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VARIATION OF ORBIT–COINCIDENCE SETS

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ABSTRACT. David Gauld [3] proved that in many familiar cases the upper semi-finite topology on the set of closed subsets of a space is the largest topology making the coincidence function continuous, when the collection of functions is given the graph topology. Considering G-spaces and taking the coincidence set to consist of points where orbits coincidence, we obtain G-version of many of his results.

1. Introduction

An *action* of a topological group G on a topological space Y is a continuous map θ from $G \times Y$ to Y satisfying $\theta(e, y) = y$ and $\theta(g_1, \theta(g_2, y)) = \theta(g_1 g_2, y)$, where $g_1, g_2 \in G$ and e is the identity of G : a topological space together with a given action is called a G-*space*. Denote $\theta(g, y)$ by $g \cdot y$. For an element y of a G-space Y, the set $\{g \cdot y \mid g \in G\}$ denoted by $G(y)$ is called the *orbit* of y. The collection Y/G of orbits together with the topology coinduced by the map $\pi : Y \to Y/G$ taking y to $G(y)$ is called the *orbit space* of Y. The map π is called the *orbit map*. It is an open map and becomes a closed map as well when G is compact. If Y is Hausdorff and G is compact, then Y/G is Hausdorff. Each $G \in G$ determines a homeomorphism $Tg : Y \to Y$ defined by $Tg(y) = g \cdot y, y \in Y$. The action θ of G on Y is called *proper* if the map from $G \times Y$ to $Y \times Y$ defined by sending (g, y) to $(Tg(y), y)$, $g \in G, y \in Y$ is proper i.e., closed with compact fibres. If G acts on Y properly, then also G/Y is Hausdorff [See 5]. If

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X and Y are G-spaces, then the action on the product space $X \times Y$ is taken to be the diagonal action.

The collection $\mathcal{F}(X,Y)$ of continuous maps from a topological space X to a topological space Y is equipped with the *graph topology* : the family $\{\langle W\rangle \mid W$ is an open set of $X \times Y\}$, where $\langle W\rangle = \{f \in \mathcal{F}(X,Y) \mid$ graph $\Gamma(f)$ of f is contained in $W\}$ forms a basis for this topology [See 2]. The collection of closed sets of X is denoted by ζX and is given the *upper semi-finite topology* : the family $\{[V] \mid V$ is an open set of $X\}$, where $[V] = \{F \in \zeta X \mid F \subset V\}$ forms a basis for this topology [See 4].

Let $h : X \to Y$ be a continuous map from a topological X to a Hausdorff space Y. Then for a continuous map $f : X \to Y$, the *coincidence set* $\Psi_h(f)$ consists of those points of X at which f and h agree. Because $\Psi_h(f) \in \zeta X$ we can define a map $\Psi_h : \mathcal{F}(X,Y) \to \zeta X$. In [3], Gauld stuied the variation of $\Psi_h(f)$ with f. Also, the continuity of the coincidence function $\Psi : \mathcal{F}(X,Y) \times \mathcal{F}(X,Y) \to \zeta X$ sending (f,h) to their coincidence set, is considered. For G-spaces X, Y with Y Hausdorff, we let the orbit coincidence set $K_{f,h}$ be the set of points of X where the orbits of $f(x)$ and $h(x)$ coincide, the orbit coincidence function K is defined in a similar way. In Section 2 of this paper, taking G to be compact Hausdorff we prove the continuity of K_h, the restriction of K to $\mathcal{F}(X,Y) \times \{h\}$ identified with $\mathcal{F}(X,Y)$. It is also noted that if the action of G on Y is proper, then the continuity of K_h continues to be true for any group G. Also the continuity of the orbit-coincidence function K is considered. Finally, we find G-version of some results of Gauld [3] in Section 3.

For terms and definitions not explained here, we refer to [1,3,5].

Unless stated otherwise, X and Y will denote G-spaces with Y Hausdorff and G will be a compact Hausdorff group.

2. Variation of orbit-coincidence sets

2.1 DEFINITION. For $f, h \in \mathcal{F}(X,Y)$, the set $K_{f,h}$ consisting of points of X where $\pi \circ f$ and $\pi \circ h$ agree is called the *orbit-coincidence set* of f and h.

