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Three-Parameter Gamma Distribution and Its Significance in Structural Reliability

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ABSTRACT

Information on the distribution of the basic random variables is essential for the accurate evaluation of structural reliability. The usual method for determining the distribution is to fit a candidate distribution to the histogram of available statistical data of the variable and perform appropriate goodness-of-fit tests. Generally, such candidate distributions would have two parameters that may be evaluated from the mean value and standard deviation of the statistical data. In the present paper, a-parameter Gamma distribution, whose parameters can be directly defined in terms of the mean value, standard deviation and skewness of available data, is suggested. The flexibility and advantages of the distribution in fitting statistical data and its significance in structural reliability evaluation are identified and discussed. Numerical examples are presented to demonstrate these advantages.

Keywords: probability distributions, histograms, structural reliability, reliability index, probability of failure, statistical moments

1. Introduction

In structural reliability evaluation, the basic random variables representing the uncertain quantities, such as loads, environmental factors, material properties, structural dimensions and variables accounting for the modeling and prediction errors, are generally assumed to have known cumulative distribution functions (CDF) or probability density functions (PDF). Determination of the distributions of these basic random variables is essential for the accurate evaluation of the reliability of a structure.

Often, the method for determining the required distribution is to fit the histogram of the statistical data of a variable with a candidate distribution (Ang and Tang, 1975), and apply statistical goodness-of-fit tests. A Bayesian approach in which the distribution is assumed to be a weighted average of all candidate distributions, where the weights represent the subjective probabilities of the respective candidates, was suggested by Der Kiureghian and Liu (1986). A method of estimating complex distributions using the B-spline functions has been proposed by Zong and Lam (1998), in which the determination of the PDF is formulated as a nonlinear programming problem and iteration is required.

Two-parameter distributions such as the normal, lognormal, Weibull and Gamma distributions are often selected as the candidate distribution, in which the parameters of the distribution are generally evaluated from the mean value and standard deviation of the available data. After the two parameters are determined, the distribution form and any higher-order moments, such as skewness, may be evaluated; quite often these higher-order moments may not be consistent with those of the available data. This is illustrated with the following: The two histograms shown in Fig. 1 represent the observed variabilities in the properties of H-shape structural steel (after Ono *et al.*, 1986). Fig. 1(a) shows the histogram of the section area and Fig.1(b) the histogram of the residual stress at the flange. From Fig. 1a, one can see that the coefficient of variation of the section area is small, 0.0514, whereas the skewness is large 0.7085. Conversely, the coefficient of variation of the residual stress from Fig. 1(b) is large, 0.7492, whereas the skewness is 0.823. The skewness of the normal, lognormal, Weibull and Gamma distributions

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that have the same mean value and standard deviation as the data in Fig.1(a) can be shown to be, respectively, 0.00, 0.1555, –0.9121 and 0.1024. Clearly, none of these two-parameter distributions has skewness that is consistent with the skewness of the data which is 0.7085. Similarly, with respect to Fig. 1(b), the skewness of these same four distributions can be shown to be 0.00, 1.7819, 0.6834 and 1.4984, respectively. Again, none of these matches the skewness of the data which is 0.823 in this case.

As illustrated with the data of Fig. 1, two-parameter distributions may not be appropriate when the skewness of the statistical data is important and must be reflected in the distribution.

For the above purpose, three-parameter distributions are required, such as the 3P lognormal distribution (Tichy, 1995) and the square normal distribution (see Appendix D). The evaluation of the parameters of these distributions, however, are complicated; furthermore, there is a limitation on the skewness for the square normal distribution. In the present paper, a three-parameter (3P) Gamma distribution is suggested, in which the three parameters can be directly defined in terms of the mean value, standard deviation and skewness. Besides being quite simple and flexible for fitting statistical data of basic random variables, the 3P Gamma distribution is convenient also for performing normal transformations needed in structural reliability analysis as described below.

