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ON CONVERGENCE THEOREMS FOR HENSTOCK INTEGRALS

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ABSTRACT. In this paper we prove a controlled convergence theorem for the Henstock integral by using the new conditions.

1. Introduction

In the 1950's J. Kurzweil and R. Henstock independently gave a Riemann Complete type integral, called the Kurzweil-Henstock integral (or KH-integral/H-integral). It has been proved that this integral is equivalent to the special Denjoy integral. Therefore the Henstock integral contains the Newton, Riemann and Lebesgue integrals. In 1985, P. Y. Lee and T. S. Chew [?, ?] gave the controlled convergence theorem. But we want to find a better convergence theorem. In this paper, using the equivalent conditions to the generalized $\mathcal{P}$-Cauchy property, we give a controlled convergence theorem.

First we introduce some necessary terms. Throughout this paper, D will denote a finite collection of non-overlapping tagged intervals in $[a, b]$. For $D = \{(t_i, [c_i, d_i]) : 1 \leq i \leq N\}$, we will write

$$f(D) = \sum_{i=1}^{N} f(t_i)(d_i - c_i), \quad F(D) = \sum_{i=1}^{N} (F(d_i) - F(c_i)),$$

$$\text{and} \quad \mu(D) = \sum_{i=1}^{N} (d_i - c_i).$$

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Let δ be a positive function defined on $[a, b]$. We say that D is subordinate to δ if $[c_i, d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for each i and that D is subordinate to δ on $[a, b]$, or D is said to be δ-fine on $[a, b]$ if in addition D is a partition of $[a, b]$. A real-valued function f is said to be Henstock integrable to A on a closed bounded interval $[a, b]$ if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that whenever a division D given by

$$a = x_0 < x_1 < \cdots < x_n = b$$

satisfies $0 \le x_i - \xi_i < \delta(\xi_i)$ and $0 \le \xi_i - x_{i-1} < \delta(\xi_i)$ for $i = 1, 2, \ldots, n$ we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right| < \varepsilon ,$$

or alternatively,

$$\left| \sum f(\xi)(v - u) - A \right| < \varepsilon$$

where $[u, v]$ denotes a typical interval in D with $\xi - \delta(\xi) < u \le \xi \le v < \xi + \delta(\xi)$. A sequence of functions $\{f_n\}$ is said to be equi-Henstock integrable on $[a, b]$ if f_n, $n = 1, 2, \ldots$, is Henstock integrable on $[a, b]$ using the same $\delta(\xi)$ in the definition.

2. Convergence theorems

DEFINITION 2.1 ([?]). (a) A function F on $[a, b]$ is ACG_δ^* on $X \subset [a, b]$ if X is the union of a sequence of subsets X_i such that F is $\mathrm{AC}_\delta^*(X_i)$ for each i, i.e., for every $\varepsilon > 0$ there are $\eta > 0$ and $\delta(\xi) > 0$ such that for any δ-fine partial division $D = \{([u, v], \xi)\}$ with $\xi \in X_i$ satisfying $(D) \sum |v - u| < \eta$ we have $(D) \sum |F(u, v)| < \varepsilon$.
(b) The sequence $\{F_n\}$ is UACG_δ^* or uniformly-ACG_δ^* on $X \subset [a, b]$ if X is the union of a sequence of subsets X_i such that $\{F_n\}$ is $\mathrm{UAC}_\delta^*(X_i)$ for each i, independent of n.

DEFINITION 2.2 ([?]). (a) A function F on $[a, b]$ is ACG* on $X \subset [a, b]$ if X is the union of a sequence of subsets X_i such that F is AC*(X_i) for each i, i.e., for every $\varepsilon > 0$ there is $\eta > 0$ such that for any finite or infinite sequence of nonoverlapping intervals $\{[a_k, b_k]\}$ with at least one of a_k, b_k belonging to X_i satisfying $\sum_k |b_k - a_k| < \eta$ we have $\sum_k |F(a_k, b_k)| < \varepsilon$, where $F(a_k, b_k)$ denotes $F(b_k) - F(a_k)$.
(b) The sequence $\{F_n\}$ is UACG* or uniformly-ACG* on $X \subset [a, b]$ if X is the union of a sequence of subsets X_i such that $\{F_n\}$ is UAC*(X_i) for each i, independent of n.

DEFINITION 2.3 ([?]). Let $\{F_n\}$ be a sequence of functions defined on $[a, b]$ and let $E \subset [a, b]$ be measurable.

