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EXPANSIVITY OF A CONTINUOUS SURJECTION

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ABSTRACT. We introduce the notion of expansivity for a continuous surjection on a compact metric space, as the positively and negatively expansive map. We also prove that some well-known properties about positively expansive maps in [2] hold by using our definition.

1. Introduction

The study of positively expansive maps is an interesting subject in topological dynamics. Definition of positively extensive maps is presented in [2,9] as the following : Let (X, d) be a metric space. A continuous surjection $f : X \to X$ is called *positively expansive* if there exists a constant $e > 0$ such that if $x \neq y$ in X then $d(f^n(x), f^n(y)) > e$ for some non-negative integer n (e is called an expansive constant for f). For compact spaces, this is independent of the compatible metrics used, although not the expansive constants. Also, a homeomorphism $f : X \to X$ is *expansive* if there is a constant $e > 0$ such that $x \neq y$ in X implies $d(f^n(x), f^n(y)) > e$ for some integer n. Expansivity is very often appearing in several branch of the theory of dynamical systems, and it is known that every axiom A diffeomorphism restricted to the non-wandering set possesses it [10].

We introduce the concept of expansivity for a continuous surjection as the positively and negatively expansive map, that is, for a compact

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metric space (X, d), we say that a continuous surjection $f : X \to X$ is *expansive* if

(i) for any $\varepsilon > 0$ and $x \in X$, there exists a $\delta > 0$ such that $d(f(x), y) < \delta$ for $y \in X$ implies $B(x, \varepsilon) \cap f^{-1}(y) \neq \varnothing$, where $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$,

(ii) there exists a constant $e > 0$ such that for $x, y \in X$,

$$d(f^n(x), f^n(y)) \leq e$$

for all $n \in \mathbb{Z}$ implies $x = y$.
We call the condition (i) as Property B.

In this paper we examine some properties about Property B and show that some well-known properties about positively expansive maps in [2] hold via our definition. Furthermore, we prove that expansivity is preserved under a topological conjugacy.

2. Property B

Let (X, d) be a metric space and $f : X \to X$ be a continuous surjection. We say that f has *Property B* if for any $\varepsilon > 0$ and $x \in X$ there exists a $\delta > 0$ such that $d(f(x), y) < \delta$ for $y \in X$ implies that $B(x, \varepsilon) \cap f^{-1}(y) \neq \varnothing$, where $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

Example. Let $X = [-1, \frac{1+\sqrt{5}}{2}]$ be a subset of the Euclidean space $\mathbb{R}$. Then the map $f : X \to X$ defined by $f(x) = x^4 - 2x^2$ does not have Property B.

Before introducing the concept of expansivity for a continuous surjection we investigate some properties about Property B.

THEOREM 2.1. *(1) Every local homeomorphism has Property B.*

(2) Every continuous surjection which has Property B is an open map.

(3) Let X be a connected compact metric space and $f : X \to X$ be any continuous map. If f has Property B, then f is a surjection.

Proof. (1) Let $f : X \to X$ be a local homeomorphism and $x \in X$. There exists a neighborhood U of x such that $f|_U : U \to f(U)$ is a homeomorphism. Also, there exists an $\nu > 0$ such that $B(x,\nu) \subset U$. For every ε with $0 < \varepsilon < \nu$ we have $B(f(x),\delta) \subset f(B(x,\varepsilon))$ for some $\delta > 0$ since $f(B(x,\varepsilon))$ is a neighborhood of $f(x)$. Since $y = f(z)$ for some $z \in B(x,\varepsilon), z \in B(x,\varepsilon) \cap f^{-1}(y)$. This implies that f has Property B.

(2) Let U be an open subset of X and $y \in f(U)$. Then $y = f(x)$ for some $x \in U$ and $B(x,\varepsilon) \subset U$ for some $\varepsilon > 0$. There exists a $\delta > 0$ such that $d(f(x),z) < \delta$ implies $B(x,\varepsilon) \cap f^{-1}(z) \neq \varnothing$ by Property B. Since $B(x,\varepsilon) \cap f^{-1}(z) \neq \varnothing$ for every $z \in B(y,\delta)$, we have $z = f(w) \in f(B(x,\varepsilon)) \subset f(U)$ for some $w \in B(x,\varepsilon) \cap f^{-1}(z)$. It follows that $B(y,\delta) \subset f(U)$ and $f(U)$ is an open set in X.

