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LINEAR OPERATORS IN BANACH MODULES OVER A C^*–ALGEBRA AND ITS UNITARY GROUP

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ABSTRACT. In this paper, we prove the generalized Hyers-Ulam-Rassias stability of linear operators in Banach modules over a unital C^*-algebra associated with its unitary group.

Let E_1 and E_2 be Banach spaces with norms $||\cdot||$ and $||\cdot||$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [6] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p}||x||^p$$

for all $x \in E_1$.

In this paper, let A be a unital C^*-algebra with norm $|\cdot|$, $\mathcal{U}(A)$ the unitary group of A, and ${}_A\mathcal{H}$ a left Hilbert A-module with norm $||\cdot||$. Let ${}_A\mathcal{B}$ and ${}_A\mathcal{C}$ be left Banach A-modules with norms $||\cdot||$ and $||\cdot||$, respectively.

We are going to prove the generalized Hyers-Ulam-Rassias stability of linear operators in Banach modules over a unital C^*-algebra associated with its unitary group.

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THEOREM 1. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping for which there exists a function $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0,\infty)$ such that*

$$\text{(i)} \qquad \widetilde{\varphi}(x,y) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) < \infty,$$
$$\|uf(x+y) - f(ux) - f(uy)\| \le \varphi(x,y)$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\text{(ii)} \qquad \|f(x) - T(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x)$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. By the Găvruta result [2], there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (ii). The mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ was given by $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ for all $x \in {}_A\mathcal{B}$.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$\|uf(2^n x) - 2f(2^{n-1}ux)\| \le \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in {}_A\mathcal{B}$. And one can show that

$$\|f(2^n ux) - 2f(2^{n-1}ux)\| \le \varphi(2^{n-1}ux, 2^{n-1}ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}\|f(2^n ux) - uf(2^n x)\| \le& \|f(2^n ux) - 2f(2^{n-1}ux)\| \\ &+ \|2f(2^{n-1}ux) - uf(2^n x)\| \\ \le& \varphi(2^{n-1}ux, 2^{n-1}ux) + \varphi(2^{n-1}x, 2^{n-1}x)\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Thus $2^{-n}\|f(2^n ux) - uf(2^n x)\| \to 0$ as $n \to \infty$ for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$T(ux) = \lim_{n\to\infty} \frac{f(2^n ux)}{2^n} = \lim_{n\to\infty} \frac{uf(2^n x)}{2^n} = uT(x)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

Now let $a \in A$ $(a \neq 0)$ and M an integer greater than $4|a|$. Then

$$|\frac{a}{M}| = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [4, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. And $T(x) = T(3 \cdot \frac{1}{3}x) = 3T(\frac{1}{3}x)$ for all $x \in {}_A\mathcal{B}$. So $T(\frac{1}{3}x) = \frac{1}{3}T(x)$ for all $x \in {}_A\mathcal{B}$. Thus

$$\begin{aligned} T(ax) &= T(\frac{M}{3} \cdot 3\frac{a}{M}x) = M \cdot T(\frac{1}{3} \cdot 3\frac{a}{M}x) = \frac{M}{3}T(3\frac{a}{M}x) \\ &= \frac{M}{3}T(u_1x + u_2x + u_3x) = \frac{M}{3}(T(u_1x) + T(u_2x) + T(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)T(x) = \frac{M}{3} \cdot 3\frac{a}{M}T(x) \\ &= aT(x) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$. So the unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is an A-linear mapping. □

THEOREM 2. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping for which there exists a function $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ satisfying* (i) *such that*

$$\|uf(x + y) - f(ux) - f(uy)\| \leq \varphi(x, y)$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $\frac{f(2^n x)}{2^n}$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (ii).

Proof. By Theorem 1, there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (ii). By the continuity of f, the uniform convergence and the definition of T, the A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is continuous, as desired. □

THEOREM 3. *Let $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a continuous mapping for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ satisfying* (i) *such that*

$$\|uh(x+y) - h(ux) - h(uy)\| \le \varphi(x, y)$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{H}$. Assume that $h(2^n x) = 2^n h(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$. Then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a bounded A-linear operator. Furthermore,

(1) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|h(x) - h^*(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a self-adjoint operator,

(2) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|h \circ h^*(x) - h^* \circ h(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a normal operator,

(3) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities*

$$\|h \circ h^*(x) - x\| \le \varphi(x, x),$$
$$\|h^* \circ h(x) - x\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a unitary operator, and

(4) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities*

$$\|h \circ h(x) - h(x)\| \le \varphi(x, x),$$
$$\|h^*(x) - h(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a projection.

