미분구적법(DQM)을 이용한 곡선보의 외평면 좌굴해석

강 기 준
호서대학교 기계설계공학과
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Out-of-Plane Buckling Analysis of Curved Beams Using DQM

Kijun Kang
Department of Mechanical Design Engineering, Hoseo University
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Abstract : The differential quadrature method (DQM) is applied to computation of the eigenvalues of out-of-plane buckling of curved beams. Critical moments including the effect of radial stresses are calculated for a single-span wide-flange beam subjected to equal and opposite in-plane bending moments with various end conditions, and opening angles. Results are compared with existing exact solutions where available. The differential quadrature method gives good accuracy even when only a limited number of grid points is used. New results are given for two sets of boundary conditions not previously considered for this problem: clamped-clamped and clamped-simply supported ends.

초 록 : I-단면 곡선보 (curved beam)의 모멘트 하중 하에서 비틀림 (warping)을 포함한 평면외 (out-of-plane)의 좌굴을 미분구적법 (DQM)을 이용하여 해석하였다. 다양한 경계조건 (boundary conditions) 및 굽힘각 (opening angles)에 따른 임계모멘트 (critical moments)를 계산하였고, DQM의 해석결과는 해석적 해답 (exact solution) 과 비교 분석하였다. DQM은 적은 요소 (grid points)를 사용하여 정확한 해석결과를 보여주었고, 두 경계조건 (고정-고정, 고정 -단순지지)하에서 새로운 결과 또한 제시하였다.
Key Words : curved beams, differential quadrature method, exact solution, critical moments, buckling, warping, in-plane bending moments

1. Introduction

The common engineering theory of flexure is based on the Bernoulli-Euler-Navier assumption that cross sections, which are perpendicular to the centroid before bending, remain plane and perpendicular to the deformed locus. In contrast, torsion was considered to be completely defined by the theory of Saint-Venant. A crucial point in the Saint-Venant theory is that warping deformations can occur freely and uniformly throughout the beam (Yang and Kuo[1]. Ojalvo et al[2]. studied the elastic stability of ring segments with a thrust or a pull directed along the chord neglecting the warping effect. Timoshenko and Gere[3] first took into account the effect of warping for a bisymmetrical I-beam. Vlasov[4] derived closed-form solutions such as for a beam, in which cross-sections are allowed to warp non-uniformly along the beam axis, subject to in-plane bending moments. Cheney[5] studied the buckling of thin-walled open-section rings including both the effect of axial stress and the effect of warping. Papangelis and Trahair[6] conducted a theoretical study of the flexural-torsional buckling of doubly symmetric arches to confirm the predictions of Timoshenko and Gere[3] for beams in uniform compression and of Vlasov[4] for beams in uniform bending. Trahair and Papangelis[7] also developed an out-of-plane buckling theory for beams of monosymmetric cross-section using the second variation of the total potential. Yand and Kuo[1] studied the static stability of curved

kjkang@office.hoseo.ac.kr

thin-walled beams using the principle of virtual displacements in a Lagrangian formulation with emphasis placed on the effect of curvature, and they presented closed-form solutions for arches in uniform bending and uniform compression. Recently, Han and Kang[8] studied the out-of-plane buckling of curved beams without warping using the differential quadrature method (DQM).

In the present work, the differential quadrature method, introduced by Bellman and Casti[9], is used to analyze the out-of-plane stability of curved beams, specifically, of a single-span, wide-flame beam including a warping contribution. Critical moments are calculated for the member subjected to equal and opposite end moments. The differential equations used to model the static elastic behavior of the curved beam, derived by Vlasov[4], are based on the assumption that the cross-sectional shape is assumed to be constant along the entire center and doubly symmetric, and the shear center and centroid coincide. The member has both ends either simply supported or clamped, or has clamped-simply supported ends. Numerical result are compared with existing exact solutions and numerical solutions by the finite element method(FEM) where available.

2. Governing Differential Equations

The x and y axes shown in Fig. 1 are the principal centroidal axes of the beam cross section; the x-axis is in the horizontal plane of curvature, and the z-axis coincides with the centroid. The horizontal radius of curvature R is constant.

