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STABILITY OF THE JENSEN'S EQUATION IN A HILBERT MODULE OVER A C^*–ALGEBRA

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ABSTRACT. We prove the generalized Hyers-Ulam-Rassias stability of linear operators in a Hilbert module over a unital C^*-algebra.

Let E_1 and E_2 be Banach spaces. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$||f(x+y) - f(x) - f(y)|| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [9] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$||f(x) - T(x)|| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in E_1$.

In this paper, let A be a unital C^*-algebra with norm $| \cdot |$, $A_1 = \{a \in A \mid |a| = 1\}$, A_1^+ the set of positive elements in A_1, $\mathbb{R}^+$ the set of nonnegative real numbers, and ${}_A\mathcal{H}$ a left Hilbert A-module with norm $\| \cdot \|$. Throughout this paper, assume that (i) $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are continuous mappings, and that (ii) $\lim_{n\to\infty} 3^{-n}F(3^n x)$ and $\lim_{n\to\infty} 3^{-n}G(3^n x)$ converge uniformly on ${}_A\mathcal{H}$.

We are going to prove the generalized Hyers-Ulam-Rassias stability of linear operators in a Hilbert module over a unital C^*-algebra.

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LEMMA 1. *Let $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ such that*

$$\text{(iii)} \qquad \widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 3^{-k}\varphi(3^k x, 3^k y) < \infty,$$

$$\|2F(\frac{ax + ay}{2}) - aF(x) - aF(y)\| \leq \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then there exists a unique bounded A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\text{(iv)} \qquad \|F(x) - T(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in {}_A\mathcal{H}$.

Proof. Put $a = 1 \in A_1^+$. By [7, Theorem 1], there exists a unique additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iv). The mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ was given by $T(x) = \lim_{n\to\infty} \frac{F(3^n x)}{3^n}$ for all $x \in {}_A\mathcal{H}$. But $F(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{H}$. By the same reasoning as the proof of [9, Theorem], the additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is $\mathbb{R}$-linear.

By the assumption, for each $a \in A_1^+ \cup \{i\}$,

$$\|2F(3^n ax) - aF(2 \cdot 3^{n-1}x) - aF(4 \cdot 3^{n-1}x)\| \leq \varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x)$$

for all $x \in {}_A\mathcal{H}$. Using the fact that there exists a $K > 0$ such that, for each $a \in A$ and each $z \in {}_A\mathcal{H}$, $\|az\| \leq K|a| \cdot \|z\|$, one can show that

$$\begin{aligned}
&\|\frac{1}{2}aF(2 \cdot 3^{n-1}x) + \frac{1}{2}aF(4 \cdot 3^{n-1}x) - aF(3^n x)\| \\
&\quad \leq \frac{1}{2}K|a| \cdot \|2F(3^n x) - F(2 \cdot 3^{n-1}x) - F(4 \cdot 3^{n-1}x)\| \\
&\quad \leq \frac{K}{2}\varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x)
\end{aligned}$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. So

$$\begin{aligned}\|F(3^n ax) - aF(3^n x)\| \le & \|F(3^n ax) - \frac{1}{2}aF(2\cdot 3^{n-1}x) - \frac{1}{2}aF(4\cdot 3^{n-1}x)\| \\ & + \|\frac{1}{2}aF(2\cdot 3^{n-1}x) + \frac{1}{2}aF(4\cdot 3^{n-1}x) - aF(3^n x)\| \\ & \le \frac{1}{2}\varphi(2\cdot 3^{n-1}x, 4\cdot 3^{n-1}x) \\ & + \frac{K}{2}\varphi(2\cdot 3^{n-1}x, 4\cdot 3^{n-1}x)\end{aligned}$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. Thus $3^{-n}\|F(3^n ax) - aF(3^n x)\| \to 0$ as $n \to \infty$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. Hence

$$T(ax) = \lim_{n\to\infty} \frac{F(3^n ax)}{3^n} = \lim_{n\to\infty} \frac{aF(3^n x)}{3^n} = aT(x)$$

for each $a \in A_1^+ \cup \{i\}$. So

$$\begin{aligned}T(ax) = |a|T(\frac{a}{|a|}x) =& |a|\frac{a}{|a|}T(x) = aT(x), \ \forall a \in A^+ (a \neq 0), \ \forall x \in {}_A\mathcal{H}, \\ T(ix) =& iT(x), \quad \forall x \in {}_A\mathcal{H}.\end{aligned}$$

