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INVERSE SHADOWING PROPERTY OF MORSE-SMALE SYSTEMS

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ABSTRACT. We consider the inverse shadowing property of a dynamical system which is an "inverse" form of the shadowing property of the system. In particular, we show that every Morse-Smale system f on a compact smooth manifold has the inverse shadowing property with respect to the class $\mathcal{T}_h(f)$ of continuous methods generated by homeomorphisms, but the system f does not have the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$ of continuous methods.

Let X be a compact metric space with a metric d, and let f be a homeomorpism (or dynamical system) mapping X onto itself. A *δ-pseudo orbit* of f is a sequence of points $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ such that

$$d(f(x_n), x_{n+1}) < \delta, \quad n \in \mathbb{Z}.$$

The notion of a pseudo orbit plays an important role in the general qualitative theory of dyamical systems. It is used to define some types of invariant sets such as the chain recurrent sets or chain prolongation sets([3],[5]).

We say that a pseudo orbit $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ is *ϵ-shadowed* by a point $x \in X$ if the inequality

$$d(f^n(x), x_n) < \epsilon, \quad n \in \mathbb{Z}$$

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holds. Thus the existence of a shadowing point for a pseudo orbit ξ means that ξ is close to a real orbit of f.

A homeomorphism f is said to have the *shadowing property* if for any $\epsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit of f is ϵ-shadowed by a point in X.

The theory of shadowing was developed intensively in recent years and became a significant part of the qualitative theory of dynamical systems containing a lot of interesting and deep results([3],[5]). Recently, S. Pilyugin showed that the shadowing property is generic (natural) ([4]).

In this paper we consider the inverse shadowing property of a dynamical system which is an "inverse" form of the shadowing property of a dynamical system ([1],[2]); and show that every Morse-Smale system f on a compact smooth manifold has the inverse shadowing property with respect to the class $\mathcal{T}_h(f)$ of continuous methods generated by homeomorphisms, but the system f does not have the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$ of continuous methods. Moreover we see that every Anosov system f has the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$.

To begin with, we recall the concepts of Anosov diffeomorphsims and Morse-Smale diffeomorphisms. For our purpose, we let M be a compact smooth manifold with a Riemannian metric r and $f : M \to M$ a C^1 diffeomorphism. Denote by T_xM the tangent space of M at $x \in M$, and let $\|v\|$ be the norm of $v \in T_xM$ induced by r. Fix $x \in M$ and define two linear subspaces of T_xM :

$$E_x^s = \{v \in T_xM : \|Df_x^n(v)\| \to 0 \text{ as } n \to \infty\}$$
$$E_x^u = \{v \in T_xM : \|Df_x^{-n}(v)\| \to 0 \text{ as } n \to \infty\},$$

where Df is the derivative of f.

A closed invariant set $\Lambda \subset M$ is said to be *hyperbolic* for f if each tangent space T_xM, $x \in \Lambda$, has a continuous direct sum $T_xM = E^s_x \oplus E^u_x$ which are invariant under the map Df_x :

$$Df_x(E^s_x) = E^s_{f(x)} \text{ and } Df_x(E^u_x) = E^u_{f(x)}$$

If M is hyperbolic for f then f is called *Anosov.*

A diffeomorphsim f is said to be *Morse-Smale* provided

(1) the nonwandering set $\Omega(f)$ is a finite set of periodic orbits, each of which is hyperbolic ; and

(2) f satisfies the *strong transversality condition* ; i.e

$$T_xM = E^s_x + E^u_x, \quad x \in M$$

On S^2, it is possible to have a Morse-Smale diffeomorphsim with one fixed point source at the north pole and one fixed point sink at the south pole. Of course, we can easily construct a Morse-Smale diffeomorphism on S^1. Moreover we can see that the set of Morse-Smale diffeomorhisms on S^1 is dense on the set of all C^1-diffeomorphisms on S^1 with the C^0-topology. However it is well known that there does not exist any Anosov diffeomorphsims on S^1 and S^2.

Now we consider the notion of the inverse shadowing property of a dynamical system (or homeomorphism) which is introduced by P. Kloeden, J. Ombach and S. Pilyugin ([1],[2]). For our purpose, we let $M^{\mathbb{Z}}$ be the compact space (with the product topology) of all two sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with components $x_n \in M$. For $\delta > 0$, we let $\Phi_f(\delta) \subset M^{\mathbb{Z}}$ be the set of all δ-pseudo orbits of f.