(3) In view of (2) $f(X)$ is an open set. Also, $f(X)$ is a closed set since f is continuous. Hence $f(X) = X$ by the connectedness of X. □

Remark. When X is a compact metric space any positively expansive open map $f : X \to X$ is a local homeomorphism [2, p. 50].

Let X be a compact metric space. A map $\alpha : X \to \mathbb{R}$ is said to be *upper semicontinuous* if for every $\varepsilon > 0$ and $x \in X$, there exists a neighborhood U of x such that $\alpha(U) \subset (-\infty, \alpha(x) + \varepsilon)$. And a map $\beta : X \to \mathbb{R}$ is said to be *lower semicontinuous* if for every $\varepsilon > 0$ and $x \in X$, there exists a neighborhood V of x such that $\beta(V) \subset (\beta(x) - \varepsilon, \infty)$.

We need the following Dowker's Lemma.

LEMMA 2.2. [7] *Suppose that α and ω are, respectively, upper and* *lower semicontinuous on a paracompact space, and that* $\alpha(x) < \omega(x)$ *for every* x. *Then there is a continuous map* $\tau(x)$ *with* $\alpha(x) < \tau(x) < \omega(x)$ *for every* x.

THEOREM 2.3. *Let* (X, d) *be a compact metric space. If* $f : X \to X$ *has Property B, then there exists a continuous map* $\delta : X \to (0, \infty)$ *such that for every* $x \in X, d(f(x), y) < \delta(x)$ *implies* $B(x, \varepsilon) \cap f^{-1}(y) \neq \varnothing$.

Proof. For any $\varepsilon > 0$, define a map $\beta : X \to (0, \infty)$ by

$$\beta(x) = \sup\{\eta > 0 : d(f(x), y) < \eta \text{ implies } B(x, \alpha) \cap f^{-1}(y) \neq \varnothing \\ \text{for some } 0 < \alpha < \varepsilon\}.$$

The map β is well-defined since there exists an $\eta > 0$ such that $d(f(x), y) < \eta$ implies $B(x, \alpha) \cap f^{-1}(y) \neq \varnothing$ by Property B, and so $\beta(x) \geq \eta > 0$.

Let $x \in X$ and $h < \beta(x)$. Then $h < \eta$ for some $\eta > 0$ and $d(f(x), y) < \eta$ implies $B(x, \alpha) \cap f^{-1}(y) \neq \varnothing$ for some $0 < \alpha < \varepsilon$. If we choose γ with $h < \gamma < \eta$, then there exists a neighborhood V of x such that $f(V) \subset B(f(x), \eta - \gamma)$. Also, we choose ξ with $\alpha < \xi < \varepsilon$ and put $U = B(x, \xi - \alpha) \cap V$. Then $B(x, \alpha) \subset B(y, \xi)$ for every $y \in U$. If $d(f(y), z) < \gamma$, then we have

$$d(f(x), z) \leq d(f(x), f(y)) + d(f(y), z) < \eta.$$

Since $B(x, \alpha) \cap f^{-1}(z) \subset B(y, \xi) \cap f^{-1}(z)$, we have $B(y, \xi) \cap f^{-1}(z) \neq \varnothing$. There exists ξ with $0 < \alpha < \xi < \varepsilon$ such that $d(f(x), z) < \gamma$ implies $B(y, \xi) \cap f^{-1}(z) \neq \varnothing$. It follows that $\beta(y) \geq \gamma > h$. Thus β is lower semicontinuous.

Now, we define a map O on X by $O(x) = 0$. Since $O(x) < \beta(x)$ for every $x \in X$, the map O is upper semicontinuous. Therefore there exists a continuous map $\delta : X \to (0, \infty)$ such that $0 < \delta(x) < \beta(x)$ by Lemma 2.2. This completes the proof. □

Let (X, d) be a compact metric space and $C(X)$ be a set of all nonempty closed subsets of X. $C(X)$ becomes a compact metric space with the Hausdorff metric H, i.e.,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

for $A, B \in C(X)$. Note that $d(x, y) = H(x, y)$ for all $x, y \in X$ if we denote one point set by a point.