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = (\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-$, where $(\frac{a+a^*}{2})^+$, $(\frac{a+a^*}{2})^-$, $(\frac{a-a^*}{2i})^+$, and $(\frac{a-a^*}{2i})^-$ are positive elements (see [2, Lemma 38.8]). So

$$\begin{aligned}T(ax) =& T((\frac{a+a^*}{2})^+ x - (\frac{a+a^*}{2})^- x + i(\frac{a-a^*}{2i})^+ x - i(\frac{a-a^*}{2i})^- x) \\ =& (\frac{a+a^*}{2})^+ T(x) + (\frac{a+a^*}{2})^- T(-x) + (\frac{a-a^*}{2i})^+ T(ix) \\ & + (\frac{a-a^*}{2i})^- T(-ix) \\ =& (\frac{a+a^*}{2})^+ T(x) - (\frac{a+a^*}{2})^- T(x) + i(\frac{a-a^*}{2i})^+ T(x) \\ & - i(\frac{a-a^*}{2i})^- T(x) \\ =& ((\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-)T(x) \\ =& aT(x)\end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{H}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{H}$. So the unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear operator.

Since $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is continuous and $\lim_{n\to\infty} 3^{-n}F(3^n x)$ converges uniformly on ${}_A\mathcal{H}$, the A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is continuous. Hence the A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is bounded (see [3, Proposition II.1.1]). So there exists a unique bounded A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iv), as desired. □

THEOREM 2. *Let $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be mappings for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ satisfying* (iii) *such that*

$$\|2F(\frac{ax + ay}{2}) - aF(x) - aF(y)\| \le \varphi(x, y),$$
$$\|2G(\frac{ax + ay}{2}) - aG(x) - aG(y)\| \le \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Assume that $F(3^n x) = 3^n F(x)$ and $G(3^n x) = 3^n G(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$. Then the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators. Furthermore,

(1) *if the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfy the inequalities*

$$\|F \circ G(x) - x\| \le \varphi(x, x),$$
$$\|G \circ F(x) - x\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping G is the inverse of the mapping F,

(2) *if the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|F(x) - F^*(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a self-adjoint operator,*

(3) *if the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequality*

$$\|F \circ F^*(x) - F^* \circ F(x)\| \leq \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a normal operator,*

(4) *if the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequalities*

$$\begin{aligned} \|F \circ F^*(x) - x\| &\leq \varphi(x, x), \\ \|F^* \circ F(x) - x\| &\leq \varphi(x, x) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a unitary operator, and*

(5) *if the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequalities*

$$\begin{aligned} \|F \circ F(x) - F(x)\| &\leq \varphi(x, x), \\ \|F^*(x) - F(x)\| &\leq \varphi(x, x) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a projection.*

Proof. By the same method as the proof of Lemma 1, one can show that there exists a unique bounded A-linear operator $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\|G(x) - L(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in {}_A\mathcal{H}$.

By the assumption,

$$\begin{aligned} T(x) &= \lim_{n \to \infty} \frac{F(3^n x)}{3^n} = F(x), \\ L(x) &= \lim_{n \to \infty} \frac{G(3^n x)}{3^n} = G(x) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$, where the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Lemma 1. Hence the bounded A-linear operators T and L are the mappings F and G, respectively. So the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators.

(1) By the assumption,

$$\|F \circ G(3^n x) - 3^n x\| \le \varphi(3^n x, 3^n x),$$
$$\|G \circ F(3^n x) - 3^n x\| \le \varphi(3^n x, 3^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$3^{-n}\|F \circ G(3^n x) - 3^n x\| \to 0,$$
$$3^{-n}\|G \circ F(3^n x) - 3^n x\| \to 0$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$F \circ G(x) = \lim_{n\to\infty} \frac{F \circ G(3^n x)}{3^n} = x,$$
$$G \circ F(x) = \lim_{n\to\infty} \frac{G \circ F(3^n x)}{3^n} = x$$

for all $x \in {}_A\mathcal{H}$. So the mapping G is the inverse of the mapping F.

