JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 15, No.1, June 2002

DYNAMICS OF RELATIONS

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ABSTRACT. Let X be a compact metric space and let f be a continuous relation on X. Let U be an attractor block for f and let A be an attractor determined by U. Then there exists a continuous function $\lambda : X \to [0,1]$ such that

$$\lambda^{-1}(0) = A,\ \lambda^{-1}(1) = X - B(A,U),\ \text{and}\ M(\lambda, f)(x) < \lambda(x)$$

for all $x \in B(A,U) - A$.

1. Introduction

Let (X,d) be a compact metric space and let $f : X \to X$ be a continuous map. The following result is well known [4].

THEOREM. *Let A be an attractor for f. Then there exists a continuous function $h : X \to [0,1]$ such that*
(1) $h^{-1}(0) = A$ and $h^{-1}(1) = X - B(A)$,
(2) $h(f(x)) < h(x)$ for all $x \in B(A) - A$,
where $B(A)$ is a basin of A.

In this paper, we extend this result to the case of a continuous relation.

2. Continuous relations

In this paper, X is a compact metric space with a metric d.

DEFINITION 2.1. Let f be a relation on X whose domain is X and let $x \in X$.

(1) f is said to be *upper semicontinuous* at x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $f(y) \subset B(f(x), \varepsilon)$.

supported by the Research Foundations of Chungnam National University, 2001.

Received by the editors on June 3, 2002 .

2000 *Mathematics Subject Classifications*: 37B25, 54H20.

Key words and phrases: attractor, attractor block, continuous relation, chain.

(2) f is said to be *lower semicontinuous* at x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $f(x) \subset B(f(y), \varepsilon)$.

(3) f is said to be *continuous* at x if f is both upper semicontinuous and lower semicontinuous at x.

Throughout this paper, f is a continuous relation on X such that $f(x)$ is a closed subset of X for all $x \in X$.

For any continuous function $\varepsilon : X \to \mathbb{R}$, define $m(\varepsilon, f), M(\varepsilon, f) : X \to \mathbb{R}$ by

$$m(\varepsilon, f)(x) = \min \varepsilon(f(x)) \text{ and } M(\varepsilon, f)(x) = \max \varepsilon(f(x)).$$

THEOREM 2.1. *$m(\varepsilon, f)$ and $M(\varepsilon, f)$ are continuous functions.*

Proof. Let $x \in X$. Since $f(x)$ is a compact subset of X, there is $z \in f(x)$ such that $\varepsilon(z) = m(\varepsilon, f)(x)$. For any $\eta > 0$ there exists $\nu > 0$ such that

$$d(z, y) < \nu \text{ implies } \varepsilon(y) < \varepsilon(z) + \eta = m(\varepsilon, f)(x) + \eta.$$

Let $\alpha \in f(x)$. Since $m(\varepsilon, f)(x) - \eta < m(\varepsilon, f)(x) \leq \varepsilon(\alpha)$, there is $\nu_\alpha > 0$ such that

$$d(\alpha, y) < \nu_\alpha \text{ implies } m(\varepsilon, f)(x) - \eta < \varepsilon(y).$$

$\bigcup_{\alpha \in f(x)} B(\alpha, \nu_\alpha)$ is a neighborhood of $f(x)$. Since $f(x)$ is a compact subset of X, there exists $\zeta > 0$ such that $B(f(x), \zeta) \subset \bigcup_{\alpha \in f(x)} B(\alpha, \nu_\alpha)$. Let $\xi = \min\{\nu, \zeta\}$. There is $\delta > 0$ such that $d(x, y) < \delta$ implies $D(f(x)), f(y)) < \xi$. Let $d(x, y) < \delta$. Since $z \in f(x) \subset B(f(y), \xi)$, there is $b \in f(y)$ such that $d(z, b) < \xi \leq \nu$. Thus $\varepsilon(b) < m(\varepsilon, f)(x) + \eta$. For every $p \in f(y)$, since

$$f(y) \subset B(f(x), \xi) \subset B(f(x), \zeta) \subset \bigcup_{\alpha \in f(x)} B(\alpha, \nu_\alpha),$$

there is an $\alpha \in f(x)$ such that $d(\alpha, p) < \nu_\alpha$. We have $m(\varepsilon, f)(x) - \eta < \varepsilon(p)$. Thus

$$m(\varepsilon, f)(x) - \eta < m(\varepsilon, f)(y) \leq \varepsilon(b) < m(\varepsilon, f)(x) + \eta.$$

Therefore $m(\varepsilon, f)$ is continuous at x.

