미분구적법을 이용한 곡선보의 내평면 진동분석

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In-Plane Extensional Vibration Analysis of Curved Beams using DQM

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Abstract : DQM(differential quadrature method) is applied to computation of eigenvalues of the equations of motion governing the free in-plane vibration for circular curved beams including mid-surface extension and the effects of rotatory inertia. Fundamental frequencies are calculated for the members with various end conditions and opening angles. The results are compared with numerical solutions by other methods for cases in which they are available. The differential quadrature method gives good accuracy even when only a limited number of grid points is used.

초 록 : 아크축(arch axis)의 연장(extensibility) 및 회전관성(rotatory inertia)을 고려한 곡선보(curved beam)의 평면내(in-plane) 자유진동을 미분구적법(DQM)을 이용하여 다양한 경계조건(boundary conditions)과 굽힘각(opening angles)에 따른 진동수(frequencies)를 계산하였다. DQM의 결과를 다른 수치해석결과와 비교하였으며, DQM은 적은 요소(grid points)를 사용하여 정확한 해석결과를 보여주었다.
Key Words : differential quadrature method(DQM), mid-surface extension, rotatory inertia, vibration, numerical solution

1. Introduction

The increasing use of curved beams in buildings, vehicles, ships, and aircraft has results in considerable effort being directed toward developing an accurate method for analyzing the dynamic behavior of such structures. Accurate knowledge of the vibration response of curved beams is of great importance in many engineering applications such as the design of machines and structures.

The early investigators into the in-plane vibration of rings were Hoppe[1] and Love[2]. Love[2] improved on Hoppe's theory by allowing for stretching of the ring. Lam[3] investigated the statics of incomplete ring with various boundary conditions and the dynamics of an incomplete free-free ring of small curvature. Den Hartog[4] used the Rayleigh-Ritz method for finding the lowest natural frequency of circular arcs with simply supported or clamped ends and his work was extended by Volterra and Morell[5] for the vibrations of arches having center lines in the form of cycloids, catenaries or parabolas. Archer[6] carried out for a mathematical study of the in-plane inextensional vibrations of an incomplete circular ring of small cross section with the basic equations of motion as given in Love[2] and gave a prescribed time - dependent displacement at the other end for the case of clamped ends. Nelson[7] applied the Rayleigh-Ritz method in conjunction with Lagrangian multipliers to the case of a circular ring segment having simply supported ends. Auciello and De Rosa[8] reviewed the free vibrations of circular arches and briefly illustrated a number of other approaches. Ojalvo[9] obtained the equations governing three-dimensional linear motions of elastic rings and results for generalized

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loadings and viscous damping making use of usual classical beam-theory assumptions for the clamped ends. Rodgers and Warner[10] calculated the frequencies of curved elastic rods with simply supported ends.

A rather efficient alternate procedure for the solution of partial differential equations is the method of differential quadrature which was introduced by Bellman and Casti[11]. This simple direct technique can be applied to a large number of cases to circumvent the difficulties of programming complex algorithms for the computer, as well as excessive use of storage. This method is used in the present work to analyze the free in-plane extensional vibrations of curved beams with various boundary conditions and opening angles. The frequencies are calculated for the member. The results are compared with other numerical solutions by a combination of a Holzer-type iterative procedure and an initial value integration procedure.

2. System and Governing Equations

The uniform curved beam considered is shown in Fig. 1. A point on the centroidal axis is defined by the angle θ, measured from the left support. The tangential and radial displacements of the arch axis are v and w, respectively. a is the radius of the centroidal axis.

Veletsos et al.[12] used a theory which accurately considers the extensibility of the arch axis and the curved beam effect but neglects the effects of rotatory inertia and shearing deformation; they considered simply supported and clamped ends. The differential equations for free vibration of the system which consider the extensibility of the arch axis but neglect the effects of rotatory inertia and shearing deformation, obtained by specializing Flugge's equations for cylindrical shells[13] and incorporating the radial and tangential inertia effects, are

$$w'''' + 2\theta_0^2 w'' + [\theta_0^4 + \theta_0^2(\frac{S}{r})^2 - \lambda^2]w + \theta_0(\frac{S}{r})^2 v' = 0 \quad (1)$$

$$v'' + \lambda^2(\frac{r}{S})^2 v + \theta_0 w' = 0 \quad (2)$$

where $S\ (= a\theta_0)$ is the length of the arch axis, θ_0 is the opening angle, r is the radius of gyration of the cross section $(=\sqrt{\frac{\text{moment of inertia}}{\text{mass}}})$, and λ is a dimensionless parameter related to the circular frequency of vibration of the system, ω, by

$$w = \frac{\lambda}{S^2}\sqrt{\frac{EI}{m}} \quad (3)$$

Here, m is the mass per unit length, E is the Young's modulus of elasticity for the material, and I is the area moment of inertia of the cross section.