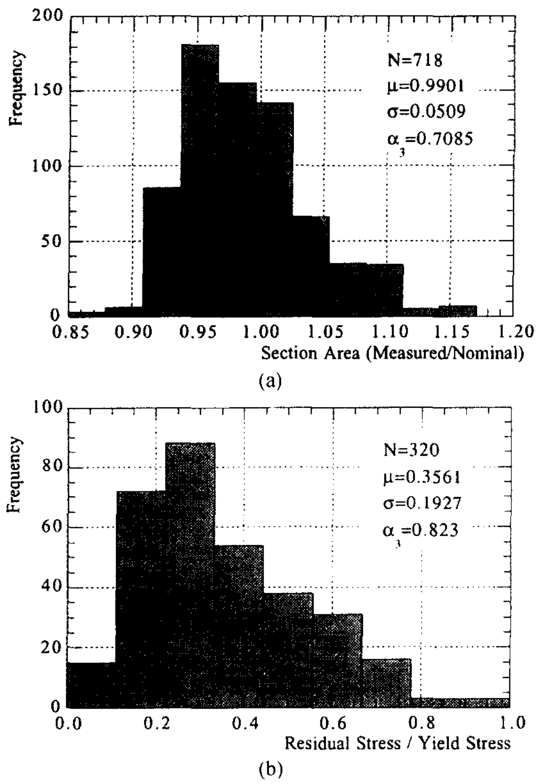

Fig. 1. Two histogram examples of practical data.

2. Three-parameter Gamma Distribution

2.1 Definition of the Distribution

In the Type III distribution of the Pearson system (Stuart and Ord, 1987), it can be shown that the PDF of the 3P Gamma distribution is (see Appendix A):

$$f(x) = \frac{|\lambda|^{\lambda^2}}{\exp(\lambda^2)\Gamma(\lambda^2)}\left|\lambda+\frac{x-\mu}{\sigma}\right|^{\lambda^2-1}\exp\left(-\lambda\frac{x-\mu}{\sigma}\right) \tag{1}$$

in which μ, σ and λ are the three parameters of the distribution.

For a standardized variable $x_s = (x-\mu)/\sigma$, the standard form of the PDF is expressed as

$$f(x_s) = \frac{|\lambda|^{\lambda^2}}{\exp(\lambda^2)\Gamma(\lambda^2)}|\lambda+x_s|^{\lambda^2-1}\exp(-\lambda x_s) \tag{2}$$

Let, $y = \lambda(\lambda + x_s)$ then the PDF in Eq. 2 becomes

$$f(y) = \frac{1}{\Gamma(\lambda^2)}y^{\lambda^2-1}\exp(-y) \tag{3}$$

which is the standard Gamma distribution with parameter λ^2. Therefore, the CDF of the distribution can be obtained as

$$F(x_s) = F_{g,\lambda^2}[\lambda(\lambda+x_s)] \text{ or } F(x) = F_{g,\lambda^2}\left[\lambda\left(\lambda+\frac{x-\mu}{\sigma}\right)\right] \tag{4}$$

where F_{g,λ^2} is the CDF of a standard gamma distribution with parameter of λ^2.

Equations (1) or (2) is the PDF of the three-parameter (3P) Gamma distribution.

2.2 Moments and Some Properties of the Distribution

The moments of x in any order are obtained as(see Appendix B):

$$E[x] = \mu \tag{5a}$$

$$E[(x-\mu)^2] = \sigma^2 \tag{5b}$$

$$E\left[\left(\frac{x-\mu}{\sigma}\right)^3\right] = \frac{2}{\lambda} \tag{5c}$$

$$E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right] = 3+\frac{6}{\lambda^2} \tag{5d}$$

$$E\left[\left(\frac{x-\mu}{\sigma}\right)^r\right] = (-\lambda)^r + \sum_{j=1}^{r-1}\frac{(-1)^j r!}{j!(r-j)!}\lambda^{2j-r}\sum_{k=1}^{r-j}(\lambda^2+r-j-k) \quad (5e)$$