By the similar method, $M(\varepsilon, f)$ is a continuous function. □

THEOREM 2.2. *f is a closed subset of $X \times X$.*

Proof. Let $(x,y) \in X \times X - f$. Since $f(x)$ is a compact subset of X, there exists $\varepsilon > 0$ such that $B(y,\varepsilon) \bigcap B(f(x),\varepsilon) = \emptyset$. Since f is continuous at x, there exists a $\delta > 0$ such that $d(x,z) < \delta$ implies $D(f(x), f(z)) < \varepsilon$. We claim that $B(x,\delta) \times B(y,\varepsilon) \bigcap f = \emptyset$. Suppose that $(u,v) \in B(x,\delta) \times B(y,\varepsilon) \bigcap f$. Since $d(x,u) < \delta$, we have $D(f(x), f(u)) < \varepsilon$. Since $v \in f(u) \subset B(f(x),\varepsilon)$ and $v \in B(y,\varepsilon)$, we have $B(y,\delta) \times B(f(x),\varepsilon) \neq \emptyset$. This is a contradiction. Thus $B(x,\varepsilon) \times B(y,\varepsilon) \bigcap f = \emptyset$. Hence f is a closed subset of $X \times X$. □

THEOREM 2.3. *For any compact subset K of X, $f(K)$ is a compact subset of X.*

Proof. Let (y_n) be a sequence in $f(K)$. There exists a sequence (x_n) in K such that $(x_n, y_n) \in f$. Let $x_n \to x \in K$ and $y_n \to y \in X$. Since $(x_n, y_n) \in f$, we have $(x,y) \in \overline{f} = f$. Thus $y \in f(x) \subset f(K)$. Therefore $f(K)$ is a compact subset of X. □

THEOREM 2.4. *For any $A \subset K$, we have $f(\overline{A}) = \overline{f(A)}$.*

Proof. Let $y \in f(\overline{A})$. There exists $x \in \overline{A}$ such that $(x,y) \in f$. For any $\varepsilon > 0$ there is $\delta > 0$ such that $d(x,z) < \delta$ implies $D(f(x), f(z)) < \varepsilon$. Since $x \in \overline{A}$, we have $B(x,\delta) \bigcap A \neq \emptyset$. Let $z \in B(x,\delta) \bigcap A$. Since $d(x,z) < \delta$, we have $D(f(x), f(z)) < \varepsilon$. Since $y \in f(x) \subset B(f(z),\varepsilon)$, there exists $w \in f(z) \subset f(A)$ such that $d(y,w) < \varepsilon$. Thus we have $B(y,\varepsilon) \bigcap f(A) \neq \emptyset$. Therefore $y \in \overline{f(A)}$. Hence $f(\overline{A}) \subset \overline{f(A)}$.

Since $f(A) \subset f(\overline{A})$ and $f(\overline{A})$ is a closed subset of X, we have $\overline{f(A)} \subset f(\overline{A})$. Thus $f(\overline{A}) = \overline{f(A)}$. □

THEOREM 2.5. *Let g be a continuous relation on X such that $g(x)$ is a closed set of X for all $x \in X$. Then $g \circ f$ is a continuous relation on X.*