A mapping $\varphi : M \to \Phi_f(\delta) \subset M^{\mathbb{Z}}$ is said to be a δ-*method* for f. Then each $\varphi(x) \in \Phi_f(\delta)$ is a δ-pseudo orbit of f. For convenience, we denote $\varphi(x)$ by

$$\{\varphi(x)_n\}_{n \in \mathbb{Z}}.$$

We say that φ is a *continuous* δ*-method* for f if φ is continuous. The set of all δ-methods (resp. continuous δ-methods) for f will be denoted by $\mathcal{T}_0(f,\delta)$ (resp. $\mathcal{T}_c(f,\delta)$), and we define $\mathcal{T}_0(f)$ and $\mathcal{T}_c(f)$ by

$$\mathcal{T}_0(f) = \bigcup_{\delta>0} \mathcal{T}_0(f,\delta) \text{ and } \mathcal{T}_c(f) = \bigcup_{\delta>0} \mathcal{T}_c(f,\delta),$$

respectively. Moreover, if $g : M \to M$ is a homeomorphism then g induces a continuous method $\varphi_g : M \to M^{\mathbb{Z}}$ for f by defining

$$\varphi_g(x) = \text{the orbit of } g \text{ through } x.$$

We can easily see that if $d_0(f,g) < \delta$ then φ_g is a continuous δ-method for f, where $d_0(f,g)=\sup\{d(f(x),g(x)) : x \in M\}$. Denote $\mathcal{T}_h(f)$ by the set of all continuous methods for f which are induced by homeomorpisms on M. Then we have the following inclusions :

$$\mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f).$$

Definition 1. A homeomorphism $f : M \to M$ is said to have the *inverse shadowing property* with respect to the class $\mathcal{T}_i(f)$ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\mathcal{T}_i(f)$ and any point $y \in M$ there exists a point $x \in M$ for which

$$d(f^n(y), \varphi(x)_n) < \epsilon, \quad \text{for all } n \in \mathbb{Z}$$

where $i = h, c$ or 0.

Remark 2. Note that even if a homeomorphism f has the inverse shadowing property with respect to the class $\mathcal{T}_h(f)$, f need not have the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$.

The fact that the hyperbolic sets, especially Anosov diffeomorphisms, have the shadowing property was proved by many peoples using the different methods, and is one of the main results in the qualitative theory of dynamical systems([3],[5]). A similar result was obtained by P. Kloeden and J. Ombach for the case of inverse shadowing property ; they showed that every Anosov diffeomorphism has the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$([2], Theorem 1).

Theorem 3. Every Anosov diffeomorphism f on a compact smooth manifold has the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$.

Recently, R. Corless and S. Pilyugin proved that every diffeomorphism f with the strong transversality condition does not have the inverse shadowing property with respect to the class $\mathcal{T}_0(f)$ ([1], Theorem 1.4). As a corollary, we obtain that every Morse-Smale diffeomorphism f also does not have the inverse shadowing property with respect to the class $\mathcal{T}_0(f)$.

However, in the following theorem, we can see that the Morse-Smale diffeomorphism f has the inverse shadowing property with respect to the class $\mathcal{T}_h(f)$.

Theorem 4. Let $f : M \to M$ be a Morse-Smale diffeomorphism. Then f has the inverse shadowing property with respect to the class $\mathcal{T}_h(f)$.

Proof. Let $\epsilon > 0$ be arbitrary. Since f is Morse-Smale, we can find $\delta > 0$ such that for any homeomorphsim g on M with $d_0(f,g) < \delta$, there exists a continuous surjective map h on M with

$$d_0(h, 1d_M) < \epsilon \text{ and } f \circ h = h \circ g.$$

Let $\varphi \in \mathcal{T}_h(f)$ be any δ-method and $y \in M$. Then there exists a homeomorphism g on M generating the method φ. Since $d_0(f,g) < \delta$, we can choose a continuous surjective map $h : M \to M$ satisfying

$$d_0(h, 1d_M) < \epsilon \quad \text{and} \quad f \circ h = h \circ g.$$

Choose $x \in M$ with $h(x) = y$. Then we have

$$\begin{aligned} d(\varphi(x)_n, f^n(y)) &= d(g^n(x), f^n(h(x))) \\ &= d(g^n(x), h(g^n(x))) \\ &< d_0(h, 1d_M) < \epsilon, \end{aligned}$$

for all $n \in \mathbb{Z}$. This means that f has the inverse shadowing property with respect to the class $\mathcal{T}_h(f)$.