Austin and Veletsos[14] studied the free vibrational characteristics of circular arches vibrating in their own planes and presented a simple approximate procedure for estimating the natural frequencies of the systems based on a theory including the effects of rotatory inertia.

The differential equations including the effects of rotatory inertia, but neglecting shearing deformation, obtained from Federhofer's system[15] are

$$w'''' + [2\theta_0^2 + \lambda^2(\frac{r}{S})^2]w'' + [\theta_0^4 + \theta_0^2(\frac{S}{r})^2 - \lambda^2]w + [(\frac{S}{r})^2 - \lambda^2(\frac{r}{S})^2]\theta_0 v' \quad (4)$$

$$v'' + [\lambda^2(\frac{r}{S})^2 + \theta_0^2\lambda^2(\frac{r}{S})^4]v + \theta_0[1 - \lambda^2(\frac{r}{S})^4]w' = 0 \quad (5)$$

The boundary conditions for simply supported and clamped ends are, respectively,

$$v = w = w'' = 0 \quad (6)$$

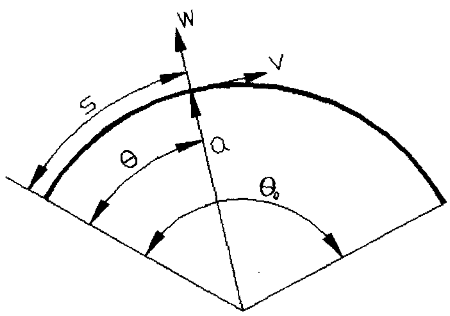

Fig. 1. Curved beam considered

$$v = w = w' = 0 \qquad (7)$$

3. Differential Quadrature Method

The Differential Quadrature Method(DQM) was introduced by Bellman and Casti.[11] By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang et al.[16]. The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident in recent publications. Kang and Han[17] and Kang[18] applied the method to the analysis of a curved beam using classical and shear deformable beam theories, and vibration analysis of curved beams. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij}\, f(x_j) \quad \text{for } i,\ j = 1, 2, \ldots, N \qquad (8)$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(x) = x^{k-1} \quad \text{for } k = 1, 2, 3, \ldots, N \qquad (9)$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1} = (k-1)(k-2)\cdots(k-n)x_i^{k-n-1}$$
$$\text{for } i,\ k = 1, 2, \ldots, N \qquad (10)$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix which always has an inverse, as described by Hamming[19].

4. Application

Applying the differential quadrature method to equations (1) and (2) gives

$$\sum_{j=1}^{N} D_{ij}\, w_j + 2\theta^2{}_0 \sum_{j=1}^{N} B_{ij}\, w_j + [\ \theta^4{}_0 + \theta^2{}_0 (\frac{S}{r})^2 - \lambda^2] w_i$$
$$+ \theta_0 (\frac{S}{r})^2 \sum_{j=1}^{N} A_{ij}\, v_j = 0 \qquad (11)$$

$$\sum_{j=1}^{N} B_{ij}\, v_j + \lambda^2 (\frac{r}{S})^2 v_i + \theta_0 \sum_{j=1}^{N} A_{ij}\, w_j = 0 \qquad (12)$$

where, A_{ij}, B_{ij}, and D_{ij} are the weighting coefficients for the first-, second-, and fourth-order derivatives, respectively, along the dimensionless axis.