Proof. Let $x \in X$ and $\varepsilon > 0$. For every $y \in f(x)$ there is $\delta_y > 0$ such that

$$d(y,z) < \delta_y \text{ implies } D(g(y), g(z)) < \frac{\varepsilon}{2}.$$

$\{B(y, \frac{\delta_y}{2} \mid y \in f(x)\}$ is an open cover of $f(x)$. Since $f(x)$ is a compact set, there exist finitely many $y_1, \cdots, y_n \in f(x)$ such that $f(x) \subset \cup_{i=1}^{n} B(y_i, \frac{\delta_{y_i}}{2})$. Let $\delta = \min\{\frac{\delta_{y_i}}{2} | i = 1, \cdots, n\}$. Since f is continuous at x, there exists $\eta > 0$ such that $d(x,w) < \eta$ implies $D(f(x), f(w)) < \delta$. Let $d(x,w) < \eta$. For every $u \in g \circ f(w)$ there exists $a \in X$ such that $(w,a) \in f$ and $(a,u) \in g$. Since $D(f(x), f(w)) < \delta$, we have $a \in f(w) \subset B(f(x), \delta)$. Thus there is $b \in f(x)$ such that $d(a,b) < \delta$. Since $b \in f(x) \subset \bigcup_{i=1}^{n} B(y_i, \frac{\delta_{y_i}}{2})$, there is an integer i such that $b \in B(y_i, \frac{\delta_{y_i}}{2})$. We have

$$d(y_i, a) \leq d(y_i, b) + d(b, a) < \frac{\delta_{y_i}}{2} + \delta \leq \frac{\delta_{y_i}}{2} + \frac{\delta_{y_i}}{2} = \delta_{y_i}.$$

Thus $D(g(y_i), g(a)) < \frac{\varepsilon}{2}$. Since $u \in g(a) \subset B(g(y_i), \frac{\varepsilon}{2})$, there is $c \in g(y_i)$ such that $d(u,c) < \frac{\varepsilon}{2}$. Since $y_i \in f(x)$, we have $c \in g \circ f(x)$. Thus $u \in B(g \circ f(x), \varepsilon)$ for all $u \in g \circ f(w)$. Therefore $g \circ f(w) \subset B(g \circ f(x), \varepsilon)$.

For every $v \in g \circ f(x)$ there is $a' \in X$ such that $(x, a') \in f$ and $(a', v) \in g$. Since $a' \in f(x) \subset \bigcup_{i=1}^{n} B(y_i, \frac{\delta_{y_i}}{2})$, there exists an integer i such that $a' \in B(y_i, \frac{\delta_{y_i}}{2})$. Since $d(y_i, a') < \frac{\delta_{y_i}}{2}$, we have $D(g(y_i), g(a')) < \frac{\varepsilon}{2}$. Since $v \in g(a') \subset B(g(y_i), \frac{\varepsilon}{2})$, there exists $c' \in g(y_i)$ such that $d(v, c') < \frac{\varepsilon}{2}$. Since $D(f(x), f(w)) < \delta$, we have $a' \in f(x) \subset B(f(w), \delta)$. Thus there exists $b' \in f(w)$ such that $d(a', b') < \delta$. Then

$$d(y_i, b') \leq d(y_i, a') + d(a', b') < \frac{\delta_{y_i}}{2} + \delta \leq \frac{\delta_{y_i}}{2} + \frac{\delta_{y_i}}{2} = \delta_{y_i}.$$

Thus we have $D(g(y_i), g(b')) < \frac{\varepsilon}{2}$. Since $c' \in g(y_i) \subset B(g(b'), \frac{\varepsilon}{2})$, there exists $p \in g(b')$ such that $d(c', p) < \frac{\varepsilon}{2}$. Then

$$d(v, p) \leq d(v, c') + d(c', p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $p \in g \circ f(w)$, we have $v \in B(g \circ f(w), \varepsilon)$. Thus $g \circ f(x) \subset B(g \circ f(w), \varepsilon)$. Therefore we have $D(g \circ f(x), g \circ f(w)) < \varepsilon$. Hence $g \circ f$ is continuous at x. □

COROLLARY 2.1. *Let $n \geq 0$. f^n is a continuous relation on X such that $f^n(x)$ is a closed subset of X for all $x \in X$.*

3. Attractors for continuous relations

DEFINITION 3.1. Let U be a nonempty open subset of X. U is called an *attractor block* for f if $\overline{f(U)} \subset U$. An attractor block determines the attractor A where A is defined as $A = \bigcap_{n \geq 0} \overline{f^n(U)}$. The *basin* of A relative to U is the open set defined by

$$\{x \in X | f^n(x) \subset U \text{ for some } n \geq 0\}$$

and is denoted as $B(A, U)$.