From Eqs. (5a) and (5b), one can see that the mean value and standard deviation of x are equal to the parameters μ and σ, respectively, of the distribution. The third dimensionless central moment, i.e., the skewness α_3 is a function of only the parameter λ and is independent of μ and σ. Therefore, the distribution can be defined by the three parameters, mean value μ, standard deviation σ, and skewness α_3 of a random variable. Using Eq. (5c), the parameter λ can be easily determined as

$$\lambda = \frac{2}{\alpha_3} \quad (6)$$

When α_3 approaches 0, λ approaches infinity, using the Stirling formula, the limit of the PDF is obtained as(see Appendix C)

$$\lim_{\lambda\to\infty}[f(x)_s] = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_s^2\right) \quad (7)$$

Therefore, the distribution approaches the normal distribution when α_3 approaches 0.

The PDFs of the standard three-parameter Gamma distribution for $\alpha_3 > 0$ and $\alpha_3 < 0$ are shown in Fig. 2. When $\alpha_3 = 0$, the PDF degenerates to that of the standard normal distribution and is depicted as a thick solid line in Fig. 2. From Fig. 2, one can see that the distribution reflects the characteristics of the skewness.

When $\alpha_3 = 2$, λ becomes 1, and the PDF becomes

$$f(x_s) = \exp(-x_s) \quad (8)$$

which is the exponential distribution.

From Eq. (2), it may be observed that the random variable x_s is defined in the following ranges:

$$-\infty \le x_s \le -\lambda \ \text{ for } \lambda>0 \quad (9a)$$

$$-\lambda \le x_s \le -\infty \ \text{ for } \lambda>0 \quad (9b)$$

When x_s approaches $-\lambda$, the limit value of the PDF is obtained as

$$\lim_{x_s\to-\lambda} f(x_s) = \lim_{x_s\to-\lambda}\frac{|\lambda|^{\lambda^2}}{\Gamma(\lambda^2)}|\lambda+x_s|^{\lambda^2-1} \quad (10)$$

From Eq. (10), it can be shown that:

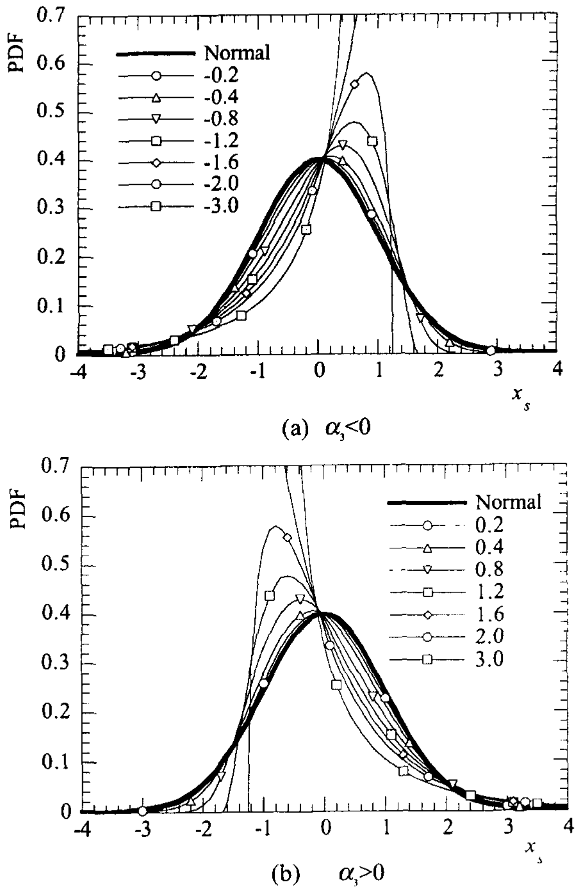

Fig. 2. PDFs of the standard 3P gamma distribution with varying α_3.