Note that $K_{f,h} = \{x \in X \mid G(f(x)) = G(h(x))\}$ is a closed set of X.

The coincidence set and the orbit coincidence set of two maps may differ : consider the action by the usual addition of the discrete group of integers Z on the real line R (hereafter termed as a Z-space R) and the maps $f_n : R \to R$ defined by $f_n(x) = nx$ where $n = 1, 2$. Then $\Psi(f_1, f_2) = \{0\}$ and $K(f_1, f_2) = Z$.

2.2 Definition. The map $K : \mathcal{F}(X,Y) \times \mathcal{F}(X,Y) \to \zeta X$ defined by $K(f,h) = K_{f,h}$ is called the *orbit-coincidence function.*

If h is fixed, then the restriction of the map K to $\mathcal{F}(X,Y) \times \{h\} \equiv \mathcal{F}(X,Y)$ is denoted by K_h. For the trivial group G, K and K_h are easily seen to be the maps Ψ and Ψ_h respectively as described in [3].

We state the following lemma without proof.

2.3 Lemma. *Let $h \in \mathcal{F}(X,Y)$ and Z be a topological space. Then the map $\hat{h} : \mathcal{F}(Z,X) \to \mathcal{F}(Z,Y)$ defined by $\hat{h}(f) = h \circ f$, $f \in \mathcal{F}(Z,X)$ is continuous.*

2.4 Remark. Let $\pi : Y \to Y/G$ be the orbit map. Then $\hat{\pi} : \mathcal{F}(X,Y) \to \mathcal{F}(X,Y/G)$ is a continuous map.

2.5 Proposition. Let $h \in \mathcal{F}(X,Y)$. Then $K_h : \mathcal{F}(X,Y) \to \zeta X$ is continuous.

Proof. The proof follows by noting that $K_h = \Psi_{\pi \circ h} \circ \hat{\pi}$. □

2.6 Remark. Since the compactness of G is required for the orbit space Y/G to be Hausdorff, Proposition 2.5 remains true for any G on Y is proper.

2.7 PROPOSITION. *If $X \times Y$ is normal, then the orbit-coincidence function K is continuous.*

Proof. Choose a basic open set $[V]$ of ζX containing $K(f,h)$. Denote by i, the inclusion of $X - V$ into X and by I_X the identity map on X. Since $I_X \times \pi$ is a perfect map, $X \times Y/G$ is normal and hence disjoint closed sets $\Gamma(\pi \circ f \circ i)$ and $\Gamma(\pi \circ h \circ i)$ can be separated by open sets U_1' and U_2'. Setting $U_i = U_i' \cup V \times Y/G$ and $V_i = (I_X \times \pi)^{-1}(U_i), i = 1,2$, we show that $(f,h) \in \langle V_1 \rangle \times \langle V_2 \rangle \subset K^{-1}[V]$. That $(f,h) \in \langle V_1 \rangle \times \langle V_2 \rangle$ is simple. Let $(k_1,k_2) \in \langle V_1 \rangle \times \langle V_2$. If $x \in X - V$, then $(x, \pi \circ k_i(x)) \in U_i'$. Since $U_1' \cap U_2' = \emptyset$, we have $\pi \circ k_1(x) \neq \pi \circ k_2(x)$. Thus $x \notin K(k_1,k_2)$. It follows that $K(k_1,k_2) \subset U$. Therefore $(k_1,k_2) \in K^{-1}[V]$. □