The differential equations governing a curved beam subjected to in-plane constant bending moment M_y can be written as (Yang and Kuo[1])

$$EI_x\left(\frac{\partial^4 v}{\partial z^4}-\frac{\partial^2\phi}{R\partial z^2}\right)-\frac{GK_T}{R}\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2 v}{R\partial z^2}\right)+M_y\left(1-\frac{r^2}{R^2}\right)\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2 v}{R\partial z^2}\right)=0 \qquad (1)$$

$$EC_w\left(\frac{\partial^4\phi}{\partial z^4}+2\frac{\partial^2\phi}{R^2\partial z^2}+\frac{\phi}{R^4}\right)-\frac{EI_x}{R}\left(\frac{\partial^2 v}{\partial z^2}-\frac{\phi}{R}\right)-GK_T\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2 v}{R\partial z^2}\right)+M_y\frac{\partial^2 v}{\partial z^2}-M_y\frac{r^2}{R}\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2 v}{R\partial z^2}\right)=0 \qquad (2)$$

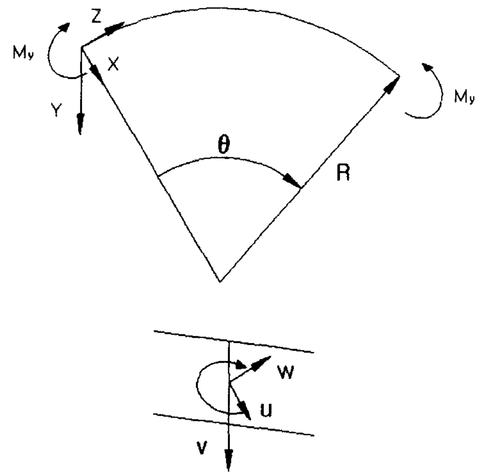

Fig. 1. Coordinate system and cross-section of curved beam

where E is the modulus of elasticity, G is the shear modulus, I_x is the moment of inertia about the x-axis (see Fig. 1), C_w is the warping constant, K_T is the Saint-Venant torsion constant, M_y is the applied constant in-plane bending moment, r is the polar radius of gyration defined as $[(I_y+I_x)/A]^{1/2}$, A is the cross-sectional area, v is the displacement of the shear center in the y-direction, and ϕ is the angle of twist of the curved beam cross-section.

Replacing z by $R\theta$, one can rewrite Eqs. (1) and (2) as

$$EI_x\left(\frac{v^{IV}}{\theta_0^4R^4}-\frac{\phi''}{\theta_0^2R^3}\right)-\frac{GK_T}{R}\left(\frac{\phi''}{\theta_0^2R^2}+\frac{v''}{\theta_0^2R^3}\right)+M_y\left(1-\frac{r^2}{R^2}\right)\left(\frac{\phi''}{\theta_0^2R^2}+\frac{v''}{\theta_0^2R^3}\right)=0 \qquad (3)$$

$$EC_w\left(\frac{\phi^{IV}}{\theta_0^4R^4}+2\frac{\phi''}{\theta_0^2R^4}+\frac{\phi}{R^4}\right)-\frac{EI_x}{R}\left(\frac{v''}{\theta_0^2R^2}-\frac{\phi}{R}\right)-GK_T\left(\frac{\phi''}{\theta_0^2R^2}+\frac{v''}{\theta_0^2R^3}\right)+M_y\frac{v''}{\theta_0^2R^2}-M_y\frac{r^2}{R}\left(\frac{\phi''}{\theta_0^2R^2}+\frac{v''}{\theta_0^2R^3}\right)=0 \qquad (4)$$

where each ($' = d/dX$) prime denotes one differentiation with respect to the dimensionless distance coordinate $X = \theta / \theta_0$, in which θ_0 is the opening angle of the member, and θ is the angle from left support to generic point.

The following boundary conditions are taken for simply supported ends (Tan and Shore[10]): (a) no out-of-plane deflection; (b) no torsional rotation; (c) no bending moment; and (d) no bimoment. The bending moment and the bimoment of the member can written as

$$M_x = EI_x\left(\frac{\phi}{R} - \frac{d^2v}{dz^2}\right),$$
$$B_w = -EC_w\left(\frac{d^2\phi}{dz^2} + \frac{1}{R}\frac{d^2v}{dz^2}\right) \quad (5)$$

For clamped ends, v, ϕ, dv/dz, and τ equal zero where τ represents the warping as defined by Vlasov[4]. It can be written as (Chaudhuri and Shore[11])

$$\tau(z) = -\left(\frac{1}{R}\frac{dv}{dz} + \frac{d\phi}{dz}\right) \quad (6)$$

