JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 15, No.1, June 2002

THE PROBABILITY DISTRIBUTION AND ITS SIMULATION ACTIVITY OF A TRIANGLE RANDOMLY DRAWN IN A CIRCLE WITH RADIUS r

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ABSTRACT. Trot(1999) considered how to calculate the expected area of a random triangle in the unit square $[0,1] \times [0,1]$. He used the Mathematica software package for the computational part. In this article, we study various aspects of the probability distribution of a triangle randomly chosen inside the circle of radius r. A simulation activity that can be conducted in statistics and probability classrooms is also considered.

1. Introduction

Consider a circle with radius r. For convenience purpose, let us assume that the center of the circle is $(0,0)$. Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) represent three coordinates randomly chosen inside the circle. A triangle is formed by joining these three points. In this article, the area of the triangle will be represented by A. Note that due to the randomness of the three coordinates, the area A has a sampling distribution. In this article we study the shape of the sampling probability distribution, the mean value, and the standard deviation of A. Due to the complexity of the structure associated with A, most of the computational part will be done through simulation activities

Received by the editors on June 1, 2002.
2000 *Mathematics Subject Classifications* : Primary 97C80 ; Secondary 97U70.
Key words and phrases: sampling distribution, mean value, standard deviation, simulation.

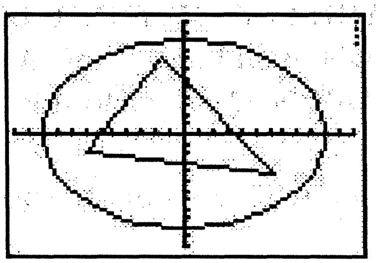

Figure 1. A triangle randomly drawn in a circle with radius r.

It is known that the area of the triangle with the three vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is determined by the formula

$$A = .5 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \text{ or } A = .5\sqrt{(x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3 - x_2y_1)^2}$$

For convenience let

$$A(x_1, y_1, x_2, y_2, x_3, y_3) = .5\sqrt{(x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3 - x_2y_1)^2}$$

Since each (x_i, y_i) is chosen from the inside the circle with radius r and center $(0, 0)$, it has the joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{\pi r^2} & \text{if } x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

The mean value of A is

$$\begin{aligned} E(A) &= E(A(X_1, Y_1, X_2, Y_2, X_3, Y_3)) \\ &= \underset{\substack{x_i^2+y_i^2 \le r^2 \\ i=1,2,3}}{\int \cdots \int} A(x_1, y_1, x_2, y_2, x_3, y_3) \left(\frac{1}{\pi r^2}\right)^3 dx_1 dy_1 dx_2 dy_2 dx_3 dy_3 \\ &= \int_{-r}^{r} \int_{-\sqrt{r^2-x_1^2}}^{\sqrt{r^2-x_1^2}} \int_{-r}^{r} \int_{-\sqrt{r^2-x_1^2}}^{\sqrt{r^2-x_1^2}} \int_{-r}^{r} \int_{-\sqrt{r^2-x_1^2}}^{\sqrt{r^2-x_1^2}} A(x_1, y_1, x_2, y_2, x_3, y_3) \\ (1) \qquad & \qquad \left(\frac{1}{\pi r^2}\right)^3 dx_1 dy_1 dx_2 dy_2 dx_3 dy_3 \end{aligned}$$

and the variance of A is

$$V(A) = E(A^2) - E^2(A)$$

where

$$E(A^2) = \underset{\substack{x_i^2+y_i^2 \le r^2 \\ i=1,2,3}}{\int \cdots \int} A(x_1, y_1, x_2, y_2, x_3, y_3) \left(\frac{1}{\pi r^2}\right)^3 dx_1 dx_2 dx_3 dy_1 dy_2 dy_3$$

2. Simulation

It requires a careful consideration to choose a point (x, y) randomly inside the circle with radius r and center $(0, 0)$, because x and y are dependent random variables subject to the constraint $x^2 + y^2 \le r$. We state and prove Theorem 1, which is a basis to generate a point (x, y) randomly inside the circle.