2.8 PROPOSITION. *If Y is separable metric, then K is continuous.*

Proof. Let V be an open set of X and $(f,h) \in K^{-1}[V]$. Note that Y/G is a metric space. Define $p : X \times Y/G \to R$ by $d(x, G(y)) = d(\pi \circ f(x), G(y)) - d(\pi \circ h(x)), G(y))$, where d is the metric on Y/G induced by that of Y. Then p is continuous. Let $U_1 = p^{-1}((-\infty, 0)) \cup V \times Y/G, U_2 = p^{-1}((0,\infty)) \cup V \times Y/G$ and let $V_i = (I_X \times \pi)^{-1}(U_i)$, $i = 1,2$. We show that $(f,h) \in \langle V_1 \rangle \times \langle V_2 \rangle \subset K^{-1}[V]$. Let $x \in X - V$. Since $x \notin K(f,h)$, we have $\pi \circ f(x) \neq \pi \circ h(x)$. Thus $p(x, \pi \circ f(x)) = d(\pi \circ f(x), \pi \circ h(x)) - d(\pi \circ h(x), \pi \circ f(x)) = -d(\pi \circ h(x), \pi \circ f(x)) < 0$ and $p(x, \pi \circ h(x)) = d(\pi \circ f(x), \pi \circ h(x)) - d(\pi \circ h(x), \pi \circ h(x)) = d(\pi \circ f(x), \pi \circ h(x)) > 0$. If $x \in V$, then we have $(x, \pi \circ f(x)), (x, \pi \circ h(x)) \in V \times Y/G$. It follows that $(I_X \times \pi)(\Gamma(f)) \subset U_1$ and $(I_X \times \pi)(\Gamma(h)) \subset U_2$. Thus we have $(f,h) \in \langle V_1 \rangle \times \langle V_2 \rangle$. By similar way of proof of Proposition 2.7, we can show that $\langle V_1 \rangle \times \langle V_2 \rangle \subset K^{-1}[V]$. Thus $K^{-1}[V]$ is open. Therefore K is continuous. □

3. Upper semi-finite topology and continuity of $K(K_h)$

3.1 DEFINITION. Let I be the closed unit interval $[0,1]$ of the real line R with the trivial action of G. Then a homotopy $H : X \times I \to Y$

is called *G-active* if $H(x,t) \notin G(H(x,0))$, for any $t \in (0,1]$.

3.2 EXAMPLES.

(a) Let f and h be constant maps from R to$R$ defined by $f(t) = 1$ and $h(t) = 2, t \in R$ and let R be acted upon by the discrete group of integers with the usual addition. Then the straight line homotopy between f and h is active but not G-active.

(b) Let G be the subgroup $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in R, a > 0 \right\}$ of $Gl(2, R)$ acting on R by $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \cdot x = ax + b$, $x \in R$. Then the straight line homotopy between the identity map on R and any nonzero translation of R is an example of an active deformation which is not G-active.

Let $H : X \times I \to Y$ be a G-active homotopy. We call the map $\tau_H : \mathcal{F}(X, I) \to \mathcal{F}(X, Y)$ defined by $\tau_H(\alpha)(x) = H(x, \alpha(x))$, where $\alpha \in \mathcal{F}(X, I)$ and $x \in X$ citeSee 3. Denote by H_0, the restriction of H to the base $X \times \{0\}$. Identify $X \times \{0\}$ with X and note that $K_{H_0} \circ \tau_H = \sigma$, where $\sigma : \mathcal{F}(X, I) \to \zeta X$maps α to its zero-set. Since for a perfectly normal space X the upper semi-finite topology is the largest topology on ζX making σ continuous [See 3; Proposition [1.4]], we conclude the following:

Let X be perfectly normal and $H : X \times I \to Y$ be a G-active homotopy. Then the upper semi-finite topology is the largest topology on ζX making K_{H_0} continuous.

Call a path f in X *orbit non-overlapping* if $f(t) \notin G(f(0))$, for any $t \in (0,1]$, and say that X has the property W (G-strong) if there exists a G-active homotopy $H : X \times I \to X$ such that $H_0 = I_X$.

The Z-space R is W (Z-strong). Also the Euclidean spaceR^2 with the action θ of R defined by $\theta(x, (y, z)), x, y, z \in R$ is W (R-strong): the straight line homotopy between the identity map on R^2 and a translation of R^2 by (a, b), $b \neq 0$ is an R-active deformation of R^2

We easily obtain G-versions of Corollaries 2.6 and 2.7 of [3] as follows:

Let X be perfectly normal. Then the upper semi-finite topology on ζX is the largest topology making K_h continuous provided either of the following holds:

(a) Y has the property W(G-strong).

(b) $h : X \to Y$ is a constant map and Y has an orbit non-overlapping path beginning from the image point of h.

Also, if X is perfectly normal and Y contains an orbit non-overlap ping path, then the upper semi-finite topology on ζX is the largest topology making K continuous.

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