The boundary conditions for both ends simply supported, both ends clamped, and for mixed clamped-simply supported ends are, respectively

$$v = \phi = v'' = \phi'' = 0 \ \text{ at } X = 0 \text{ and } 1 \quad (7)$$
$$v = \phi = v' = \phi' = 0, \ \text{ at } X = 0 \text{ and } 1 \quad (8)$$
$$v = \phi = v'' = \phi'' = 0, \text{ at } X = 0$$
$$v = \phi = v' = \phi' = 0 \ \text{ at } X = 1 \quad (9)$$

3. Differential Quadrature Method

The differential quadrature method(DQM) was introduced by Bellman and Casti[9]. By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. However, the method is limited with increasing number of grid points and boundary adjacent δ points. The accuracy of the quadrature solutions is dictated by the choice of sampling points and by the choice of adjacent δ points. It was applied for the first time to static analysis of structural components by Jang et al.[12]. The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the related publications of recent years. Kang and Han[13] applied the method to the analysis of a curved beam using classical and shear deformable beam theories, and Kang[14] studied the vibration analysis of curved beams using DQM. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij}\, f(x_j) \text{ for } i,\ j = 1, 2, \ldots, N \quad (10)$$

where L denotes a differential operator, x_i are the discrete points considered in the domain, i are the row vectors of the N values, $f(x_i)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(x) = x^{k-1} \text{ for } k = 1, 2, 3, \ldots, N \quad (11)$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1} = (k-1)(k-2)\cdots(k-n)x_i^{k-n-1} \text{ for}$$
$$i,\ k = 1, 2, \ldots, N \quad (12)$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix which always has an inverse.

4. Application

Here DQM is applied to the out-of-plane buckling analysis of curved beams. The differential quadrature approximations governing the beams subjected to an applied in-plane constant moment M_y and the boundary conditions are shown below.

Applying the DQM to Eqs. (3) and (4) gives

$$EI_x\left(\frac{1}{\theta_0^4R^4}\sum_{j=1}^{N}D_{ij}v_j-\frac{1}{\theta_0^2R^3}\sum_{j=1}^{N}B_{ij}\phi_j\right)$$
$$-\frac{GK_T}{R}\left(\frac{1}{\theta_0^2R^2}\sum_{j=1}^{N}B_{ij}\phi_j+\frac{1}{\theta_0^2R^3}\sum_{j=1}^{N}B_{ij}v_j\right)$$
$$+M_y\left(1-\frac{r^2}{R^2}\right)\left(\frac{1}{\theta_0^2R^2}\sum_{j=1}^{N}B_{ij}\phi_j+\frac{1}{\theta_0^2R^3}\sum_{j=1}^{N}B_{ij}v_j\right)$$
$$=0 \tag{13}$$

$$EC_w\left(\frac{1}{\theta_0^4R^4}\sum_{j=1}^{N}D_{ij}\phi_j+2\frac{1}{\theta_0^2R^4}\sum_{j=1}^{N}B_{ij}\phi_j+\frac{\phi_i}{R^4}\right)$$
$$-\frac{EI_x}{R}\left(\frac{1}{\theta_0^2R^2}\sum_{j=1}^{N}B_{ij}v_j-\frac{\phi_i}{R}\right)$$
$$-GK_T\left(\frac{1}{\theta_0^2R^2}\sum_{j=1}^{N}B_{ij}\phi_j+\frac{1}{\theta_0^2R^3}\sum_{j=1}^{N}B_{ij}v_j\right)$$
$$+M_y\frac{1}{\theta_0^2R^2}\sum_{j=1}^{N}B_{ij}v_j$$
$$-M_y\frac{r^2}{R}\left(\frac{1}{\theta_0^2R^2}\sum_{j=1}^{N}B_{ij}\phi_j+\frac{1}{\theta_0^2R^3}\sum_{j=1}^{N}B_{ij}v_j\right)=0 \tag{14}$$

where B_{ij} and D_{ij} are the weighting coefficients for the second and fourth-order derivatives, respectively, along the dimensionless axis.

The boundary conditions for both ends simply supported, given by Eq. (7), can be expressed in differential quadrature form as

$$V_1 = 0 \text{ at } X=0 \tag{15}$$
$$\Phi_1 = 0 \text{ at } X=0 \tag{16}$$
$$\sum_{j=1}^{N}B_{2j}V_j = 0 \text{ at } X = 0+\delta \tag{17}$$
$$\sum_{j=1}^{N}B_{2j}\Phi_j = 0 \text{ at } X = 0+\delta \tag{18}$$
$$\sum_{j=1}^{N}B_{(N-1)j}V_j = 0 \text{ at } X = 1-\delta \tag{19}$$
$$\sum_{j=1}^{N}B_{(N-1)j}\Phi_j = 0 \text{ at } X = 1-\delta \tag{20}$$
$$V_N = 0 \text{ at } X = 1 \tag{21}$$
$$\Phi_N = 0 \text{ at } X = 1 \tag{22}$$

