ, $\underset{\sim}{\xi}$ and ARL.

(Table 1) ARL values for multivariate EWMA charts based on residual vectors with $\Phi = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}$ and $\Theta = \begin{pmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \end{pmatrix}$

$\underset{\sim}{\xi}'$	ρ	λ=0.05	λ=0.1	λ=0.3
(1 0)	0.2	38.458	30.265	30.840
	0.5	26.303	19.002	16.663
	0.8	11.602	7.172	4.014
(0 1)	0.2	25.249	18.105	16.022
	0.5	17.270	11.622	8.681
	0.8	7.619	4.740	2.634
(-1 1)	0.2	9.362	5.794	3.192
	0.5	5.778	3.317	1.595
	0.8	1.842	1.077	1.001
(2 0)	0.2	10.957	6.706	3.672
	0.5	6.843	3.786	1.719
	0.8	2.002	1.093	1.001

(Table 2) ARL values for multivariate EWMA charts based on residual vectors with $\Phi = \begin{pmatrix} 0.3 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}$ and $\Theta = \begin{pmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \end{pmatrix}$

ξ'	ρ	λ=0.05	λ=0.1	λ=0.3
(1 0)	0.2	72.258	64.940	71.774
	0.5	8.682	5.926	4.013
	0.8	26.107	18.670	15.979
(0 1)	0.2	50.372	42.186	45.680
	0.5	8.202	5.629	3.792
	0.8	17.419	11.636	8.568
(-1 1)	0.2	21.365	14.729	11.673
	0.5	6.532	4.278	2.615
	0.8	5.247	2.748	1.276
(2 0)	0.2	24.354	17.044	14,059
	0.5	6.298	4.072	2.465
	0.8	6.191	3.079	1.347

(Table 3) ARL values for multivariate EWMA charts based on residual vectors with $\Phi = \begin{pmatrix} 0.8 & 0.5 \\ 0.5 & 0.8 \end{pmatrix}$ and $\Theta = \begin{pmatrix} 0.3 & 0.2 \\ 0.3 & 0.2 \end{pmatrix}$

ξ'	ρ	λ=0.05	λ=0.1	λ=0.3
(1 0)	0.2	77.216	70.730	77.999
	0.5	70.593	63.350	70.376
	0.8	35.140	26.912	26.482
(0 1)	0.2	88.769	82.840	91.036
	0.5	86.426	80.816	89.249
	0.8	44.801	36.334	37.736
(-1 1)	0.2	35.255	27.084	26.465
	0.5	25.545	18.019	15.231
	0.8	10.343	5.945	2.800
(2 0)	0.2	31.204	23.363	22.042
	0.5	22.625	15.638	12.412
	0.8	9.214	5.296	2.524

(Table 4) ARL values for multivariate EWMA charts based od residual vectors with $\Phi = \begin{pmatrix} 0.9 & 0.8 \\ 0.8 & 0.9 \end{pmatrix}$ and $\Theta = \begin{pmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \end{pmatrix}$

ξ'	ρ	λ=0.05	λ=0.1	λ=0.3
(1 0)	0.2	25.953	18.799	16.696
	0.5	17.675	12.037	9.150
	0.8	7.648	4.818	2.735
(0 1)	0.2	38.794	30.267	30.665
	0.5	26.355	18.931	16.480
	0.8	10.648	6.377	3.370
(-1 1)	0.2	9.280	5.741	3.163
	0.5	5.760	3.308	1.588
	0.8	1.818	1.072	1.001
(2 0)	0.2	7.783	4.921	2.788
	0.5	5.008	3.068	1.601
	0.8	1.935	1.091	1.001

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