THEOREM 2.4. *Every continuous surjection $f : X \to X$ has Property B if and only if $f^{-1} : C(X) \to C(X)$ is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be given. Then, in view of Theorem 2.3, there exists a continuous map $\delta : X \to (0, \infty)$ such that $d(f(x), y) < \delta(x)$ implies $B(x, \varepsilon) \cap f^{-}(y) \neq \varnothing$. Put $\delta = \min\{\delta(x) > 0 : x \in X\}$. Let $A, B \in C(X)$ with $H(A, B) < \delta$. Then $A \subset B(B, \delta)$ and $B \subset B(A, \delta)$. Let $x \in f^{-1}(A)$. Then $f(x) \in A \subset B(B, \delta)$. There exists an $y \in B$ such that $d(f(x), y) < \delta \leq \delta(x)$. Since f has Property B, we have $B(x, \varepsilon) \cap f^{-1}(y) \neq \varnothing$, say $z \in B(x, \varepsilon) \cap f^{-1}(y)$. Then $d(x, z) < \varepsilon$ and $z \in f^{-1}(y) \subset f^{-1}(B)$. Hence $x \in B(f^{-1}(B), \varepsilon)$ and so $f^{-1}(A) \subset B(f^{-1}(B), \varepsilon)$. Similarly, we have $f^{-1}(B) \subset B(f^{-1}(A), \varepsilon)$. Therefore $H(f^{-1}(A), f^{-1}(B)) < \varepsilon$, which implies f^{-1} is uniformly continuous.

For the converse let $\varepsilon > 0$ and $x \in X$. Then there exists a $\delta > 0$ such that $H(A, B) < \delta$ implies $H(f^{-1}(A), f^{-1}(B)) < \varepsilon$. If we let $d(f(x), y) = H(f(x), y) < \delta$, then $H(f^{-1}(f(x), f^{-1}(y)) < \varepsilon$. Since $x \in f^{-1}(f(x)) \subset B(f^{-1}(y), \varepsilon)$, $d(x, z) < \varepsilon$ for some $z \in f^{-1}(y)$, and so $z \in B(x, \varepsilon) \cap f^{-1}(y)$. It follows that $B(x, \varepsilon) \cap f^{-1}(y) \neq \varnothing$. This means that f has Property B. □

3. Expansivity

We define the concept of expansivity for a continuous surjection.

Definition. Let (X,d) be a compact metric space. A continuous surjection $f : X \to X$ is *expansive* if

(i) f has Property B.

(ii) There exists a constant $e > 0$ such that for $x, y \in X$,

$$d(f^n(x), f^n(y)) \leq e$$

for all $n \in \mathbb{Z}$ implies $x = y$. Such a number e is called an expansive constant for f.

There are well-known properties about positive expansivity for a continuous surjection in [2, Chapter 2]. We prove some of these properties by using the above definition. Throughout this section (X,d) is a compact metric space and $f : X \to X$ is a continuous surjection.

THEOREM 3.1. *If $f : X \to X$ is expansive, then $f^k : X \to X$ is also expansive for all integer $k > 1$.*

Proof. By Theorem unithm, $f^{-1} : C(X) \to C(X)$ is continuous and so $(f^k)^{-1} = (f^{-1})^k : C(X) \to C(X)$ is continuous. Thus f^k has Property B by Theorem 2.4.

For the converse let e_f be an expansive constant for f. There exists $0 < e < e_f$ such that $d(f^i(a), f^i(b)) \leq e_f$ for all $i = 1, 2, \cdots, k-1$ when $d(a,b) \leq e$. Suppose that $d(f^{nk}(x), f^{nk}(y)) \leq e$ for all $n \in \mathbb{Z}$. When $n \geq 0$, we have

$$d(f^i(f^{nk}(x)), f^i(f^{nk}(y))) = d(f^{nk+i}(x), f^{nk+i}(y)) \leq e_f$$

for all $i = 1, 2, \cdots, k-1$. When $n < 0$,

$$d(a,b) = d(f^{nk}(x), f^{nk}(y))$$

for some $a \in f^{nk}(x)$ and $b \in f^{nk}(y)$. We obtain

$$d(f^{nk+i}(x), f^{nk+i}(y)) \leq d(f^i(a), f^i(b)) \leq e_f$$

since $f^i(a) \in f^{nk+i}(x)$ and $f^i(b) \in f^{nk+i}(y)$ for $0 < i < k$. Therefore $d(f^n(x), f^n(y)) \leq e_f$ for all $n \in \mathbb{Z}$. This implies that $x = y$. □

Let (X_1, d_1) and (X_2, d_2) be compact metric spaces. We can give a metric d on $X_1 \times X_2$, which is compatible with the product topology of $X_1 \times X_2$, defined by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$.