DEFINITION 3.2. Let $\varepsilon > 0$. An *ε-chain* for f is any finite nonempty sequence $\Psi = (x_0, x_1, \cdots, x_n)$ of points of X with the property that $d(x_{i+1}, f(x_i)) < \varepsilon$ for all $0 \leq i \leq n-1$.

A *p-chain* for f is any finite nonempty sequence $\Psi = ((x_0, y_0), \cdots, (x_n, y_n))$ of ordered pairs of points of X with the property that $x_{i+1} \in f(y_i)$ for all $0 \leq i \leq n-1$.

Let $\varepsilon > 0$. Then Ψ is said to be an *ε-p-chain* for f if $d(x_i, y_i) < \varepsilon$ for all $0 \leq i \leq n$.

LEMMA 3.1. *For every $\varepsilon > 0$ there exists $\delta > 0$ with the property that if $((x_i, y_i))$ is a δ-p-chain for f then (x_i) is an ε- chain for f.*

Proof. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $D(f(x), f(y)) < \varepsilon$. Since $d(x_i, y_i) < \delta$, we have $D(f(x_i), f(y_i)) < \varepsilon$. Since $x_{i+1} \in f(y_i) \subset B(f(x_i), \varepsilon)$, we have $d(x_{i+1}, f(x_i)) < \varepsilon$. Thus (x_i) is an ε- chain for f. □

For any p-chain $\Psi = ((x_0, y_0), \cdots, (x_n, y_n))$ define

$$\Gamma(\Psi) = \sum_{i=0}^{n} d(x_i, y_i).$$

Let Y be a nonempty closed subset of X and let $Z(x, Y, f)$ be the set of all p-chains for f that begin in Y and end at $x \in X$. Define $L(x, Y, f) = \inf\{\Gamma(\Psi) | \Psi \in Z(x, Y, f)\}$.

Let $x \in Y$. Since $((x, x)) \in Z(x, Y, f)$, we have $L(x, Y, f) = 0$.

LEMMA 3.2. *If $(x, y) \in f$, then $L(y, Y, f) \leq L(x, Y, f)$.*

Proof. Let $\Psi = ((x_0, y_0), \cdots, (x_n, x)) \in Z(x, Y, f)$ with $x_0 \in Y$. Define $\Psi' = ((x_0, y_0), \cdots, (x_n, x), (y, y))$. Then $\Psi' \in Z(x, Y, f)$ and

$$\begin{aligned} \Gamma(\Psi') &= d(x_0, y_0) + \cdots + d(x_n, x) + d(y, y) \\ &= d(x_0, y_0) + \cdots + d(x_n, x) \\ &= \Gamma(\Psi). \end{aligned}$$

Thus we have

$$\begin{aligned} L(y,Y,f) &\leq \inf\{\Gamma(\Psi')|\Psi \in Z(x,Y,f)\} \\ &= \inf\{\Gamma(\Psi)|\Psi \in Z(x,Y,f)\} \\ &= L(x,Y,f). \end{aligned}$$

□

LEMMA 3.3. *The map $x \to L(x,Y,f)$ is continuous.*

Proof. Let $x \in Y$. Then $L(x,Y,f) = 0$. For any $\varepsilon > 0$, $B(x,\varepsilon)$ is a neighborhood of x. For any $y \in B(x,\varepsilon)$, let $\Psi = ((x,y)) \in Z(y,Y,f)$. Then $L(y,Y,f) \leq \Gamma(\Psi) = d(x,y) < \varepsilon$. Thus $L(,Y,f)$ is continuous at x.