Similarly, applying the differential quadrature method to equations (4) and (5) gives

$$\sum_{j=1}^{N} D_{ij}\, w_j + [\ 2\theta^2{}_0 + \lambda^2 (\frac{r}{S})^2] \sum_{j=1}^{N} B_{ij} w_j$$
$$+ [\ \theta^4{}_0 + \theta^2{}_0 (\frac{S}{r})^2 - \lambda^2] w_i \qquad (13)$$
$$+ [(\frac{S}{r})^2 - \lambda^2 (\frac{r}{S})^2] \sum_{j=1}^{N} A_{ij}\ \ v_j = 0$$

$$\sum_{j=1}^{N} B_{ij}\, v_j + [\lambda^2 (\frac{r}{S})^2 + \theta^2{}_0\ \lambda^2{}_0 (\frac{r}{S})^4] v_i$$
$$+ \theta_0 [1 - \ lamba^2{}_0 (\frac{r}{S})^4] \sum_{j=1}^{N} A_{ij}\, w_j = 0 \qquad (14)$$

The boundary conditions for simply supported ends, given by equations (6), can be expressed in differential quadrature form as follows:

$$v_1 = 0 \text{ at } X = 0$$
$$w_1 = 0 \text{ at } X = 0$$
$$\sum_{j=1}^{N} B_{2j}\, w_j = 0 \text{ at } X = 0 + \delta$$
$$\sum_{j=1}^{N} B_{(N-1)j}\, w_j = 0 \text{ at } X = 1 - \delta$$
$$w_N = 0 \text{ at } X = 1 \qquad (15)$$

Similarly, the boundary conditions for clamped ends, given by equations (7), can be expressed in differential quadrature form as follows:

$$\begin{aligned} v_1 &= 0 \text{ at } X = 0 \\ v_N &= 0 \text{ at } X = 1 \\ w_1 &= 0 \text{ at } X = 0 \\ \sum_{j=1}^{N} A_{2j}\, w_j &= 0 \text{ at } X = 0+\delta \\ \sum_{j=1}^{N} A_{(N-1)j}\, w_j &= 0 \text{ at } X = 1-\delta \\ w_N &= 0 \text{ at } X = 1 \end{aligned} \tag{16}$$

Here, δ denotes a very small distance measured along the dimensionless axis from the boundary ends. This set of equations together with the appropriate boundary conditions can be solved for the in-plane extensional free vibrations.

5. Numerical Results and Comparisons

The natural frequencies of vibration are calculated by the differential quadrature method and presented together with results from another method: using a combination of a Holzer-type iterative procedure and an initial value integration procedure by Veletsos et al.[12] and Austin and Veletsos[14].

Tables 1 and 2 present the results of convergence studies relative to the number of grid points N and δ parameter, respectively comparing the exact solution of inextensional vibration of curved beams with DQM. Table 1 shows that the accuracy of the numerical solution increases with increasing N and passes through a maximum. Then, numerical instabilities arise if N becomes too large. The optimal value for N is found to be 11 to 13.

Table 1. Fundamental frequency parameters $\Omega = \omega(ma^4/EI)^{1/2}$, for in-plane inextensional vibration of thin curved beams with clamped ends including a range of grid point, θ_0=180°

Archer[6] (Exact)	Kang[18] Number of grid points (DQM)			
$\Omega = \omega(ma^4/EI)^{1/2}$	7	9	11	13
4.3841	5.0586	4.1740	4.3975	4.3844

Table 2. Fundamental frequency parameters, $\Omega = \omega(ma^4/EI)^{1/2}$, for in-plane inextensional vibration of thin curved beams with clamped ends including a range of δ, θ_0=180°

Archer[6] (Exact)	Kang[18] δ(DQM)				
$\Omega = \omega(ma^4/EI)^{1/2}$	1×10^{-2}	1×10^{-3}	1×10^{-4}	1×10^{-5}	1×10^{-6}
4.3841	4.8845	4.4301	4.3885	4.3844	4.3840

Table 2 shows the sensitivity of the numerical solution to the choice of δ. The optimal value for δ is found to be 1×10^{-5} to 1×10^{-6}, which is obtained from trial-and-error calculations. The solution accuracy decreases due to numerical instabilities if δ becomes too small (see Kang[18]).

The values λ corresponding to the fundamental frequencies (n=1) including higher frequencies (n=2, 3, 4) neglecting the effects of rotatory inertia have been evaluated for hinged and fixed arches having angles of opening 45°, 90°, and 180° for a wide range of the slenderness ratio (= S/r) which is the total length of the member over the length of the cross section. Representative data are tabulated in Tables 3 and 4. Tables 3 and 4 show that the frequency increases as the slenderness ratio increases, and the opening angle decreases. The value at 45° opening angle with 50 slenderness ratio is less than that of 90° in Table 4 because the values of 45° and 90° are symmetric mode and antisymmetric mode, respectively. When the slenderness ratio is great-