Proof. By Theorem 1, there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (ii). By the assumption,

$$T(x) = \lim_{n\to\infty} \frac{h(2^n x)}{2^n} = h(x)$$

for all $x \in {}_A\mathcal{H}$, where the mapping T is given in the proof of Theorem 1. Thus the A-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is the mapping h. But the sequence $\frac{h(2^n x)}{2^n}$ converges uniformly on ${}_A\mathcal{H}$. So the A-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is continuous. Hence the A-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is bounded (see [1, Proposition II.1.1]). So the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a bounded A-linear operator.

(1) By the assumption,

$$\|h(2^n x) - h^*(2^n x)\| \le \varphi(2^n x, 2^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$2^{-n}\|h(2^n x) - h^*(2^n x)\| \to 0$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$h(x) = \lim_{n\to\infty} \frac{h(2^n x)}{2^n} = \lim_{n\to\infty} \frac{h^*(2^n x)}{2^n} = h^*(x)$$

for all $x \in {}_A\mathcal{H}$. So the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a self-adjoint operator.

The proofs of the others are similar to the proof of (1). □

So the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an element of the C^*-algebra $\mathcal{L}({}_A\mathcal{H})$ of all bounded A-linear operators on ${}_A\mathcal{H}$.

Now we are going to prove the generalized Hyers-Ulam-Rassias stability of the Jensen's functional equation in Banach modules over a unital C^*-algebra associated with its unitary group.

THEOREM 4. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping for which there exists a function $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ such that*

$$\text{(iii)} \qquad \widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) < \infty,$$

$$\|2uf(\frac{x+y}{2}) - f(ux) - f(uy)\| \leq \varphi(x, y)$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\text{(iv)} \qquad \|f(x) - T(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. By [3, Theorem 1], there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iv). The mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ was given by $T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$ for all $x \in {}_A\mathcal{B}$.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$\|2uf(3^n x) - f(2 \cdot 3^{n-1} ux) - f(4 \cdot 3^{n-1} ux)\| \leq \varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x)$$

for all $x \in {}_A\mathcal{B}$. And one can show that

$$\|\frac{1}{2} f(2\cdot 3^{n-1} ux) + \frac{1}{2} f(4\cdot 3^{n-1} ux) - f(3^n ux)\| \leq \frac{1}{2}\varphi(2\cdot 3^{n-1} ux, 4\cdot 3^{n-1} ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}
\|f(3^n ux) - uf(3^n x)\| \leq & \|f(3^n ux) - \frac{1}{2} f(2 \cdot 3^{n-1} ux) - \frac{1}{2} f(4 \cdot 3^{n-1} ux)\| \\
& + \|\frac{1}{2} f(2 \cdot 3^{n-1} ux) + \frac{1}{2} f(4 \cdot 3^{n-1} ux) - uf(3^n x)\| \\
\leq & \frac{1}{2}\varphi(2 \cdot 3^{n-1} ux, 4 \cdot 3^{n-1} ux) \\
& + \frac{1}{2}\varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x)
\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Thus $3^{-n}\|f(3^n ux) - uf(3^n x)\| \to 0$ as $n \to \infty$ for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$T(ux) = \lim_{n\to\infty} \frac{f(3^n ux)}{3^n} = \lim_{n\to\infty} \frac{uf(3^n x)}{3^n} = uT(x)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 1. □

THEOREM 5. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping for which there exists a function $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0,\infty)$ satisfying* (iii) *such that*

$$\|2uf(\frac{x+y}{2}) - f(ux) - f(uy)\| \le \varphi(x,y)$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $\frac{f(3^n x)}{3^n}$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iv).

Proof. By Theorem 4, there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iv). By the continuity of f, the uniform convergence and the definition of T, the A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is continuous, as desired. □

THEOREM 6. *Let $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a continuous mapping for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0,\infty)$ satisfying* (iii) *such that*

$$\|2uh(\frac{x+y}{2}) - h(ux) - h(uy)\| \le \varphi(x,y)$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{H}$. Assume that $h(3^n x) = 3^n h(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$. Then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a bounded A-linear operator. Furthermore, the properties, given in the statement of Theorem 3, hold.

Proof. By Theorem 3, there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iv).

The rest of the proof is similar to the proof of Theorem 3. □

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