Let $x \in X - Y$. For any $h > 0$, since $L(x,Y,f) < L(x,Y,f) + \frac{h}{2}$, there exists $\Psi \in Z(x,Y,f)$ such that $\Gamma(\Psi) > L(x,Y,f) + \frac{h}{2}$. Let $\Psi = ((x_0,y_0),\cdots,(x_n,x))$. $B(x,\frac{h}{2})$ is a neighborhood of x. For any $y \in B(x,\frac{h}{2})$, let $\Psi' = ((x_0,y_0),\cdots,(x_n,y)) \in Z(y,Y,f)$. Since

$$|\Gamma(\Psi) - \Gamma(\Psi')| = |d(x_n,x) - d(x_n,y)| \leq d(x,y) < \frac{h}{2},$$

we have

$$\Gamma(\Psi') - \Gamma(\Psi) \leq |\Gamma(\Psi) - \Gamma(\Psi')| < \frac{h}{2}.$$

Thus

$$L(y,Y,f) \leq \Gamma(\Psi') < \Gamma(\Psi) + \frac{h}{2} < L(x,Y,f) + h.$$

Therefore $L(\ ,Y,f)$ is upper semicontinuous at x.

Suppose that $L(\ ,Y,f)$ is not lower semicontinuous at x. There exists $\beta > 0$ such that for every $\eta > 0$ there is $y \in B(x,\eta)$ such that $L(y,Y,f) < L(x,Y,f) - \beta$. For each i, there exists $z_i \in B(x,\frac{1}{i})$ such that $L(z_i,Y,f) < L(x,Y,f) - \beta$. There is $\Psi_i \in Z(z_i,Y,f)$ such that $\Gamma(\Psi_i) < L(x,Y,f) - \beta$. Let $\Psi_i = ((x_0^i,y_0^i),\cdots,(x_{n_i}^i,z_i))$. Define $\Psi_i' = ((x_0^i,y_0^i),\cdots,(x_{n_i}^i,x)) \in Z(x,Y,f)$. Since $z_i \to x$, there is i such that $d(z_i,x) < \frac{\beta}{2}$. Since

$$|\Gamma(\Psi_i) - \Gamma(\Psi_i')| = |d(x_{n_i}^i,z_i) - d(x_{n_i}^i,x)| \leq d(z_i,x) < \frac{\beta}{2},$$

we have

$$\Gamma(\Psi_i') - \Gamma(\Psi_i) \leq |\Gamma(\Psi_i) - \Gamma(\Psi_i')| < \frac{\beta}{2}.$$

Thus

$$L(x, Y, f) \leq \Gamma(\Psi_i') < \Gamma(\Psi_i) + \frac{\beta}{2} < L(x, Y, f) - \frac{\beta}{2}.$$

This is a contradiction. Thus $L(\ ,Y,f)$ is lower semicontinuous at x. Hence $L(\ ,Y,f)$ is continuous at x. □

REMARK 3.1. *Let $x \in A$. For every $k \geq 1$, since $x \in \overline{f^k(U)}$, we have $L_k'(x) = 0$.*

REMARK 3.2. *Let $\Psi = ((x_0, y_0), \cdots, (x_n, y_n))$ be an $\varepsilon - p$-chain for f^k with $x_0 \in \overline{f^k(U)}$. Then $y_0 \in B(x_0, \varepsilon) \subset B(\overline{f^k(U)}, \varepsilon) \subset B(\overline{f(U)}, \varepsilon) \subset U$. Since $x_1 \in f^k(y_0) \subset f^k(U) \subset \overline{f^k(U)}$, we have $y_1 \in U$. By induction, we have $y_n \in U$.*

LEMMA 3.4. *If $x \in X - U$, then $L_k'(x) \geq \varepsilon$ for every k.*

Proof. Let $\Psi = ((x_0, y_0), \cdots, (x_n, x)) \in Z(x, \overline{f^k(U)}, f^k)$. By the above Remark, Ψ is not $\varepsilon - p$-chain. Thus there is i such that $d(x_i, y_i) \geq \varepsilon$. We have $\Gamma(\Psi) \geq d(x_i, y_i) \geq \varepsilon$.