where δ denotes a small distance measured along the dimensionless axis from the boundary ends. In their work on the applications of DQM to the static analysis of beams and plates, Jang et al.[12] proposed the so-called δ-technique wherein adjacent to the boundary points of the differential quadrature grid points are chosen at a small distance (in dimensionless value). This δ approach is used to apply more than one boundary conditions for clamped ends, given by Eq. (8), can be expressed in differential quadrature form as

$$V_1 = 0 \text{ at } X = 0 \tag{23}$$
$$\Phi_1 = 0 \text{ at } X = 0 \tag{24}$$
$$\sum_{j=1}^{N}A_{2j}V_j = 0 \text{ at } X=0+\delta \tag{25}$$
$$\sum_{j=1}^{N}A_{2j}\Phi_j = 0 \text{ at } X=0+\delta \tag{26}$$
$$\sum_{j=1}^{N}A_{(N-1)j}V_j = 0 \text{ at } X=1-\delta \tag{27}$$
$$\sum_{j=1}^{N}A_{(N-1)j}\Phi_j = 0 \text{ at } X=1-\delta \tag{28}$$
$$V_N = 0 \text{ at } X = 1 \tag{29}$$
$$\Phi_N = 0 \text{ at } X = 1 \tag{30}$$

where A_{ij} are the weighting coefficients for the first-order derivative.

Similarly, the boundary conditions for one clamped end and one simply supported end, given by Eq. (9), can be expressed in differential quadrature form as

$$V_1 = 0 \text{ at } X = 0 \tag{31}$$
$$\Phi_1 = 0 \text{ at } X = 0 \tag{32}$$
$$\sum_{j=1}^{N}A_{2j}V_j = 0 \text{ at } X=0+\delta \tag{33}$$
$$\sum_{j=1}^{N}A_{2j}\Phi_j = 0 \text{ at } X=0+\delta \tag{34}$$
$$\sum_{j=1}^{N}B_{(N-1)j}V_j = 0 \text{ at } X=1-\delta \tag{35}$$
$$\sum_{j=1}^{N}B_{(N-1)j}\Phi_j = 0 \text{ at } X=1-\delta \tag{36}$$
$$V_N = 0 \text{ at } X = 1 \tag{37}$$

$$\Phi_N = 0 \text{ at } X = 1 \tag{38}$$

Mixed boundaries can be easily accommodated by combining these equations; simply change the weighting coefficients.

5. Numerical results and comparisons

The critical bending moments subjected to an applied in-plane constant bending moment are calculated by the differential quadrature method, and are presented together with existing exact solutions. The critical values are evaluated for the case of a single-span, wide-flange beam with various end conditions and opening angles.

The example considered here has a constant length of 10.24m (403.32 in.) and a variety of opening angles ranging from 10° and 90°. Cross-sectional properties of the beam are : A = 92.9cm^2(14.4 in.^{2}), I_x = 11,360cm^4 (273 in. ^{4}), I_y = 3870cm^4 (93 in. ^{4}), C_w = 555,900 cm^6(2,070 . ^{6}), K_T = 58.9cm^4(1.141 in. ^{4}), and r = 12.81cm(5.042 in.). Values used for the elastic modulus and shear modulus are E = 200. GN/m^2(29,000 *ksi*) and G = 77.2 GN/m^2(11,200 *ksi*).

Table 1 presents the results of convergence studies relative to the number of grid points N and the δ parameter with θ_0 and 30°. The data show that the accuracy of the numerical solution increases with increasing N. Then numerical instabilities arise if N becomes too large(possible greater than approx. 19).

Table 1. Critical moment M_{ycr} of out-of-plane buckling of curved beams with both ends simply supported for a range of grid points N and δ, θ_0 = 30° and 90°

θ_0	Exact(Yang and Kuo1)	N	M_{ycr}(kN-cm) δ: 1×10^{-9}	1×10^{-10}	1×10^{-11}	1×10^{-12}
30°	-4821	7	-4815	-4864	-4822	-4822
		9	-4820	-4821	-4821	-4821
		11	-4821	-4821	-4821	-4821
		13	-4821	-4821	-4821	-4843
90°	-1099	7	-1089	-1096	-1115	-1099
		9	-1098	-1098	-1098	-1117
		11	-1099	-1098	-1098	-1098
		13	-1103	-1097	-1098	-1098