THEOREM 3.2. *Let $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ be continuous surjections. If f_1 and f_2 are expansive, then $f_1 \times f_2$ is also expansive.*

Proof. To show that $f_1 \times f_2$ has Property B let $\varepsilon > 0$ and $(x_1, x_2) \in X_1 \times X_2$. There exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$d_1(f_1(x_1), y_1) < \delta_1 \text{ implies } B_{d_1}(x_1, \varepsilon) \cap f_1^{-1}(y_1) \neq \varnothing$$

and

$$d_2(f_2(x_2), y_2) < \delta_2 \text{ implies } B_{d_2}(x_2, \varepsilon) \cap f_2^{-1}(y_2) \neq \varnothing.$$

Take $\delta = \min\{\delta_1, \delta_2\} > 0$. Let $d((f_1 \times f_2)(x_1, x_2), (y_1, y_2)) < \delta$. Then

$$d_1(f_1(x_1), y_1) \leq d((f_1 \times f_2)(x_1, x_2), (y_1, y_2)) < \delta \leq \delta_1,$$
$$d_2(f_2(x_2), y_2) \leq d((f_1 \times f_2)(x_1, x_2), (y_1, y_2)) < \delta \leq \delta_2.$$

Thus we have

$$z_1 \in B_{d_1}(x_1, \varepsilon) \cap f_1^{-1}(y_1), z_2 \in B_{d_2}(x_2, \varepsilon) \cap f_2^{-1}(y_2)$$

for some $z_1 \in X_1$ and $z_2 \in X_2$. Since $d_1(x_1, z_1) < \varepsilon$ and $d_2(x_2, z_2) < \varepsilon$, we obtain $d((x_1, x_2), (z_1, z_2)) < \varepsilon$. Since $f_1(z_1) = y_1$ and $f_2(z_2) = y_2$, $(f_1 \times f_2)(z_1, z_2) = (y_1, y_2)$. Hence $(z_1, z_2) \in B_{d_1}((x_1, x_2), \varepsilon) \cap (f_1 \times f_2)^{-1}(y_1, y_2)$ and so $f_1 \times f_2$ has Property B by Theorem 2.4

Now, let $e_1 > 0$ and $e_2 > 0$ be expansive constant for f_1 and f_2, respectively. Suppose that

$$d((f_1 \times f_2)^n(x_1, x_2), (f_1 \times f_2)^n(y_1, y_2)) \leq e \text{ for all } n \in \mathbb{Z},$$

where $e = \min\{e_1, e_2\} > 0$. When $n \geq 0$, we have

$$\begin{aligned} d_1(f_1^n(x_1), f_1^n(y_1)) &\leq d((f_1^n(x_1), f_2^n(x_2)), (f_1^n(y_1), f_2^n(y_2)) \\ &= d((f_1 \times f_2)^n(x_1, x_1), (f_1 \times f_2^n)(y_1, y_2))) \\ &\leq e \leq e_1. \end{aligned}$$

Similarly, we have $d_2(f_2^n(x_2), f_2^n(y_2)) \leq e_2$.

Suppose $n < 0$. There exist $(a_1, a_2) \in (f_1 \times f_2)^n(x_1, x_2)$ and $(b_1, b_2) \in (f_1 \times f_2)^n(y_1, y_2)$ such that

$$d((a_1, a_2), (b_1, b_2)) = d_1((f_1 \times f_2)^n(x_1, x_2), (f_1 \times f_2)^n(y_1, y_2)).$$

Since $x_1 = f_1^{-n}(a_1)$ and $x_2 = f_2^{-n}(a_2)$, we obtain

$$(x_1, x_2) = (f_1^{-n}(a_1), f_2^{-n}(a_2)) = (f_1 \times f_2)^{-n}(a_1, a_2).$$

It follows that $(a_1, a_2) \in (f_1 \times f_2^n)(x_1, x_2)$. Also, we have $(b_1, b_2) \in (f_1 \times f_2)^n(y_1, y_2)$ by the same manner. Note that

$$\begin{aligned} d_1(f_1^n(x_1), f_1^n(y_1)) &\leq d_1(a_1, b_1) \leq d((a_1, a_2), (b_1, b_2)) \\ &= d((f_1 \times f_2)^n(x_1, x_2), (f_1 \times f_2)^n(y_1, y_2)) \\ &\leq e \leq e_1 \end{aligned}$$

and

$$d_2(f_2^n(x_2), f_2^n(y_2)) \leq e_2.$$

Therefore we have

$$d_1(f_1^n(x_1), f_1^n(y_1)) \leq e \text{ and } d_2(f_2^n(x_2), f_2^n(y_2)) \leq e_2$$

for all $\in \mathbb{Z}$. Consequently, we have $(x_1, x_2) = (y_1, y_2)$ and the expansivity of $f_1 \times f_2$ follows. □