Thus $L_k'(x) = L(x, \overline{f^k(U)}, f^k) \geq \varepsilon$. □

LEMMA 3.5. $(L_k')^{-1}(0) = \overline{f^k(U)}$ *for all k.*

Proof. Since $L_k' = 0$ on $\overline{f^k(U)}$, we have $\overline{f^K(U)} \subset (L_k')^{-1}(0)$. It suffices to show that $x \in X - \overline{f^k(U)}$ implies $x \in X - (L_k')^{-1}(0)$. Let

$$\Psi = ((x_0, y_0), \cdots, (x_n, x)) \in Z(x, \overline{f^k(U)}).$$

If Ψ is not $\varepsilon - p$-chain, then $\Gamma(\Psi) \geq \varepsilon$. Let Γ be an $\varepsilon - p$-chain. Since

$$y_0 \in B(x_0, \varepsilon) \subset B(\overline{f^k(U)}, f^k) \subset U,$$

we have $x_1 \in f^k(y_0) \subset f^k(U) \subset \overline{f^k(U)}$. By induction, we have $x_n \in \overline{f^k(U)}$. Thus

$$L_k'(x) = L(x, \overline{f^k(U)}, f^k) \geq \min\{\varepsilon, d(x, \overline{f^k(U)})\} > 0.$$

□

For each $k \geq 1$, define $L_k''(x) = \frac{1}{k} \sum_{i=0}^{k-1} M(L_k', f^i)(x)$. L_k'' is a nonnegative continuous function.

LEMMA 3.6. *If* $(x, y) \in f$, *then* $L''_k(y) \leq L''_k(x)$.

Proof. Let $(x, y) \in f$. Since $y \in f(x)$, we have

$$M(L'_k, f^0)(y) = L'_k(y) \leq M(L'_k, f)(x).$$

For $1 \leq i \leq k-2$, since $f^i(y) \subset f^{i+1}(x)$, we have $M(L'_k, f^i)(y) \leq M(L'_k, f^{i+1})(x)$. Since

$$M(L'_k, f^{k-1})(y) \leq M(L'_k, f^k)(x) \leq L'_k(x) = M(L'_k, f^0)(x),$$

we have

$$\begin{aligned} L''_k(y) &= \frac{1}{k}\sum_{i=0}^{k-1} M(L'_k, f^i)(y) \\ &\leq \frac{1}{k}\sum_{i=0}^{k-1} M(L'_k, f^i)(x) = L''_k(x). \end{aligned}$$

□

For $k \geq 1$, define $L_k(x) = \min\{\frac{1}{\varepsilon}L''_k(x), 1\}$.

REMARK 3.3.
(1) L_k *is continuous and nonincreasing along* f *orbits.*
(2) For every $x \in X$, $0 \leq L_k(x) \leq 1$.
(3) $L_k^{-1}(0) = \overline{f^k(U)}$.

Define $L(x) = \sum_{k=1}^{\infty} \frac{L_k(x)}{2^k}$ and $\lambda(x) = \sum_{i=0}^{\infty} \frac{M(L, f^i)(x)}{2^{i+1}}$.

REMARK 3.4.
(1) Since the infinite sums in the above definition are uniformly convergent, L *and* λ *are continuous.*
(2) L *is nonincreasing along the orbits of* f.
(3) For all $x \in X$, $0 \leq L(x), \lambda(x) \leq 1$.

LEMMA 3.7. $\lambda^{-1}(0) = A$.

Proof. Let $x \in \lambda^{-1}(0)$. Then $\lambda(x) = 0$. So that $L(x) = 0$. Thus $L_k(x) = 0$ for all k. Therefore $x \in \bigcap_{k=1}^{\infty} \overline{f^k(U)} = A$. Let $x \in A$. Then $L'_k(x) = 0$ for all k. So that $L''_k(x) = 0$ for all k. Thus $L_k(x) = 0$ for all k and so $L(x) = 0$. Thus $\lambda(x) = 0$. □