$$\lim_{x_s\to-\lambda} f(x_s) = \begin{cases} \infty & \text{for} |\lambda|<1 \text{ or } |\alpha_3|>2 \\ 1 & \text{for} |\lambda|=1 \text{ or } |\alpha_3|=2 \\ 0 & \text{for} |\lambda|>1 \text{ or } |\alpha_3|<2 \end{cases} \quad (11)$$

2.3 Comparison with Other 3-Parameter Distributions

The PDFs of the 3P Gamma, the square normal, and the 3P lognormal distributions (see Appendix D) are depicted in Fig. 3 for skewness of α_3=0.4, 0.8, 1.2, 1.6, 2.0 and 3.0. From Fig. 3, one can see that for small α_3, e.g., α_3=0.4, 0.8, there is no significant difference among the PDFs of the three different distributions. When α_3=3.0, the square normal distribution does not exist because α_3 is limited to the range of $-2.828 < \alpha_3 < 2.828$ for this distribution (Zhao and Ono, 2000).

The relationship between α_3 and α_4 for the three 3P distributions are shown in Fig. 4, from which one can observe that all the three distributions approach the normal distribution when α_3 approaches 0. For small skewness, e.g., $\alpha_3 < 1$, the three lines in Fig. 4 are quite close; this is the reason why there is no significant difference among the PDFs of the three distributions as described above.