Table 3. Fundamental frequency parameter, $\lambda = wS^2(m/EI)^{1/2}$, for simply supported thin arches including mid-surface extension

S/r	θ_0	Veletsos et al.[12]	DQM			
		n=1	n=1	n=2	n=3	n=4
11.78	90°	18.08	18.08	71.53	89.78	148.6
17.28		25.25	25.25	83.19	113.8	215.1
23.56		33.32	33.32	81.49	153.9	226.0
47.12		33.82	33.82	144.9	171.5	351.4
117.8		33.94	33.94	151.9	345.6	414.4
251.3		33.96	33.96	152.2	349.5	652.7
377.0		33.96	33.96	152.3	349.8	627.0
7.85	180°	18.26	18.26	39.18	74.73	-
15.71		21.37	21.36	66.53	133.2	167.5
47.12		22.27	22.27	133.6	205.8	339.9

Table 4. Fundamental frequency parameter, $\lambda = wS^2(m/EI)^{1/2}$, for clamped thin arches including mid-surface extension

S/r	θ_0	Veletsos et al.[12] n=1	DQM n=1	n=2	n=3	n=4
25	45°	27.33	27.33	119.9	158.6	297.0
50		39.03	39.04	120.9	296.0	318.4
100		60.08	60.10	197.6	321.4	417.8
12.5	90°	26.35	26.35	78.78	119.1	158.7
50		55.37	55.36	162.0	204.4	412.7
100		55.73	55.72	191.6	338.2	420.5
150		55.78	55.79	192.6	404.5	520.9
200		55.81	55.80	192.8	408.1	665.9
250		55.81	55.82	192.9	409.1	699.9
300		55.82	55.82	193.0	409.4	703.5
500		55.84	55.83	193.1	409.8	705.7

er than 200, the difference of fundamental frequency values is less than 0.1 percent.

The values of λ corresponding to the fundamental frequencies including higher frequencies and the effects of rotatory inertia but neglecting shearing deformation have been calculated for hinged and fixed arches θ_0=90° for a wide range of the slenderness ratio. The results are summarized in Tables 5 and 6. Tables 5 and 6 show that the values of simply supported ends are slightly higher than those of clamped ends. From Tables 4 and 6 the values of mid-surface extension vibration are slightly higher than those of mid-surface extension vibration including the effects of rotatory inertia. All results are computed with thirteen points, $\delta = 1 \times 10^{-5}$(see Kang[18]), and show the excellent agreements each other.

Table 5. Fundamental frequency parameter, $\lambda = wS^2(m/EI)^{1/2}$, for simply supported thin arches including mid-surface extension and effects of rotatory inertia

S/r	θ_0	Austin and Veletsos[14] n=1	DQM n=1	n=2	n=3	n=4
23.56	90°	32.55	32.54	80.51	137.2	223.7
47.12		33.60	33.60	141.8	169.1	327.2
70.69		33.80	33.89	149.0	244.8	341.7
94.25		33.87	33.87	150.5	317.5	354.6
141.4		33.92	33.93	151.5	345.3	491.3
188.5		33.94	33.94	151.9	347.5	611.4
251.3		33.95	33.96	152.1	348.7	622.7
314.2		33.95	33.96	152.1	349.1	624.7
377.0		33.96	33.96	152.2	349.4	625.8

Table 6. Fundamental frequency parameter, $\lambda = wS^2(m/EI)^{1/2}$, for clamped thin arches including mid-surface extension and effects of rotatory inertia

S/r	θ_0	Austin and Veletsos[14] n=1	DQM n=1	n=2	n=3	n=4
25	90°	37.81	37.82	109.9	160.2	248.9
50		54.98	54.97	160.9	198.9	385.9
100		55.63	55.62	190.0	337.2	413.4
150		55.74	55.73	191.9	401.7	520.3
200		55.79	55.78	192.4	406.4	663.7
250		55.80	55.78	192.7	407.8	696.7
300		55.81	55.80	192.8	408.7	701.2
350		55.83	55.81	192.9	409.1	702.9
400		55.83	55.81	193.0	409.4	704.2
500		55.84	55.81	193.1	409.7	704.9

6. Conclusions

The differential quadrature method was used to compute the eigenvalues of the equations of motion governing the free in-plane extensional vibrations and of curved beams. The results agree very well with the numerical solutions by other methods for the cases treated with only a limited number of grid points.

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