Note : 1 kip-in = 11.3 kN-cm

Table 2. First four critical moments of out-of-plane buckling of curved beams with ends simply supported for a range of grid points N and $\delta = 1\times10^{-11}$; θ_0 = 30° and 90°

θ_0	M_{ycr}	M_{ycr}(kN-cm) Exact(Yang and Kuo[1])	DQM 11	13	15
30°	M_{ycr1}	-4821	-4821	-4821	-4823
	M_{ycr2}	-28246	-28243	-28246	-28246
	M_{ycr3}	-84052	-83858	-84061	-84053
	M_{ycr4}	-176449	-185173	-175522	-176551
90°	M_{ycr1}	-1098	-1098	-1098	-1236
	M_{ycr2}	-9549	-9548	-9549	-1237
	M_{ycr3}	-34942	-34860	-34947	-34950
	M_{ycr4}	-89497	-89281	-88891	-89564

Table 3. Critical moment M_{ycr} of out-of-plane buckling of curved beams with both ends simply supported ; N=11 and $\delta = 1\times10^{-11}$

θ_0(degree)	M_{ycr}(kN-cm) Exact(Yang and Kuo[1])	DQM
10	-13114	-13114
30	-4821	-4821
50	-2709	-2709
70	-1706	-1709
90	-1098	-1098

Note : 1 kip-in = 11.3 kN-cm

Table 1 also shows the sensitivity of the numerical solution to the choice of δ. The optimal value for δ is found to be 1×10^{-10} to 1×10^{-11}, which is obtained from trial-and-error calculations. The solution accuracy decreases due to numerical instabilities if δ becomes too big(possibly greater than approx. 1×10^{-6} for this case). Table 2 presents the first four critical bending moments, denote by M_{ycr1} through M_{ycr4} for the number of grid points and $\delta=1\times10^{-11}$ with θ_0=30° and 90°. The data show that the accuracy of the numerical solution increases with increasing N for the lower mode critical moments. The remainder of the numerical results are computed with 11 discrete points along the dimensionless X-axis and $\delta = 1\times10^{-11}$.

In Table 3, the critical bending moments determined by the differential quadrature method are compared with the exact solution by Yang and Kuo[1)] for the case of simply supported ends. Table 4 shows the numerical results by the DQM for the case of both ends clamped and clamped-simply supported ends without comparison since no data are available. From Tables 3 and 4, it is seen that the critical loads of the member with clamped ends are much higher than those of the member with simply supported ends and those of the member with mixed clamped-simply supported ends. In Table 5, the critical bending moments determined by the DQM are compared with the solution by Yang and kuo[1] for the case of simply supported ends neglecting warping ($C_w = 0$). It is observed that the critical loads of the member with warping are much higher than those of the member without warping, and thus warping can have a significant effect on the critical loads. The results by DQM also show that the case of both ends simply supported is more affected by the warping than any other boundary conditions, and as the torsion constant of a beam cross-section becomes smaller, the warping stiffness of the cross-section becomes more significant. The critical loads can be increased by decreasing the opening angle θ_0. As can be seen, the numerical results by the differential quadrature method show good to excellent agreement with the exact solutions.

Table 4. Critical moment M_{ycr} of out-of-plane buckling of curved beams with both ends clamped and clamped-simply supported ends ; N=11 and $\delta = 1 \times 10^{-11}$

θ_0(degree)	M_{ycr}(kN-cm)	
	Both ends clamped (DQM)	Clamped-simply supported ends (DQM)
10	-69687	-33976
30	-75629	-24603
50	-67357	-17371
70	-55758	-12854
90	-46257	-9959

Note : 1 kip-in = 11.3 kN-cm

Table 5. Critical moment Mycr of out-of-plane buckling of curved beams with both ends simply supported neglecting warping ; N=11 and $\delta = 1 \times 10^{-11}$

θ_0(degree)	M_{ycr}(kN-cm)	
	Exact(Yang and Kuo[1])	DQM
10	-10924	-10924
30	-3933	-3933
50	-2205	-2205
70	-1390	-1390
90	-893	-893

Note : 1 kip-in = 11.3 kN-cm

6. Conclusions

The differential quadrature method was applied to the computation of the eigenvalues of the equations governing the out-of-plane buckling of curved beams including a warping contribution. The present approach gives excellent results for the cases treated while requiring only a limited number of grid points: only eleven discrete points were used for the evaluation. New results are given for two sets of boundary conditions not considered by previous investigators for the out-of-plane buckling: clamped-clamped and clamped-simply supported ends.

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