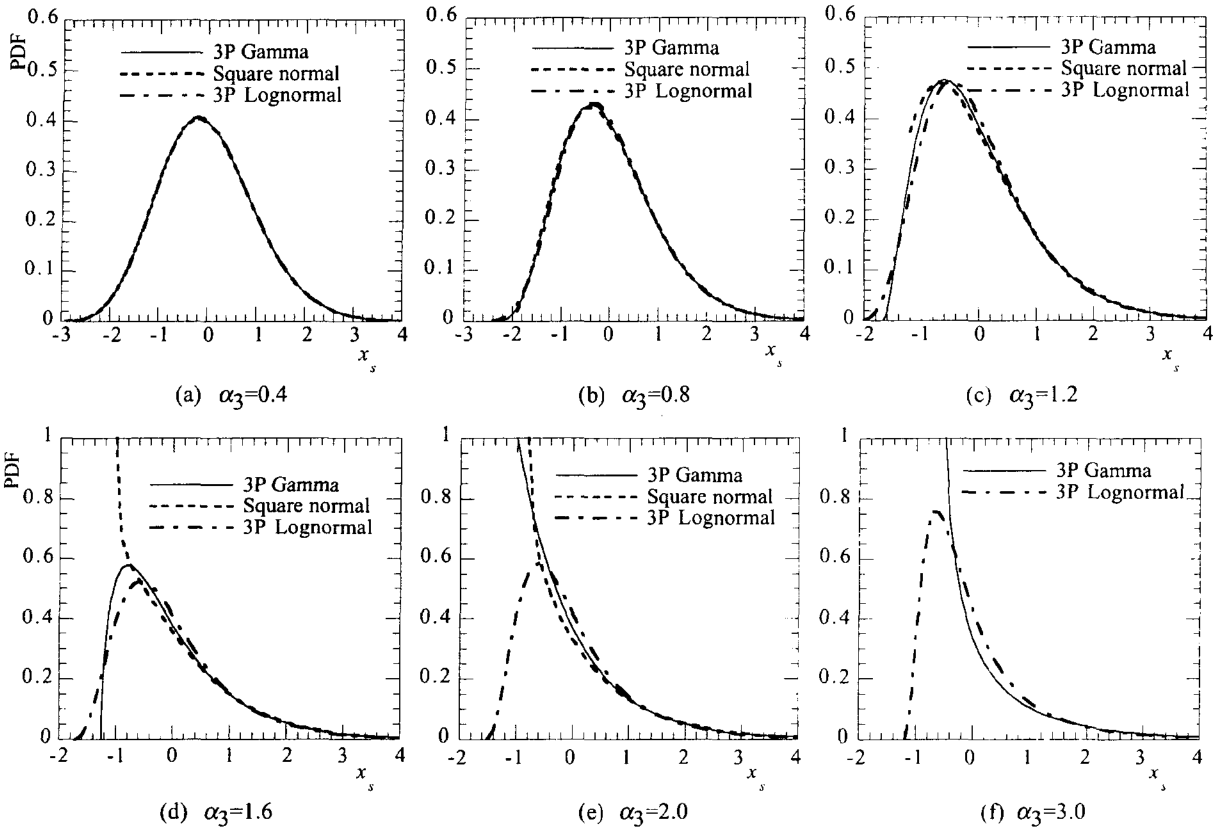

Fig. 3. Comparisons of 3-parameter distributions.

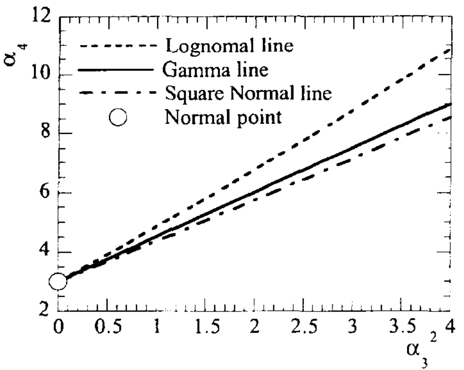

Fig. 4. Relationship between α_3 and α_4 of the 3P distributions.

3. Applications in Data Analysis and Structural Reliability

3.1 Statistical Data Analysis

The proposed three-parameter Gamma distribution is often appropriate for fitting statistical data of a random variable. For example, consider the measured data of H-shape structural steel described earlier. The histogram of the section area is shown in Fig. 5(a), in which the PDFs of the normal and lognormal distributions, with the same mean value and standard deviation as the data, and the PDF of the 3P Gamma distribution whose mean value, standard deviation and skewness are equal to those of the data, are depicted in Fig. 5(a), revealing the following:

(1) The normal distribution is clearly not appropriate as its skewness is zero, whereas the skewness of the data is 0.7085. (2) The skewness of the lognormal distribution is a function of the coefficient of variation (c.o.v.). In the present case, the c.o.v. of the lognormal distribution is 0.051, thus, its skewness is 0.1555 which is much smaller than that of the data (0.7085). (3) The skewness of the three-parameter Gamma distribution can be equated to that of the statistical data, and thus can fit the histogram much better than the normal or the lognormal distribution.

Results of the Chi-square tests of the three distributions are listed in Table 1, in which the goodness-of-fit tests were obtained using the following equation (Ang and Tang, 1975):