Theorem 1. Suppose that ν and θ are independent random variables with the uniform probability distribution on $(0, r)$ and $(0, 2\pi)$, respectively. Then, $(\sqrt{r\nu}\cos\theta, \sqrt{r\nu}\sin\theta)$ has the joint uniform probability distribution on $\{(x, y) : x^2 + y^2 \le r^2\}$.

Proof. First of all, since ν and θ are two independent random variables with the uniform probability distribution on $(0, r)$ and $(0, 2\pi)$, respectively, the joint probability density function of ν and θ is

$$f(\nu, \theta) = \begin{cases} \frac{1}{2\pi} & \text{on } \{(\nu, \theta) : 0 < \nu < r, 0 < \theta < 2\pi\} \\ 0 & \text{otherwise} \end{cases}$$

Let $X = \sqrt{r\nu}\cos\theta$ and $Y = \sqrt{r\nu}\sin\theta$. Then, the characteristic function $\varphi_{(X,Y)}(t_1, t_2)$ of (X, Y) is

$$\begin{aligned} \varphi_{(X,Y)}(t_1, t_2) &= E(e^{i(t_1,t_2)-(\sqrt{\nu r},\sqrt{\nu r})}) \\ &= \int_0^{2\pi}\int_0^r e^{i(t_1\sqrt{\nu r}\cos\theta + t_2\sqrt{\nu r}\sin\theta)} f(\nu, \theta)\, d\nu d\theta \\ &= \int_0^{2\pi}\int_0^r e^{i(t_1\sqrt{\nu r}\cos\theta + t_2\sqrt{\nu r}\sin\theta)} \frac{1}{2\pi}\, d\nu d\theta \qquad (2) \end{aligned}$$

Suppose (V, W) has a joint uniform probability distribution on $\{(x, y) : x^2 + y^2 \le r^2\}$. Then, the characteristic function, say $\phi(t_1, t_2)$, of (V, W) is

$$\begin{aligned} \phi(t_1, t_2) &= E(e^{i(t_1 V + t_2 W)}) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1 v + t_2 w)} f(v, w)\, dv dw \\ &= \iint_{v^2 + w^2 \le r^2} e^{i(t_1 v + t_2 w)} \frac{1}{\pi r^2}\, dv dw \qquad (3) \end{aligned}$$

Now for substitution purposes, let $v = \sqrt{\nu r}\cos\theta$ and $w = \sqrt{\nu r}\sin\theta$. Then,

$$(3) = \int_0^{2\pi} \int_0^r e^{i(t_1\sqrt{\nu r}\cos\theta + t_2\sqrt{\nu r}\sin\theta)} d\nu d\theta,$$

which is identical to (1). By the uniqueness theorem (Billingsley, p. 346), (X, Y) has a uniform probability distribution jointly on $\{(x, y) : x^2 + y^2 \leq r^2\}$.

□

Let ν and θ be the two independent random variables of uniform probability distribution on $(0, r)$ and $(0, 2\pi)$, respectively. As Theorem 1 indicates, a point (x, y) can be obtained randomly inside the circle with radius r and center $(0, 0)$ provided $x = \sqrt{r\nu}\cos\theta$ and $y = \sqrt{r\nu}\sin\theta$. Most graphing device carries a built in function to generate a random number between 0 and 1. For instance, $rand(n)$ in TI-83 calculator or $.rand83(n)$ in TI-89 calculator generates a data set of size n from the uniform distribution on $(0, 1)$. Therefore, one can use such function to simulate a three (x, y) points randomly inside the circle according tot the rule, $x = \sqrt{r\nu}\cos\theta$ and $y = \sqrt{r\nu}\sin\theta$, that was discussed in Theorem 1. Based on the three (x, y) points chosen, an area A will be obtained. As the process is repeated many times, say n, a simulated data of A of size n can be obtained.

The Figure 2 is the frequency histogram of a simulated As of size $n = 300$ when $r = 1$. The domain of the frequency histogram ranges from 0 to 12, and its class width is 1.0. As the histogram shows, the distribution of A is highly skewed to the right. Table 1 summarizes the sample means and standard deviations for the sample of A of various r and n. A TI-83 graphing calculator and Matlab v 5.3 were used for the computational part. As it shows, the expected value of A for $r = 1$ is approximately .231.