Let (X, d) and (Y, ρ) be compact metric spaces. A continuous surjection $f : X \to X$ is said to be *topologically conjugate* to a continuous surjection $g : Y \to Y$ if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

We prove that expansivity is preserved under a topological conjugacy.

THEOREM 3.3. *Suppose that f is topologically conjugate to g. If f is expansive, then g is also expansive.*

Proof. Let $\varepsilon > 0$ and $y \in Y$. Since there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$ and h is uniformly continuous, there exists a $\delta > 0$ such that $d(a, b) < \delta$ implies $\rho(h(a), h(b)) < \varepsilon$. Putting $x = h^{-1}(y)$, we have $B_d(x, \delta) \cap f^{-1}(z) \neq \varnothing$ when $d(f(x), z) < \eta$ for some $\eta > 0$. Also, $d(h^{-1}(a), h^{-1}(b)) < \eta$ when $\rho(a, b) < \xi$ for some $\xi > 0$ by the uniform continuity of h^{-1}.

Now, let $\rho((g(y), w)) < \xi$ for $w \in Y$. Then

$$\begin{aligned} d(h^{-1}(g(y)), h^{-1}(w)) &= d(f(h^{-1}(y), h^{-1}(w)) \\ &= d(f(x), h^{-1}(w)) < \eta \end{aligned}$$

and thus $B_d(x, \delta) \cap f^{-1}(h^{-1}(w)) \neq \varnothing$, say $u \in B_d(x, \delta) \cap f^{-1}(h^{-1}(w))$. Hence $h(u) \in B_\rho(y, \varepsilon) \cap g^{-1}(w)$ since $h(f(u)) = g(h(u)) = w$ and

$\rho(h(x), h(u)) = \rho(y, h(u)) < \varepsilon$ when $d(x, u) < \delta$. This implies that g has Property B.

Now, let e_f be an expansive constant for f. There exists an $e > 0$ such that $d(h^{-1}(a), h^{-1}(b)) \leq e_f$ if $\rho(a, b) \leq e$, by the uniform continuity of h^{-1}. Suppose that $\rho(g^n(x), g^n(y)) \leq e$ for all $n \in \mathbb{Z}$. We claim that

$$d(f^n(h^{-1}(x)), f^n(h^{-1}(y))) \leq e_f$$

for all $z \in \mathbb{Z}$. For the case $n \geq 0$, it easily follows since

$$d(f^n(h^{-1}(x)), f^n(h^{-1}(y))) = d(h^{-1}(g^n(x)), h^{-1}(g^n(y))) \leq e_f.$$

Let $n < 0$. There exist $a \in g^n(x)$ and $b \in g^n(y)$ such that $\rho(a, b) = \rho(g^n(x), g^n(y))$. Since

$$h^{-1}(x) = h^{-1}(g^{-n}(a)) = f^{-n}(h^{-1}(a))$$

and

$$h^{-1}(y) = h^{-1}(g^{-n}(b)) = f^{-n}(h^{-1}(b)),$$

we have $h^{-1}(a) \in f^n(h^{-1}(x))$ and $h^{-1}(b) \in f^n(h^{-1}(y))$. Now, we get

$$d(f^n(h^{-1}(x)), f^n(h^{-1}(y))) \leq d(h^{-1}(a), h^{-1}(b)) \leq e_f$$

since $\rho(a, b) = \rho(g^n(x), g^n(y)) \leq e$ and $d(h^{-1}(a), h^{-1}(b)) \leq e_f$.

Therefore $h^{-1}(x) = h^{-1}(y)$ and so $x = y$. This means that g is expansive. This completes the proof. □

For a continuous surjection $f : X \to X$, we define a set

$$X_f = \{(x_i) \in \prod_{-\infty}^{\infty} X : f(x_i) = x_{i+1}\}$$

and give a metric ρ on X_f defined by

$$\rho(x,y) = \sum_{i=-\infty}^{i=\infty} \frac{d(x_i, g_i)}{2^{|i|}}$$

for $x = (x_i), y = (y_i) \in X_f$.