$$T = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \tag{12}$$

where O_i and E_i are the observed and theoretical frequen-

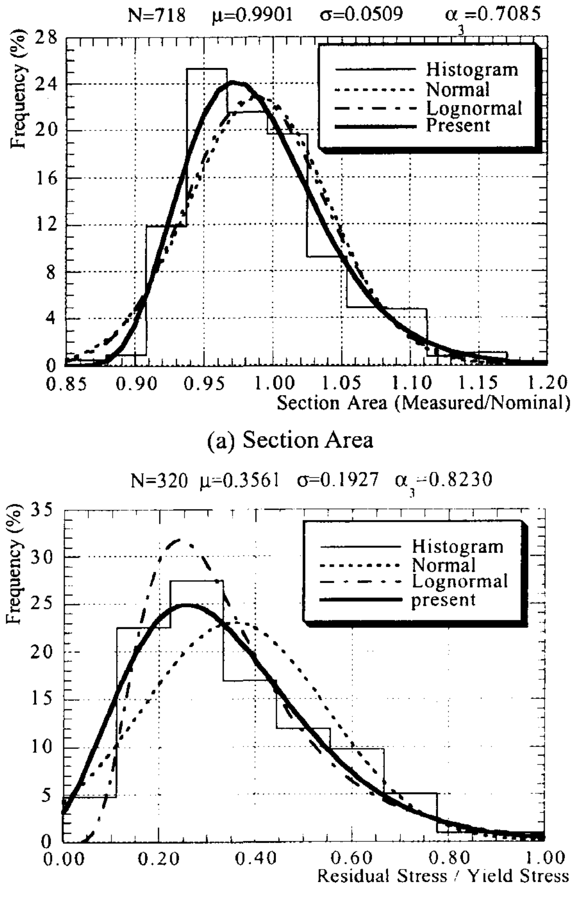

Fig. 5. Fitting results for data of H-shapes structural steel.

cies, respectively, k is the number of intervals used, and T is a measure of the respective goodness of fit. From Table 1, one can see that the goodness of fit of the proposed distribution is $T = 24.8$ which is much smaller than those of the normal with $T = 91.8$ or the lognormal with $T = 70.8$.

Similarly, the histogram of measured residual stress and the PDFs of the three distributions are shown in Fig. 5(b); the corresponding results of the Chi-square tests are listed in Table 2. From Fig. 5(b), one can observe that the lognormal distribution with a skewness of 1.7819 is much larger than that of the data of 0.7492. Clearly, therefore, the proposed 3P Gamma distribution fits the histogram much better than the normal or lognormal distribution. Also, from Table 2, the goodness-of-fit tests verify that the proposed distribution has a better fit, with $T = 13.4$, than the normal with $T = 47.3$ or the lognormal with $T = 26.7$.

The above examples clearly demonstrate the advantages of the 3P Gamma distribution for fitting statistical data with significant skewness.

3.2 Representations of One and Two-Parameter Distributions

The 3P Gamma distribution, as defined in Eq. (1) or (2), can be used to represent or approximate any one or two-parameter distributions by equating the respective three moments. This is illustrated with the Gamma, Weibull, lognormal and Exponential distributions, all of which are one- or two-parameter distributions. Fig. 6 shows the PDFs of the above distributions depicted as thin solid lines; in these same figures the respective 3P Gamma distributions with the same first three moments as those of the corresponding one or two-parameter distributions, are depicted as thick dash lines. In these figures, all the two-parameter distributions are shown with mean values of $\mu = 25$, 30, 35 and 40, and coefficients of variation

Table 1. Results of test for section area

Intervals	Freq.	Predicited Frequency			Goodness of fit		
		Nor.	Log.	Pres.	Nor.	Log.	Pres.
<0.908	9	38.7	35.3	18.0	22.8	19.6	4.46
0.908-0.938	85	69.3	72.2	83.7	3.54	2.27	0.02
0.928-0967	181	123	129	154	26.8	20.8	4.61
0.967-0.996	155	160	162	168	0.13	0.28	1.03
0.996.1.025	141	150	146	132	0.54	0.17	0.63
1.025-1.054	66	102	97.5	82.6	12.9	10.2	3.36
1.054-1.083	35	50.5	49.1	44.1	4.77	4.06	1.87
1.083-1.113	34	18.2	19.3	20.8	13.6	11.2	8.38
>1.113	5	4.75	5.95	14.5	6.67	2.28	0.43
Sum	718	718	718	718	91.8	70.8	24.8

Note: Nor.=Normal, Log.=Lognormal, Pres.=Present

Table 2. Results of test for residual stress

Intervals	Freq.	Predicited Frequency			Goodness of fit		
		Nor.	Log.	Pres.	Nor.	Log.	Pres.
<0.111	15	32.6	6.54	23.2	9.48	10.9	2.88
0.111-0.222	72	45.4	73.2	61.0	15.6	0.02	1.97
0.222-0.333	88	67.0	95.9	78.1	6.57	0.66	1.25
0.333-0.444	54	71.6	66.0	66.5	4.33	2.17	2.35
0.444-0.555	38	55.4	37.1	44.2	5.44	0.02	0.87
0.555-0.667	31	31.0	19.5	24.9	0.00	6.75	1.51
0.667-0.778	16	12.5	10.1	12.4	0.96	3.40	1.03
0.778-0.889	3	3.67	5.29	5.68	0.12	0.99	1.27
>0.889	3	0.91	6.32	4.00	4.81	1.75	0.25
Sum	320	320	320	320	47.3	26.7	13.4

