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STABILITY OF A JENSEN'S FUNCTIONAL EQUATION ON A NORMED ALMOST LINEAR SPACE

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ABSTRACT. In this paper, we prove the stability of a Jensen's functional equation on a normed almost linear space.

In 1940, S. M. Ulam ([9]) posed the following question on the stability of homomorphisms: Given a group G_1, a metric group G_2 with a metric d and a number $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with

$$d(f(x), h(x)) < \epsilon$$

for all $x \in G_1$? This question became a source of the stability theory in the Hyers-Ulam sense. The case of approximately additive mappings was solved by D. H. Hyers ([3]) under the assumption that G_1 and G_2 are Banach spaces.

In [1], G. Godini introduced a normed almost linear space. We recall definitions and notations from [2], [4] and [6].

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An *almost linear space* is a set X together with two mappings $s : X \times X \to X$ and $m : \mathbb{R} \times X \to X$ satisfying the conditions $(L_1)-(L_8)$ given below. For $x, y \in X$ and $\lambda \in \mathbb{R}$ we denote $s(x,y)$ by $x+y$ and $m(\lambda, x)$ by λx, when these will not lead to misunderstandings. Let $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$. (L_1) $x+(y+z) = (x+y)+z$; (L_2) $x+y = y+x$; (L_3) There exists an element $0 \in X$ such that $x+0 = x$ for each $x \in X$; (L_4) $1x = x$; (L_5) $\lambda(x+y) = \lambda x + \lambda y$; (L_6) $0x = 0$; (L_7) $\lambda(\mu x) = (\lambda\mu)x$; (L_8) $(\lambda+\mu)x = \lambda x + \mu x$ for $\lambda \geq 0$, $\mu \geq 0$. We denote $-1x$ by $-x$, and $x-y$ means $x+(-y)$. The sets $V_X = \{x \in X : x - x = 0\}$ and $W_X = \{x \in X : x = -x\}$ are almost linear subspaces of X.

A *norm* on an almost linear space X is a functional $|||\cdot||| : X \to \mathbb{R}$ satisfying the conditions $(N_1)-(N_3)$ below. Let $x, y, z \in X$ and $\lambda \in \mathbb{R}$. (N_1) $|||x-z||| \leq |||x-y|||+|||y-z|||$; (N_2) $|||\lambda x||| = |\lambda|\ |||x|||$; (N_3) $|||x||| = 0$ iff $x = 0$. An almost linear space X together with $|||\cdot||| : X \to \mathbb{R}$ satisfying $(N_1)-(N_3)$ is called a *normed almost linear space*.

In contrast with the case of a normed linear space, a norm of a normed almost linear space $(X, |||\cdot|||)$ does not generate a metric on X (for $x \in X \setminus V_X$, $|||x-x||| \neq 0$).

A metric (semi-metric) d on a normed almost linear space $(X, |||\cdot|||)$ is called a *metric (semi-metric) induced by a norm* if d satisfies (1), (2), (3), (4) and (5).

$$d(x+z, y+z) = d(x,y) \quad (x, y, z \in X), \tag{1}$$

$$d(\alpha x, \alpha y) = |\alpha| d(x,y) \quad (x, y \in X,\ \alpha \in \mathbb{R}), \tag{2}$$

$$d(x,v) = |||x-v||| \quad (x \in X,\ v \in V_X), \tag{3}$$

$$(4) \qquad | \, |||x||| - |||y||| \, | \leq d(x,y) \leq |||x-y||| \quad (x,y \in X),$$

$$(5) \qquad \lim_{\lambda \to \lambda_0} d(\lambda x, x) = d(\lambda_0 x, x) \quad (x \in X, \ \lambda_0 > 0).$$

In this paper, we prove the stability of a Jensen's functional equation on a normed almost linear space.

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Recall that a mapping $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is homogeneous of degree p if it satisfies $H(tu, tv) = t^p H(u, v)$ for all nonnegative real numbers t, u and v. Throughout this paper, we may assume that $\delta \geq 0$, $\theta \geq 0$ and $p \in \mathbb{R} \setminus \{1\}$ are fixed. Assume that X and Y are real normed almost linear spaces

THEOREM 1. *Let d be a metric induced by a norm on a normed almost linear space Y and Y a d-complete real normed almost linear space. Assume that $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a homogeneous function of degree p. Let a function $f : X \to Y$ satisfy the following inequality*

$$(6) \qquad d\left(2f\left(\frac{x+y}{2}\right), f(x)+f(y)\right) \leq \delta + H(|||x|||, |||y|||)$$

for all $x, y \in X$. Furthermore, assume $f(0) = 0$ and $\delta = 0$ in (6) for the case of $p > 1$. Then there exists a unique additive mapping $F : X \to Y$ such that

$$(7) \ d(F(x), f(x)) \leq \delta + |||f(0)||| + H(|||x|||, 0)\frac{1}{2^{1-p}-1} \quad (for \ p < 1)$$

or

$$(8) \qquad d(F(x), f(x)) \leq H(|||x|||, 0)\frac{1}{1-2^{1-p}} \quad (for \ p > 1)$$

for all $x \in X$.

Proof. Putting $y = 0$ in (6) we have

$$d\left(2f\left(\frac{x}{2}\right), f(x) + f(0)\right) \leq \delta + H(|||x|||, 0) \tag{9}$$

for all $x \in X$. Using the triangle inequality and (1), we have

$$\begin{aligned} d\left(2f\left(\frac{x}{2}\right), f(x)\right) &\leq d\left(2f\left(\frac{x}{2}\right), f(x) + f(0)\right) + d(f(x) + f(0), f(x)) \\ &\leq \delta + |||f(0)||| + H(|||x|||, 0) \end{aligned} \tag{10}$$

for all $x \in X$.