(2) By the assumption,

$$\|F(3^n x) - F^*(3^n x)\| \le \varphi(3^n x, 3^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus $3^{-n}\|F(3^n x) - F^*(3^n x)\| \to 0$ as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$F(x) = \lim_{n\to\infty} \frac{F(3^n x)}{3^n} = \lim_{n\to\infty} \frac{F^*(3^n x)}{3^n} = F^*(x)$$

for all $x \in {}_A\mathcal{H}$. So the mapping F is a self-adjoint operator.

The proofs of the other items are similar to the proofs of (1) and (2). □

From now on, we denote by A_{in} the set of invertible elements in A, and assume that A has real rank 0, which means that the set of invertible self-adjoint elements in A is dense in the set of self-adjoint elements in A (see [1, 4]).

THEOREM 3. *Let* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *be a mapping for which there exists a function* $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0,\infty)$ *satisfying* (iii) *such that*

$$\|2F(\frac{ax+ay}{2}) - aF(x) - aF(y)\| \le \varphi(x,y)$$

for all $a \in (A_{in} \cap A_1^+) \cup \{i\}$ *and all* $x, y \in {}_A\mathcal{H}$. *Then there exists a unique bounded* A*-linear operator* $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfying* (iv).

Proof. By the same reasoning as the proof of Lemma 1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iv).

By the same method as the proof of Lemma 1, one can show that

$$(1) \qquad T(ax) = \lim_{n\to\infty} 3^{-n}F(3^n ax) = \lim_{n\to\infty} 3^{-n}aF(3^n x) = aT(x)$$

for all $a \in (A_{in} \cap A_1^+) \cup \{i\}$ and all $x \in {}_A\mathcal{H}$.

Let $b \in A_1^+ \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exists a sequence $\{b_m\}$ in $A_{in} \cap A_{sa}$ such that $b_m \to b$ as $m \to \infty$, where A_{sa} denotes the set of self-adjoint elements in A. Put $c_m = \frac{1}{|b_m|}b_m$. Then $c_m \to \frac{1}{|b|}b = b$ as $m \to \infty$ and $c_m \in A_{in} \cap A_1$. Put $a_m = \sqrt{c_m{}^* c_m}$. Then $a_m \to \sqrt{b^* b} = b$ as $m \to \infty$ and $a_m \in A_{in} \cap A_1^+$. Thus there exists a sequence $\{a_m\}$ in $A_{in} \cap A_1^+$ such that $a_m \to b$ as $m \to \infty$, and so

$$(2) \qquad \begin{aligned} \lim_{m\to\infty} T(a_m x) &= \lim_{m\to\infty}\lim_{n\to\infty} 3^{-n}F(3^n a_m x) \\ &= \lim_{n\to\infty}\lim_{m\to\infty} 3^{-n}F(3^n a_m x) \text{ by (ii)} \\ &= \lim_{n\to\infty}(3^{-n}F(3^n \lim_{m\to\infty} a_m x)) \text{ by (i)} \\ &= \lim_{n\to\infty} 3^{-n}F(3^n bx) \\ &= T(bx) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$. By (1),

$$(3) \quad \|T(a_m x) - bT(x)\| = \|a_m T(x) - bT(x)\| \to \|bT(x) - bT(x)\| = 0$$

as $m \to \infty$. By (2),

$$\|3^{-n}F(3^n a_m x) - T(a_m x)\| \to \|3^{-n}F(3^n bx) - T(bx)\| \tag{4}$$

as $m \to \infty$. By (3) and (4),

$$\begin{aligned}\|T(bx) - bT(x)\| \leq & \|T(bx) - 3^{-n}F(3^n bx)\| \\ & + \|3^{-n}F(3^n bx) - 3^{-n}F(3^n a_m x)\| \\ & + \|3^{-n}F(3^n a_m x) - T(a_m x)\| \\ & + \|T(a_m x) - bT(x)\| \\ \to & \|T(bx) - 3^{-n}F(3^n bx)\| \\ & + \|3^{-n}F(3^n bx) - T(bx)\| \text{ as } m \to \infty \\ \to & 0 \text{ as } n \to \infty\end{aligned} \tag{5}$$

for all $x \in {}_A\mathcal{H}$. By (1) and (5),

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{H}$.

The rest of the proof is similar to the proof of Lemma 1. So the unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a bounded A-linear operator satisfying (iv).

□

Replacing $A_1^+ \cup \{i\}$ in the statement of Theorem 2 by $(A_{in} \cap A_1^+) \cup \{i\}$, one can obtain the same results as Theorem 2, under the assumption that A has real rank 0.

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