(a) The sequence $\{F_n\}$ is $\mathcal{P}$-Cauchy on E if $\{F_n\}$ converges pointwise on E and if for each $\varepsilon > 0$ there exist a positive function δ on E and a positive integer N such that $|F_n(\mathcal{P}) - F_m(\mathcal{P})| < \varepsilon$ for all $m, n \geq N$ whenever $\mathcal{P}$ is E-subordinate to δ.
(b) The sequence $\{F_n\}$ is generalized $\mathcal{P}$-Cauchy on E if E can be written as a countable union of measurable sets on each of which $\{F_n\}$ is $\mathcal{P}$-Cauchy.

DEFINITION 2.4 ([?]). A sequence $\{F_n\}$ is said to satisfy the uniformly strong Lusin condition, or briefly USL, if for every $\varepsilon > 0$ and every set Z of measure zero there exists $\delta(\xi) > 0$, independent of n, such that for any δ-fine partial division $D = \{([u, v], \xi)\}$ with $\xi \in Z$ for all n we have $(D)\sum |F_n(u, v)| < \varepsilon$.

THEOREM 2.1. *Let f_n, $n = 1, 2, \ldots$, be Henstock integrable on $[a, b]$ and $f_n(x) \to f(x)$ everywhere in $[a, b]$. If $\{f_n\}$ is equi-Henstock integrable on $[a, b]$. Then f is Henstock integrable on $[a, b]$ and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ as $n \to \infty$.*

Proof. See [?]. □

THEOREM 2.2. *Let f_n, $n = 1, 2, \ldots$, be Henstock integrable on $[a, b]$ with primitive F_n, and $f_n(x) \to f(x)$ almost everywhere in $[a, b]$. If $\{f_n\}$ is equi-Henstock integrable on $[a, b]$ and $\{F_n\}$ satisfies USL. Then f is Henstock integrable on $[a, b]$ and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ as $n \to \infty$.*

Proof. We can choose g_n and g such that $g_n(x) = f_n(x)$ almost everywhere in $[a, b]$ for all n, $g(x) = f(x)$ almost everywhere in $[a, b]$, and $g_n(x) \to g(x)$ everywhere in $[a, b]$, and $\{F_n\} = \{G_n\}$. Since $\{F_n\}$ satisfies USL, the sequence $\{g_n\}$ is still equi-Henstock integrable on $[a, b]$. Hence the result follows from Theorem ??. □

THEOREM 2.3. *Let $\{f_n\}$ be a sequence of Henstock integrable functions on $[a, b]$ with primitive of f_n by F_n. Suppose $\{f_n\}$ converges to a function f almost everywhere on $[a, b]$. The following conditions (a), (b), (c), (d) are mutually equivalent:*

(a) $\{F_n\} \in UACG^*_\delta[a, b]$.

(b) (1) $\{F_n\}$ *satisfies USL on* $[a, b]$.

(2) *For every* $\varepsilon > 0$, *there exists a closed set* $X \subset [a, b]$, $|[a, b] \backslash X| < \varepsilon$ *such that* $\{F_n\} \in UAC^*(X)$.

(c) *The sequence* $\{F_n\}$ *is generalized* $\mathcal{P}$*-Cauchy on* $[a, b]$.

(d) (1) $\{F_n\}$ *satisfies USL on* $[a, b]$.

(2) $\{F_n\}$ *is equi-Henstock integrable on* $[a, b]$.

Proof. (a) $\Rightarrow$ (b): First we show that (b)(1) holds. Since $\{F_n\} \in$ UACG$^*_\delta$, then there exists a sequence of pairwise disjoint set $\{X_i\}$ with $[a, b] = \cup_i X_i$ such that $\{F_n\} \in$ UAC$^*_\delta$ on X_i. Let $Z \in [a, b]$ with $|Z| = 0$. Let $Z_i = Z \cap X_i$. Then $|Z_i| = 0$. Let $\varepsilon > 0$ and G_i an open set such that $Z_i \subset G_i$ and $|G_i| < \eta_i$ where η_i come from the definition of $AC^*_\delta(X_i)$ for given $\varepsilon 2^{-i}$, i.e., for every $\varepsilon_i > 0$, there are $\eta_i > 0$ and $\delta(\xi_i) > 0$ such that for any δ-fine partial division $D_i = \{([u, v], \xi)\}$ with $\xi \in X_i$ satisfy $(D_i)\sum |v - u| < \eta_i$ we have $(D_i)\sum |F_n(u, v)| < \varepsilon 2^{-i}$.

Suppose that D is Z-subordinate to δ and D_i is Z_i-subordinate to δ_i. Let $D = \cup_i D_i$. Since $\mu(D_i) \leq |G_i| < \eta_i$ for each i. We have

$$(D)\sum |F_n(u,\, v)| \leq (D_i)\sum_i |F_n(u,\, v)| < \sum_i \varepsilon 2^{-i} \leq \varepsilon$$

for all n. Hence we have $\{F_n\} \in \mathrm{USL}[a,\, b]$. Next (b)(2) holds by [?, Lemma 3].