Thus we have seen that every Morse-Smale diffeomorphism f on M does not have the inverse shadowing property with respect to the class $\mathcal{T}_0(f)$, but it has the inverse shadowing property with respect to the class $\mathcal{T}_h(f)$. Hence we have the following question ; does any Morse-Smale diffeomorphism f on M have the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$?

Theorem 5. Let $f : M \to M$ be a Morse-Smale diffeomorphism. Then f does not have the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$.

To prove the theorem we need the following lemmas.

Lemma 6. Let f and g be any two homeomorphsims on M which are topologically conjugate. If f has the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$ then g has the inverse shadowing property with respect to the class $\mathcal{T}_c(g)$.

Proof. Let $\epsilon > 0$ be arbitrary. Since f and g are topologically conjugate, there exists a homeomorphism $h : M \to M$ such that $h \circ g = f \circ h$. Choose $0 < \epsilon_0 < \epsilon$ such that

$$d(a, b) < \epsilon_0 \quad \text{implies} \quad d(h^{-1}(a), h^{-1}(b)) < \epsilon.$$

Since f has the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$, for given $\epsilon_0 > 0$ there is $\delta_0 > 0$ such that for any δ_0-method ψ in $\mathcal{T}_c(f)$ and any y in M there exists a point x in M for which

$$d(f^n(y), \psi(x)_n) < \epsilon_0, \quad \text{for all} \quad n \in \mathbb{Z}.$$

Take $0 < \delta < \delta_0$ such that

$$d(a,b) < \delta \quad \text{implies} \quad d(h(a), h(b)) < \delta_0.$$

Then it is enough to show that for any δ-method φ in $\mathcal{T}_c(g)$ and any y in M there exists a point x in M such that

$$d(g^n(y), \varphi(x)_n) < \epsilon, \quad \text{for all} \quad n \in \mathbb{Z}.$$

Let φ be a continuous δ-method for g and $y \in M$. We now define a map $\psi : M \to M^{\mathbb{Z}}$ by

$$\psi(x)_n = h(\varphi(x)_n), \quad \text{for all} \quad n \in \mathbb{Z}.$$

Then it is easy to see that $\psi(x)$ is a δ_0-pseudo orbit for f and ψ is continuous. Thus ψ is a continuous δ_0-method for f. Since f has the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$ and $h(y)$ in M, we can find a point x in M for which

$$d(f^n(h(y)), \varphi(x)_n) < \epsilon_0, \quad \text{for all} \quad n \in \mathbb{Z}.$$

This means that g has the inverse shadowing property with respect to the class $\mathcal{T}_c(g)$. In fact, the inequality

$$d(f^n(h(y)), \psi(x)_n) = d(h(g^n(y)), h(\varphi(x)_n)) < \epsilon_0$$

implies

$$d(g^n(y), \varphi(x)_n) < \epsilon, \quad \text{for all} \quad n \in \mathbb{Z}.$$

Lemma 7. Let $f : M \to M$ be a homeomorphism. Then f has the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$ if and only if f^k has the inverse shadowing property with respect to the class $\mathcal{T}_c(f^k)$, where $f^k = \overbrace{f \circ \cdots \circ f}^{k}$.

Proof. Assume that f^k has the inverse shadowing property with respect to the class $\mathcal{T}_c(f^k)$, and let $\epsilon > 0$ be arbitrary. Choose $0 < \epsilon_1 < \frac{1}{2}\epsilon$ such that

$$d(x, y) < \epsilon_1 \text{ implies } d(f(x), f(y)) < \frac{1}{2}\epsilon.$$

Moreover, we can choose $0 < \epsilon_k < \epsilon_{k-1} < \cdots < \epsilon_3 < \epsilon_2 < \epsilon_1$ such that

$$d(x, y) < \epsilon_i \text{ implies } d(f(x), f(y)) < \epsilon_{i-1},$$

where $i = 2, 3, \cdots, k$. Since f^k has the inverse shadowing property with respect to the class $\mathcal{T}_c(f^k)$, for given $\epsilon_k > 0$ there is $\delta > 0$ such that for any δ-method ψ in $\mathcal{T}_c(f^k)$ and any y in M there exists a point x in M for which

$$d((f^k)^n(y), \psi(x)_n) < \epsilon_k, \quad \text{for all } n \in \mathbb{Z}.$$