We divide the remaining proof by two cases. (I) The case $p < 1$. By induction on n, we prove

$$d(2^{-n} f(2^n x), f(x)) \leq (\delta + |||f(0)|||) \sum_{k=1}^{n} 2^{-k} + H(|||x|||, 0) \sum_{k=1}^{n} 2^{(p-1)k} \tag{11}$$

for all $x \in X$. By substituting $2x$ for x in (10) and dividing by 2 both sides of (10) we see the validity of (11) for $n = 1$. Now, assume that the inequality (11) holds true for some $n \in \mathbb{N}$. If we replace x by $2^{n+1}x$ in (10) and divide by 2 both sides of (10), then it follows from (11) that

$$\begin{aligned} &d(2^{-(n+1)} f(2^{n+1} x), f(x)) \\ &\leq 2^{-n} d(2^{-1} f(2^{n+1} x), f(2^n x)) + d(2^{-n} f(2^n x), f(x)) \\ &\leq (\delta + |||f(0)|||) \sum_{k=1}^{n+1} 2^{-k} + H(|||x|||, 0) \sum_{k=1}^{n+1} 2^{(p-1)k}. \end{aligned}$$

This completes the proof of the inequality (11). For $n > m$, we use

(11) to get

$$
\begin{aligned}
&d(2^{-n}f(2^n x), 2^{-m}f(2^m x)) \\
&= 2^{-m}d(2^{-(n-m)}f(2^{n-m}2^m x), f(2^m x)) \\
&\le 2^{-m}\left((\delta + |||f(0)|||)\sum_{k=1}^{n-m} 2^{-k} + H(|||2^m x|||, 0)\sum_{k=1}^{n-m} 2^{(p-1)k}\right) \\
&\le 2^{-m}\left((\delta + |||f(0)|||) + H(|||x|||, 0)2^{mp}\sum_{k=1}^{n-m} 2^{(p-1)k}\right) \\
&\le 2^{-m}(\delta + |||f(0)|||) + H(|||x|||, 0)\frac{2^{m(p-1)}}{2^{(1-p)} - 1} \\
&\to 0
\end{aligned}
$$

as $m \to \infty$. Therefore $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is a d-complete normed almost linear space, we can define

$$
F(x) = \lim_{n\to\infty} 2^{-n}f(2^n x) \tag{12}
$$

for all $x \in X$. From (6) and (12), we have

$$
\begin{aligned}
&d(F(x+y), F(x) + F(y)) \\
&= \lim_{n\to\infty} d(2^{-n}f(2^n(x+y)), 2^{-(n+1)}f(2^{n+1}x) + 2^{-(n+1)}f(2^{n+1}y)) \\
&= \lim_{n\to\infty} 2^{-(n+1)}d\left(2f\left(\frac{2^{n+1}(x+y)}{2}\right), f(2^{n+1}x) + f(2^{n+1}y)\right) \\
&\le \lim_{n\to\infty} (2^{-(n+1)}\delta + 2^{(n+1)(p-1)}H(|||x|||, |||y|||)) \\
&= 0.
\end{aligned}
$$

Hence, F is an additive mapping. Using (11) and (12) we have

$$
d(F(x), f(x)) \le \delta + |||f(0)||| + H(|||x|||, 0)\frac{1}{2^{1-p} - 1}
$$

for all $x \in X$.

Now, let $G : X \to Y$ be another additive mapping satisfying the inequality (7). Then we have

$$\begin{aligned} d(F(x), G(x)) &= 2^{-n} d(F(2^n x), G(2^n x)) \\ &\leq 2^{-n}(d(F(2^n x), f(2^n x)) + d(G(2^n x), f(2^n x))) \\ &\leq 2^{1-n}(\delta + |||f(0)|||) + 2H(|||x|||, 0)\frac{2^{n(p-1)}}{2^{1-p} - 1} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. This proves the uniqueness of F.

(II) The case $p > 1$. Let $f(0) = 0$ and $\delta = 0$ in (9). Instead of (10), we get

$$d\left(2f\left(\frac{x}{2}\right), f(x)\right) \leq H(|||x|||, 0)$$

for all $x \in X$. By induction, we have

$$d(2^n f(2^{-n}x), f(x)) \leq H(|||x|||, 0) \sum_{k=0}^{n-1} 2^{(1-p)k}$$

instead of (11). The rest of the proof is similar to the corresponding part of the case $p > 1$. □

For $p \in \mathbb{R} \setminus \{1\}$, define $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $H(0,0) = 0$ and $H(a,b) = (a+b)^p H(1,0)$ for all $a \neq 0$ or $b \neq 0$ in $\mathbb{R}_+$, where $H(1,0) = \theta \geq 0$. Then H is a homogeneous of degree p. Thus we have the following corollary.

COROLLARY 2. *Let d be a metric induced by a norm on a normed almost linear space Y and Y a d-complete real normed almost linear space. Let a function $f : X \to Y$ satisfy the following inequality*

$$d\left(2f\left(\frac{x+y}{2}\right), f(x) + f(y)\right) \leq \delta + \theta(|||x||| + |||y|||)^p \tag{13}$$

for all $x \neq 0, y \in X$. Furthermore, assume $f(0) = 0$ and $\delta = 0$ in (13) for the case of $p > 1$. Then there exists a unique additive mapping $F : X \to Y$ such that

$$d(F(x), f(x)) \leq \delta + |||f(0)||| + \frac{\theta}{2^{1-p} - 1}|||x|||^p \quad (for\ p < 1)$$

or

$$d(F(x), f(x)) \leq \frac{1}{1 - 2^{1-p}}\theta|||x|||^p \quad (for\ p > 1)$$

for all $x \neq 0$ in X.

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