(b) $\Rightarrow$ (c): By (b)(2), for any given $\varepsilon > 0$, there exists a closed set $X \subset [a,\, b]$, $|[a,\, b]\backslash X| < \varepsilon/2$ such that $F_n \in \mathrm{UAC}^*(X)$ and by Egoroff's theorem, we have a closed set $H \subset [a,\, b]$, $|[a,\, b]\backslash H| < \varepsilon$ such that $f_n \to f$ uniformly on H. Hence there is a sequence of closed subsets $\{X_k\}$ of $[a,\, b]$ such that $|[a,\, b]\backslash \cup_{i=1}^{\infty} X_i| = 0$, and $F_n \in \mathrm{UAC}^*(X_i)$ and $f_n \to f$ uniformly on X_i for every $i \in N$. Hence for every $i \in N$ and $\varepsilon^* > 0$ there is $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon^*$ for any $x \in X_i$ when $n, m \geq N$. And since f_n is Henstock integrable function on X_i, then $\left|\sum F_n(u,\, v) - \int_{X_i} f_n(x)\, dx\right| < \varepsilon$ for $n = 1, 2, \ldots$. Hence for any $n, m \geq N$, we have

$$\begin{aligned}\left|\sum (F_m(u,\, v) - F_n(u,\, v))\right| &\leq \left|\sum F_m(u,\, v) - \int_{X_i} f_m(x)\, dx\right| \\ &\quad + \left|\sum F_n(u,\, v) - \int_{X_i} f_n(x)\, dx\right| \\ &\quad + \left|\int_{X_i} (f_m(x) - f_n(x)\, dx\right| \\ &< 2\varepsilon + \varepsilon^* \, |X_i|\end{aligned}$$

Since the inequality is for all δ-fine partial divisions,

$$\sum |F_m(u,\, v) - F_n(u,\, v)| < (4 + 2\,|X_i|)\varepsilon^* .$$

Next, since $K = [a,\, b]\backslash \cup X_i$ is measure zero, by condition (b)(1), $\{F_n\}$ satisfy USL on K. Hence for any partial division $\{([u,\, v],\, \xi)\}$ of K and any $m, n \in N$, we have

$$\sum |(F_m - F_n)(u, v)| \le \sum |F_m(u, v)| + \sum |F_n(u, v)| < 2\varepsilon^* .$$

By [?, Exercise 4.4], $\{F_n\}$ is generalized $\mathcal{P}$-Cauchy on $[a, b]$.
(c) $\Rightarrow$ (d): (d)(1) holds by [?, Lemma 3] and (d)(2) holds by[?, Lemma 13.29].
(d) $\Rightarrow$ (a): By (d)(2) and the Saks-Henstock Lemma, for every $\varepsilon > 0$ there is $\delta(\xi)$, independent of n, such that for any δ-fine division D of $[a, b]$ we have

$$(D)\sum |f_n(\xi)(v-u) - F_n(u, v)| < \varepsilon .$$

Let X_i denote the set of all $x \in [a, b]$ such that $|f_n(x)| \le i$ for all n. Then $\cup_{i=1}^{\infty} X_i = [a, b]$. For every $\varepsilon > 0$ there is $0 < \eta < \varepsilon/i$ such that for any δ-fine partial division $D = \{([u, v], \xi)\}$ with $\xi \in X_i$ satisfying $(D)\sum |v-u| < \eta$ we have

$$\begin{aligned}(D)\sum |F_n(u, v)| &\le (D)\sum |F_n(u, v) - f_n(\xi)(v-u)| \\ &\quad + (D)\sum |f_n(\xi)(v-u)| \\ &< 2\varepsilon .\end{aligned}$$

Therefore $\{F_n\} \in \mathrm{UAC}^*_\delta$ on X_i. Hence $\{F_n\} \in \mathrm{UACG}^*_\delta$ on $[a, b]$. □

3. The equivalence of Convergence theorems for the Henstock integral

DEFINITION 3.1 ([?]). A sequence $\{F_n\}$ of functions is called $\mathrm{UAC}^{**}(X)$ or uniformly-$\mathrm{AC}^{**}(X)$ whenever to each $\varepsilon > 0$ there exist $\eta > 0$ and a positive function $\delta : X \to (0, \infty)$ such that

$$(D_1 \backslash D_2)\sum |v-u| < \eta \quad \text{implying} \quad (D_1 \backslash D_2)\sum |F_n(v) - F_n(u)| < \varepsilon.$$

A sequence $\{F_n\}$ of functions is uniformly-ACG** on $[a,b]$ or UACG** on $[a,b]$ if $[a,b] = \cup_i X_i$ where X_i are measurable sets and $\{F_n\}$ is uniformly-AC**(X_i) or UAC** on each X_i.