A homeomorphism $\sigma : X_f \to X_f$ defined by $\sigma((x_i)) = (f(x_i))$ is called the *shift map* determined by f.

LEMMA 3.4. *The shift map $\sigma : X_f \to X_f$ is expansive if and only if there exists an $e > 0$ such that for $(x_i), (y_i) \in X_f$, if $d(x_i, y_i) \leq e$ for all $i \in \mathbb{Z}$, then $(x_i) = (y_i)$.*

Proof. Let $e > 0$ be an expansive constant for σ. Let $(x_i), (y_i) \in X_f$ with $d(x_i, y_i) \leq \frac{e}{3}$ for all $i \in \mathbb{Z}$. Then, for all $n \in \mathbb{Z}$

$$\begin{aligned} \rho(\sigma^n(x_i), \sigma^n(y_i)) &= \sum_{-\infty}^{\infty} \frac{d(\sigma^n((x_i))_k, \sigma^n((y_i))_k)}{2^{|k|}} \\ &= \sum_{-\infty}^{\infty} \frac{d(x_{n+k}, y_{n+k})}{2^{|k|}} \\ &\leq \sum_{-\infty}^{\infty} \frac{(\frac{e}{3})}{(2^{|k|})}. \end{aligned}$$

Hence $(x_i) = (y_i)$.

Assume that there exists an $e > 0$ such that for $(x_i), (y_i) \in X_f$, $d(x_i, y_i) \leq e$ implies $(x_i) = (y_i)$. Let $(x_i), (y_i) \in X_f$ with $(x_i) \neq (y_i)$. Then there exists an integer k such that $d(x_k, y_k) > e$. Thus we have

$$\rho(\sigma^k((x_i)), \sigma^k((y_i))) \geq d(\sigma^k((x_i))_0, \sigma^k((y_i))_0) = d(x_k, y_k) > e.$$

Therefore σ is expansive with an expansive constant e. □

LEMMA 3.5. *If f is expansive, then σ is expansive.*

Proof. Since f is expansive, there exists an $e > 0$ such that if $d(f^n(x), f^n(y)) < e$ for all $z \in \mathbb{Z}$, then $x = y$. Let $(x_i), (y_i) \in X_f$ with $d(x_i, y_i) \leq e$ for all $i \in \mathbb{Z}$. For every $k \in \mathbb{Z}$ and $n \geq 0$, since $f^n(x_{k-n}) = x_k$ and $f^n(y_{k-n}) = y_k$, we get $x_{k-n} \in f^{-n}(x_k)$ and $y_{k-n} \in f^{-n}(y_k)$. Thus

$$d(f^{-n}(x_k), f^{-n}(y_k)) \leq d(x_{k-n}, y_{k-n}) \leq e$$

and

$$d(f^n(x_k), f^n(y_k)) = d(x_{k+n}, y_{k+n}) \leq e.$$

Hence $(x_i) = (y_i)$. Therefore σ is expansive by Lemma 3.4. □

Let

$$\varprojlim(X, f) = \{(x_i) \in \prod_0^\infty X : f(x_{i+1}) = x_i \text{ for all } i \geq 0\}$$

and define a map $\tau : \varprojlim(X, f) \to \varprojlim(X, f)$ by

$$\tau(x_0, x_1, \cdots) = (f(x_0), x_0, x_1, \cdots).$$

Then the map τ is a homeomorphism.

Define a map $\varphi : X_f \to \varprojlim(X, f)$ by

$$\varphi(\cdots, x_{-1}, x_0, x_1, \cdots) = (x_0, x_{-1}, x_{-2}, \cdots).$$

Then φ is also a homeomorphism since

$$\varphi^{-1}(x_0, x_1, x_2, \cdots) = (\cdots, x_2, x_1, x_0, f(x_0), f^2(x_0), \cdots).$$

Since $\tau \circ \phi = \phi \circ \sigma$, σ is topologically conjugate to τ. Hence, by Lemma 3.5, τ is expansive if f is expansive.

THEOREM 3.6. *The closed interval $I = [0,1]$ does not admit expansive maps.*

Proof. Let $f : I \to I$ be a continuous surjection and suppose that f is expansive. Then τ is expansive. Since f is a surjection, we have the following properties : There exist $a, b \in I$ with $0 \le a \le b \le 1$ such that either

(1) $f(a) = 0, f(b) = 1$, or

(2) $f(a) = 1, f(b) = 0$.