Table 1. The sample mean and the sample standard deviation of A.

n	$r=1$		$r=2$		$r=3$		$r=4$	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
100	.227	.194	.987	.842	2.050	1.787	3.338	3.152
1000	.238	.204	.928	.824	2.078	1.663	33.701	3.189
10000	.234	.200	.923	.794	2.071	1.791	3.722	3.212
100000	.231	.197	.934	.795	2.084	1.782	3.693	3.170

n	$r=5$		$r=6$		$r=7$	
	Mean	SD	Mean	SD	Mean	SD
100	6.396	4.923	8.718	8.109	12.555	9.623
1000	5.775	4.934	8.233	7.174	11.215	9.577
10000	5.797	4.960	8.355	7.204	11.279	9.760
100000	5.762	4.936	8.447	7.322	11.318	9.626

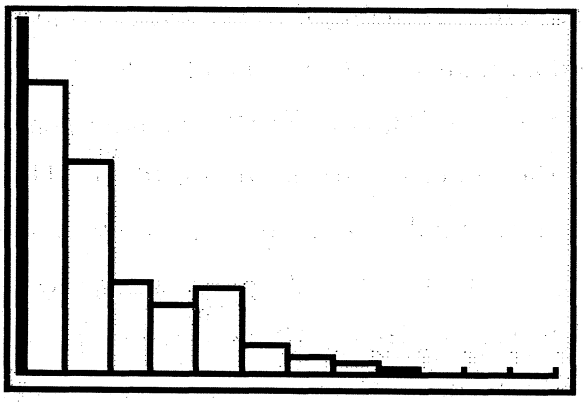

Figure 2. The frequency histogram of the 200 simulated A when $r=1$

3. Concluding remarks.

In this article we studied various aspects of the probability distribution of a triangle randomly chosen inside a circle with radius r. The integrals of the mean value and the variance of A seem to demonstrate extreme infeasibility due to the complexity of the structure associated with the three vertices. For this reason, most of the computational part was estimated by simulation work. Although these simulation works are not perfect solutions to the asked questions in this article, its spirit allows us to approach the problems in tractable mode. The content presented in this article perhaps may be classified into a geometric probability. It gives undergraduate students an opportunity to think about a geometry problem from a probabilistic and statistical stand point. The simulation work can be accomplished without so much difficulty. As a demonstration, the simulation code set for the TI-83/89 is provided in the appendix of this article.

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Appendix: TI-codes for the Simulation

```
: ClrHome
: ClrDraw
: ClrList ⌊THETA, ⌊RADI, ⌊A
: Prompt N,R
: -R-.2→ Xmin
: R+.2→ Xmax
: -R-.2→ Ymin
: R+.2→ Ymax
: Circle(0,0,R)
: For(I,1,N)
```

: For(I,1,N)

: ClrList $\lfloor$X,$\lfloor$Y

: rand(3)×2π → $\lfloor$THETA

: rand(3)×R → $\lfloor$RADI

: $\sqrt{}$(($\lfloor$RADI× R)cos($\lfloor$THETA))→ $\lfloor$X

: $\sqrt{}$(($\lfloor$RADI× R)sin ($\lfloor$THETA))→ $\lfloor$X

: .5×$\sqrt{}$(($\lfloor X_3(I)\lfloor Y_1(I)+\lfloor X_1(I)\lfloor Y_2(I)-\lfloor X_3(I)\lfloor Y_2(I)-\lfloor X_1(I)\lfloor Y_3(I)+$
$\lfloor X_2(I)\lfloor Y_3(I)-\lfloor X_2(I)\lfloor Y_1(I))^2) \to \lfloor A(I)$

: Line($\lfloor$X(1),$\lfloor$Y(1), $\lfloor$X(2), $\lfloor$Y(2))

: Line($\lfloor$X(2), $\lfloor$Y(2), $\lfloor$X(3), $\lfloor$Y(3))

: Line($\lfloor$X(3), $\lfloor$Y(3), $\lfloor$X(1), $\lfloor$Y(1))

: End

: Disp mean($\lfloor$A),stdDev($\lfloor$A)

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