Choose $\alpha > 0$ satisfying $k\alpha < \min\{\delta, \frac{1}{2}\epsilon\}$. Then there is $\beta > 0$ such that

$$d(x, y) < \beta \text{ implies } d(f^i(x), f^i(y)) < \alpha,$$

where $i = 0, 1, \cdots, k$. Let φ be a continuous β-method for f and $y \in M$. We define a map $\psi : M \to M^{\mathbb{Z}}$ by

$$\psi(x)_n = \varphi(x)_{nk}, \quad \text{for all } n \in \mathbb{Z}.$$

Then $\psi(x)$ is a δ-pseudo orbit for f^k and ψ is continuous. In fact, we have

$$\begin{aligned} d(f^k(\psi(x)_n),\psi(x)_{n+1}) &= d(f^k(\varphi(x)_{nk}),\varphi(x)_{(n+1)k}) \\ &\le d(\varphi(x)_{nk+k}, f(\varphi(x)_{nk+(k-1)})) \\ &+ d(f(\varphi(x)_{nk+(k-1)}), f^2(\varphi(x)_{nk+(k-2)})) \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots \\ &+ d(f^{k-1}(\varphi(x)_{nk+1}), f^k(\varphi(x)_{nk})) < k\alpha < \delta, \end{aligned}$$

for all $n \in \mathbb{Z}$. Hence ψ is a continuous δ-method for f^k. Since f^k has the inverse shadowing property with respect to the class $\mathcal{T}_c(f^k)$, for given $y \in M$ we can find a point $x \in M$ for which

$$d((f^k)^n(y), \psi(x)_n) < \epsilon_k, \quad \text{for all } \; n \in \mathbb{Z}.$$

Then we have

$$\begin{aligned} d(f^i(f^{nk}(y)),\varphi(x)_{nk+i}) &\le d(f^i(f^{nk}(y)), f^i(\varphi(x)_{nk})) \\ &+ d(f^i(\varphi(x)_{nk}), f^{i-1}(\varphi(x)_{nk+1})) \\ &+ d(f^{i-1}(\varphi(x)_{nk+1}), f^{i-2}(\varphi(x)_{nk+2})) \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots \\ &+ d(f^2(\varphi(x)_{nk+i-2}), f(\varphi(x)_{nk+i-1})) \\ &+ d(f(\varphi(x)_{nk+i-1}), \varphi(x)_{nk+i}) \\ &< \epsilon_{k-i} + i\alpha < \epsilon_{k-i} + k\alpha < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon, \end{aligned}$$

for any $0 \le i \le k-1$.

The proof of the converse is easily checked, so we omit it. This completes the proof.

Proof of Theorem 5. Let f be a Morse-Smale diffeomorphism on M. Then there exists a periodic orbit $\underline{P}$ of f which is a source.

Let $k > 0$ be the period of $\underline{P}$, and let $\underline{P}=\{p, f(p), \cdots, f^{k-1}(p)\}$. Put $F = \overbrace{f \circ \cdots \circ f}^{k}$. Then the point $p \in M$ is a fixed point of F which is a source.

By applying Lemma 7, it is enough to show that F does not have the inverse shadowing property with respect to the class $\mathcal{T}_c(F)$. Consequently we will find $a > 0$ such that for any $\delta > 0$ there exists a δ-method $\varphi \in \mathcal{T}_c(F)$ for which

$$\varphi(y) \not\subset N_a(O(F, p)),$$

for all $y \in M$, where $N_a(B) = \{x \in M : d(x, B) < a\}$ for $B \subset M$.