For the case (1), we have a contradiction as follows : Since f has at least two fixed points and the set of all fixed points of f is not dense in I, there are two points $u, v \in I$ with $u < v$ such that $f(u) = v, f(u) = v$ and $f(x) \neq x$ for $u < x < v$. It follows two cases

(1-1) $f(x) > x$ for $u < x < v$, or

(1-2) $f(x) < x$ for $u < x < v$.

We consider the cases (1-1). Note that $f|_{[u,v]}$ cannot be a bijection. Thus there are two points x^1, x^2 such that $u < x^1 < x^2 < v$ and $f(x^1) = f(x^2)$. If we take $y^0 = f(x^0) = \max\{f(x) : u < x \le x^2\}$, then $u < x^0 < x^2 < y^0$. Thus we can construct a sequence

$$x_0^0 > x_1^1 > \cdots > u, \; x_0^0 = x^0,$$

such that $f(x_i^0) = x_{i-1}^0$ for $i \ge 1$ and $x_n^0 \to u$ as $n \to \infty$.

For $0 < \varepsilon < \min\{x^2 - x^0, x^0 - u\}$, there exists an $N > 0$ such that $u < x_N^0 < u + \frac{\varepsilon}{2}$. Also, there exists $0 < \delta < x_N^0 - u$ such that for $x, y \in I$

$$|x - y| < 2\delta \text{ implies } |f^n(x) - f^n(y)| < \frac{\varepsilon}{2}$$

for all $0 \le n \le N$. Putting

$$c^1 = x_N^0 - \delta, \; c^2 = x_N^0 + \delta,$$

we have

$$|f^N(c^i) - f^N(x_n^0)| = |f^N(c^i) - x^0| < \frac{\varepsilon}{2}, i = 1, 2$$

since $u < c^1 < c^2 < x^2$ and $|c^i - x_N^0| = \delta < 2\delta$, Thus we obtain

$$u < x^0 - \frac{\varepsilon}{2} < f^N(c^i) < x^0 + \frac{\varepsilon}{2} < x^2$$

and

$$f(f^N(c^i)) = f^{N+1}(c^i) \leq y^0, \ i = 1, 2.$$

Now, we have the following three cases :
(i) $f^{N+1}(c^1)) = y^0$ and $f^{N+1}(c^2) < y^0$, or
(ii) $f^{N+1}(c^1) < y^0$ and $f^{N+1}(c^2) = y^0$, or
(iii) $f^{N+1}(c^1) < y^0$ and $f^{N+1}(c^2) < y^0$.
We first consider for the cases (i) and (ii). If $f^{N+1}(c^i) = y^0$ for some $i = 1, 2$, then we can find a sequence

$$c_0^i > c_1^i > c_2^i > \cdots > u, \ c_0^i = c^i$$

such that $f(c_j^i) = c_{j-1}^i$ for $j \geq 1$. We construct a sequence $\{b_j\}$ as $b_j = f^{N-j}(c^i)$ for $0 \leq j \leq N$ and $b_{N+j} = c_j^i$ for $j > 0$. Since $|c^i - x_N^0| = \delta < 2\delta$, we have

$$|f^n(c^i) - f^n(x_N^0)| = |b_{N-n} - x_{N-n}^0| < \frac{\varepsilon}{2}$$

for $0 \leq n \leq N$. Note that

$$u + \frac{\varepsilon}{2} > x_N^0 > x_{N+1}^0 > \cdots > u \ \text{ and } u + \varepsilon > x_N^0 + \delta \geq c_0^i > c_1^i > \cdots > u.$$

Thus we obtain

$$|b_j - x_j^0| = |c_{j-N} - x_j^0| < \varepsilon$$

for $j \geq N$. This implies that $|b_j - x_j^0| < \varepsilon$ for all $j \geq 0$. Furthermore, we have

$$f^j(b_0) = f^j(f^N(c^i)) = f^{j-1}(f^{N+1}(c^i)) = f^{j-1}(f(x_0^0)) = f^j(x_0^0)$$

for $j > 0$, since $f^{N+1}(c^i) = y^0 = f(x^0)$.