DEFINITION 3.2 ([?]). A sequence $\{F_n\}$ of functions is called UAC$^\nabla(X)$ or uniformly-AC$^\nabla(X)$ if for every $\varepsilon > 0$ there exist $\delta : X \to (0,1)$ and $\eta > 0$ such that

$$|F_n(\Delta_1) - F_n(\Delta_2)| \leq \varepsilon$$

for any two η-close δ-fine X-tagged systems Δ_1, Δ_2 and all $n \in N$. A sequence $\{F_n\}$ of functions is uniformly-ACG$^\nabla$ or UACG$^\nabla$ on $[a, b]$ if $[a, b] = \cup_i X_i$ where X_i are measurable sets and $\{F_n\}$ is UAC$^\nabla$ on each X_i.

THEOREM 3.1. *Let $\{F_n\}$ be a sequence of functions on $[a, b]$. Then the following conditions are equivalent:*

(a) *$\{F_n\}$ is UACG$_\delta^*$ on $[a, b]$.*
(b) *$\{F_n\}$ is UACG** on $[a, b]$.*
(c) *$\{F_n\}$ is UACG$^\nabla$ on $[a, b]$.*

Proof. (a) $\Rightarrow$ (b): See [?, Theorem 2].
(b) $\Rightarrow$ (c): Since $\{F_n\}$ is UACG** on $[a, b]$, then F_n is UAC** on X_i such that $[a, b] = \cup_i X_i$. Put $X_i = X$, $F_n = F$. Since F is AC** on X, then by the definition of AC** we have

$$(1)\quad (D_1 \backslash D_2) \sum |v - u| < \eta \quad \text{implying} \quad (D_1 \backslash D_2) \sum |F(v) - F(u)| < \varepsilon\,.$$

Let E_1 denote the union of intervals $[u, v]$ in D_1 and E_2 the union of interval $[s, t]$ in D_2. Then by (??)

$$|E_1 \backslash E_2| < \eta \quad \text{implying} \quad |F(E_1) - F(E_2)| < \varepsilon.$$

Hence F is AC^∇ on X. Hence F_n is UACG$^\nabla$.
(c) $\Rightarrow$ (a): Let $[a, b] = \cup_i X_i$ be such that $\{F_n\} \in$ UAC$^\nabla$ on each X_i. So to each $\varepsilon > 0$ there exists a constant $\eta > 0$ and a positive function δ on each X_i such that $|F_n(D_1) - F_n(D_2)| \leq \varepsilon$ whenever $|(D_1)\Delta(D_2)| < \eta$

for any two η-closed δ-fine X_i-tagged system D_1, D_2 and for all $n \in N$. Take $D = \{[c_k, d_k], x_k\}_{k=1}^p$ with $\sum_k |d_k - c_k| < \eta$ and put

$$D_1 = |([c_k, d_k], x_k) : F_n([c_k, d_k]) \geq 0| ,$$
$$D_2 = |([c_k, d_k], x_k) : F_n([c_k, d_k]) < 0| .$$

So $|(\cup D_1)\triangle(\cup D_2)| = |\cup D| = \sum_{k=1}^p |d_k - c_k| < \eta$, then

$$\sum_{k=1}^{p} |F_n[c_k, d_k]| = \left| \sum_{(c_k, d_k)\in D_1} F_n([c_k, d_k]) - \sum_{(c_k, d_k)\in D_2} F_n([c_k, d_k]) \right| < \varepsilon .$$

Hence $\{F_n\}$ is UAC$_\delta^*$ on X_i and so $\{F_n\}$ is UACG$_\delta^*$ on $[a, b]$. □

COROLLARY 3.2. *The following conditions be satisfied:*

(a) $f_n(x) \to f(x)$ *almost everywhere in* $[a, b]$ as $n \to \infty$ *where each* f_n *is Henstock integrable on* $[a, b]$.

(b) *The primitives* F_n *of* f_n *are one of the condition* (a)-(c) *of Theorem* ??.

(c) *The sequence* F_n *converges uniformly to a continuous function* F *on* $[a, b]$.

Then f *is Henstock integrable on* $[a, b]$ *with the primitive* F.

COROLLARY 3.3. *Let* $\{f_n\}$ *be a sequence of Henstock integrable functions such that* $f_n \to f$ *everywhere in* $[a, b]$. *Then* $\{f_n\}$ *is equi-Henstock integrable on* $[a, b]$ *if and only if the sequence* $\{F_n(x) = \int_a^x f_n(x)\, dx\}$ *satisfies one of the condition* (a)-(c) *of Theorem* ??.

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