LEMMA 3.8. *λ is nonincreasing along the orbits of f.*

Proof. Let $(x,y) \in f$. For every $z \in f^{i+1}(x) = f(f^i(x))$, there exists $a \in f^i(x)$ such that $(a,z) \in f$. Then $L(z) \le L(a) \le M(L, f^i)(x)$. Since z is arbitrary, we have

$$M(L, f^{i+1})(x) \le M(L, f^i)(x).$$

Since $y \in f(x)$, we have $f^i(y) \subset f^{i+1}(x)$. Thus

$$M(L, f^i)(y) \le M(L, f^{i+1})(x) \le M(L, f^i)(x).$$

Therefore

$$\lambda(y) = \sum_{i=0}^{\infty} \frac{M(L, f^i)(y)}{2^{i+1}} \le \sum_{i=0}^{\infty} \frac{M(L, f^i)(x)}{2^{i+1}},$$

which completes the proof. □

LEMMA 3.9. *If $x \in B(A,U) - A$, then $\lambda(y) < \lambda(x)$ for all $y \in f(x)$.*

Proof. Let $(x,y) \in f$. Since each term in the series defining λ is no larger at y than it is at x, it is enough to show that there is one of these terms that is actually smaller at y that at x. There are two cases. In the first case, $x \in f(U) - A$. Since the definition of A, there is a smallest integer k such that $x \notin \overline{f^k(U)}$. Then $k \ge 2$ and $L_k(x) > 0$. Since $x \in \overline{f^i(U)}, L_i(x) = 0$ for all k. Since $y \in f(x)$, we have

$$y \in f(\overline{f^{k-1}(U)}) \subset \overline{f^k(U)}.$$

Then $L_k(y) = 0$. Thus $L_k(y) < L_k(x)$. Hence $\lambda(y) < \lambda(x)$.

In the remaining case $x \in B(A,U) - f(U)$, there is a natural number i with the property that $f^i(x) \subset f(U) - A$. By the first case,

$$M(L, f^i)(y) \le M(L, f^{i+1})(x) < M(L, f^i)(x).$$

Hence $\lambda(y) < \lambda(x)$. □

LEMMA 3.10. $\lambda^{-1}(1) = X - B(A,U)$.

Proof. If $x \in X - B(A,U)$, then $f^i(x) \not\subseteq U$ for all i. Thus there is $y_i \in f^i(x) - U$ for every i. Since $y_i \notin U$, we have $L'_k(y_i) \geq \varepsilon$ for all k. Then for every k,

$$L''_k(y_i) = \frac{1}{k}\sum_{j=0}^{k-1} M(L'_k, f^j)(y_i) \geq \varepsilon.$$

We have $L_k(y_i) = 1$ for all k. Thus $L(y_i) = 1$ and so $M(L, f^i)(x) = 1$. Therefore $\lambda(x) = 1$. Hence $X - B(A,U) \subset \lambda^{-1}(1)$.

Let $x \in \lambda^{-1}(1)$. Then $M(L, f^i)(x) = 1$ for all i. Thus there is $y_i \in f^i(x)$ such that

$$L(y_i) = M(L, f^i)(x) = 1.$$

So $L_k(y_i) = 1$ for all k. In particular, we have $L_1(y_i) = 1$. Suppose that $x \in B(A,U)$. Then $f^i(x) \subset U$ for some i. Since $y_{i+1} \in f^{i+1}(x) \subset f(U) \subset \overline{f(U)}$, we have $L_1(y_{i+1}) = 0$. This is a contradiction. Thus $x \notin B(A,U)$. Therefore $\lambda^{-1}(1) \subset X - B(A,U)$. Hence $\lambda^{-1}(1) = X - B(A,U)$. □

From the above statements, we obtain the following theorem.

THEOREM 3.1. *Let U be an attractor block for f and let A be an attractor determined by U. Then there exists a continuous function $\lambda : X \to [0,1]$ such that*

(1) $\lambda^{-1}(0) = A$,

(2) $\lambda^{-1}(1) = X - B(A,U)$,

(3) $M(\lambda, f)(x) < \lambda(x)$ for all $x \in B(A,U) - A$.

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