When $n \geq 0$, we have

$$\tau^n(b_0, b_1, \cdots) = (f^n(b_0), \cdots, f(b_0), b_0, b_1, \cdots)$$

and

$$\tau^n(x_0^0, x_1^0, \cdots) = (f^n(x_0^0), \cdots, f(x_0), x_0^0, x_1^0, \cdots).$$

When $0 \neq j < n$, we have

$$\tau^n(b_0, b_1, \cdots)_j = f^{n-j}(b_0) \text{ and } \tau^n(x_0^0, x_1^0, \cdots)_j = f^{n-j}(x_0^0).$$

Thus $f^{n-j}(b_0) = f^{n-j}(x_0^0)$ since $f^j(b_0) = f^j(x_0)$. When $j \geq n$,

$$\tau^n(b_0, b_1, \cdots)_j = b_{j-n} \text{ and } \tau^n(x_0^0, x_1^0, \cdots)_j = x_{j-n}^0$$

implies that $|b_{j-n} - x_{j-n}^0| < \varepsilon < e$.

When $n < 0$, we have

$$\tau^n(b_0, b_1, \cdots) = (b_{-n}, b_{1-n}, \cdots)$$

and

$$\tau^n(x_0^0, x_1^0, \cdots) = (x_{-n}^0, x_{1-n}^0, \cdots).$$

Thus

$$\rho(\tau^n(b_0, b_1, \cdots), \tau^n(x_0^0, x_1^0, \cdots)) < \varepsilon$$

for all $n \in \mathbb{Z}$. It follows that $(b_0, b_1, \cdots) = (x_0^0, x_1^0, \cdots)$ which is a contradiction since $b_N \neq x_N^0$.

Next, we consider for the case (iii). If we assume that $f^{N+1}(c^1) < y^0$ and $f^{N+1}(c^2) < y^0$, then

$$f^{N+1}(x_N^0) = f(f^N(x_N^0)) = f(x_0^0) = y^0.$$

Since there are c^3, c^4 such that

$$c^1 < c^3 < x_N^0 < c^4 < c^2 \text{ and } f^{N+1}(c^3) = f^{N+1}(c^4),$$

we can find a sequence for $i = 3, 4$,

$$c_0^i > c_1^i. \cdots > u, c_0^i = c^i, f(c_j^i) = c_{j-1}^i \text{ for } j \geq 1.$$

Then we also get a contradiction by the same manner for the cases (i) and (ii).

For the case (1-2), we can prove similarly by the method for the case (1-1).

Finally, we consider the case (2). Since there exist a_1, a_2 with $a < a_1 < b_1 < b$ such that $f(a_1) = b$ and $f(b_1) = a$, we have $f^2(a_1) = 0$ and $f^2(b_1) = 1$. Since $\tau : \varprojlim(X, f^2) \to \varprojlim(X, f^2)$ is conjugate to $\tau^2 : \varprojlim(X, f) \to \varprojlim(X, f)$, τ^2 is expansive. Hence the remaining proof is the same as the proof for the case (1). Consequently, there is no expansive map on the closed interval I. □

REFERENCES

[1] N Aoki , *Topological Dynamics, Topics in General Topology(K. Morita and J. Nagata, eds.)*, North-Holland , Amsterdam , 1989 , pp. 625-740 .

[2] N. Aoki and K. Hiraide , *Topological Theory of Dynamical Systems*, North-Holland , Amsterdam , 1994 .

[3] E. Coven and W. Reddy , *Positively expansive maps of compact manifolds, Lecture Notes in Math. 819*, Springer-Verlag , 1980, pp. 96-110 .

[4] K. Hirade , *Expansive homeomorphisms of compact surfaces are pseudo Anosov*, Osaka J. Math. **27** (1990), 117–162 .

[5] K. Hirade , *Positively expansive open maps of Peano spaces*, Topology and its Appl. **37** (1990), 213–220 .

[6] K. Hirade , *Nonexistence of positively expansive maps on compact connected manifolds with boundary*, Proc. Amer. Math. Soc. **110** (1990), 565-568 .
[7] J. L. Kelly , *General Topology*, Van Nostrand, Princeton, N. J, , 1987 .
[8] L. Lewowicz , *Expansive homeomorphisms of surfaces*, Bull. Brazil Math. Soc. **20** (1989), 113–133 .
[9] W. Reddy , *Expanding maps on compact metric spaces*, Topology and its Appl. **13** (1982), 327–334 .
[10] K. Sakai , *Shadowing properties of $\mathfrak{L}$-hyperbolic homeomorphisms* , Topology and its Appl. **112** (2001), 229–243 .

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