To obtain the result, we let

$$a = \frac{1}{2}\min\{d(x, y) : x, y \in \Omega(F) \text{ and } x \neq y\}.$$

Let $\delta > 0$ be arbitrary, and choose a small neighborhood U_p of p satisfying

$$U_p \subset N_a(p) \cap N_{\frac{1}{2}\delta}(p) \cap W^u(p),$$

where $W^u(p) = \{x \in M : d(F^n(x), F^n(p)) \to 0 \text{ as } n \to -\infty\}$. Let $V_p \subset M$ be a small neighborhood of p such that

(1) $F(V_p) \subset U_p$, and
(2) $F^n(x) \notin \overline{V_p}$ for any $x \in \partial V_p$ and all $n \geq 1$.

Choose two distinct points y_1 and y_2 in $V_p - \{p\}$. Let G_i be any diffeomorphisms on M such that

(1) $G_i(y_i) = y_i$,
(2) $G_i = F$ on $M - V_p$,
(3) $G_i(V_p) \subset U_p$,
(4) $V_p \subset W^u(y_i, G_i)$ and $W^s(y_i, G_i) = \{y_i\}$,
(5) $G_i^n(x) \notin \overline{V_p}$ for any $x \in \partial V_p$ and all $n \geq 1$,

where $i = 1,\ 2$ and $W^s(y_i, G_i) = \{x \in M : d(G_i^n(x), G_i^n(y_i)) \to 0$ as $n \to \infty\}$. Select three pairwise disjoint neighborhoods W_1, W_2 and W_3 in V_p of y_1, y_2 and p, respectively. For any $x \in V_p - W_i$, we let

$$n_i(x) = \inf\{n \in \mathbb{N} : G_i^n(x) \in M - V_p\},$$

where $i = 1,\ 2,\ 3$ and $G_3 = F$.

Put

$$N = \max\{\ \sup\{n_i(x) : x \in V_p - W_i\} : i = 1,\ 2,\ 3\}.$$

Now we define a map $\varphi : M \to M^{\mathbb{Z}}$ by

$$\varphi(x)_n = \begin{cases} F^n(x), & n \leq 0, \\ G_1^{n-a_k}(H_k(x)), & a_k \leq n \leq b_k, \\ G_2^{n-b_k} G_1^{N+k}(H_k(x)), & b_k \leq n \leq c_k, \\ F^{n-c_k} G_2^N G_1^{N+k}(H_k(x)), & c_k \leq n \leq a_{k+1}, \end{cases}$$

where

$$\begin{cases} H_0(x) = x, \\ H_k(x) = F^N G_2^N G_1^{N+(k-1)} F^N G_2^N G_1^{N+(k-2)} \cdots \\ \qquad \cdots\cdots F^N G_2^N G_1^{N+1} F^N G_2^N G_1^N(x), \\ a_k = 3kN + \frac{k(k-1)}{2}, \\ b_k = (3k+1)N + \frac{k(k+1)}{2}, \\ c_k = (3k+2)N + \frac{k(k+1)}{2}, \quad k \geq 0. \end{cases}$$

Then it is easy to show that φ is a continuous δ-method for F. To complete the proof, we will show that for all $y \in M$

$$\varphi(y) \not\subset N_a(\{p\})\ .$$

If $y \in M - V_p$ then we have $G_i^k(y) = F^k(y)$ for all $k \geq 0$. Hence we get

$$\varphi(y)_n \to \Omega(F) - \{p\} \quad \text{as} \quad n \to \infty.$$

Otherwise we arrive at a contradiction by the fact that

$$y \in W^s(p, F) = \{p\}.$$

If $y \in V_p - W_1$ then we have $G_1^N(y) \in V_p^c$. Thus we get

$$\varphi(y)_n \to \Omega(F) - \{p\} \quad \text{as} \quad n \to \infty.$$

If $y \in W_1 - G_1^{-N}(W_2)$ then $G_1^N(y) \in W_2^c$, and so $G_2^N G_1^N(y) \in V_p^c$. Similarly we have $G_1^{N+1} F^N G_2^N G_1^N(y) \in V_p^c$ if $y \in G_1^{-N} G_2^{-N}(W_3) - G_1^{-N} G_2^{-N} F^{-N}(W_1)$.

By continuing this process and applying Lemma 6, if necessary, we can see that for any point $y \in M$ we get

$$\varphi(y)_n \to \Omega(F) - \{p\} \quad \text{as} \quad n \to \infty.$$

This means that for any $y \in M$

$$\varphi(y) \not\subset N_a(\{p\}),$$

and see we completes the proof.

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