Note: Nor.=Normal, Log.=Lognormal, Pres.=Present

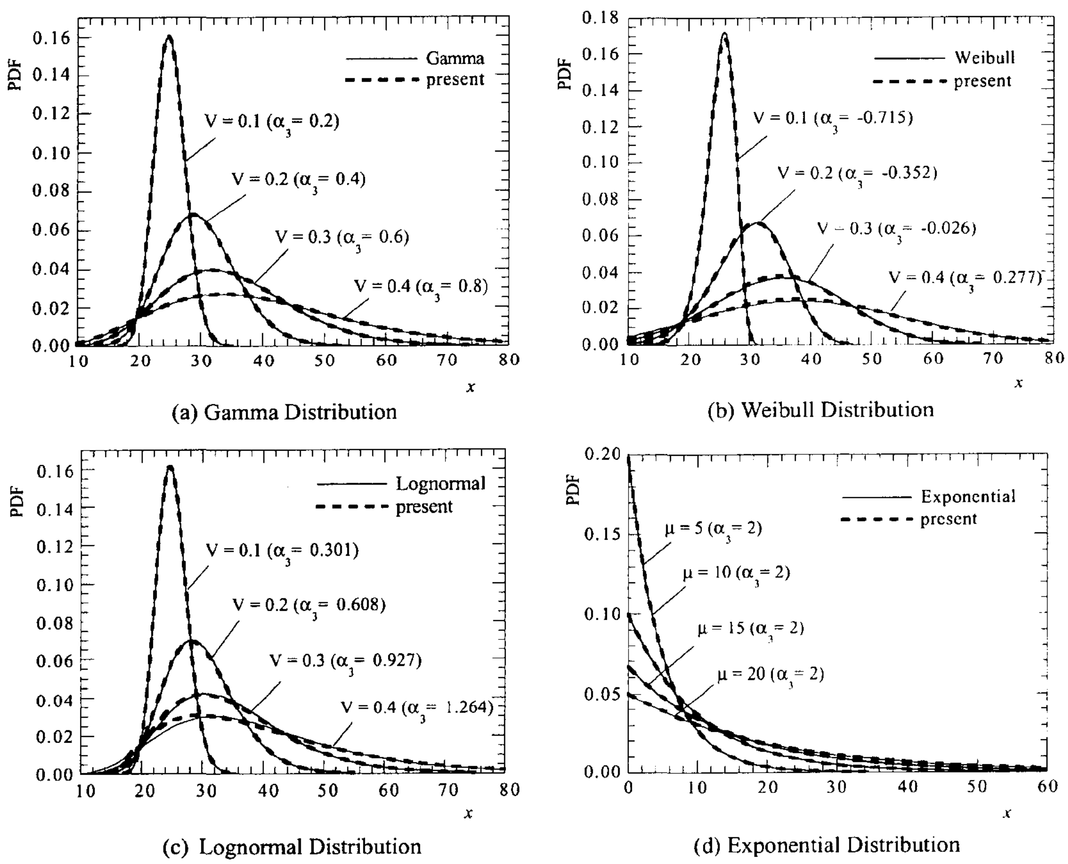

Fig. 6. PDF comparisons with some two parameter distributions.

$V = 0.1$, 0.2, 0.3 and 0.4.

Fig. 6 shows that the thick dash lines coincide closely with the thin solid lines, demonstrating the flexibility of the 3P Gamma distribution for representing one or two-parameter distributions. This flexibility can be useful in the structural reliability analysis as described below in Sect. 3.3.

3.3 Significance to Structural Reliability Assessment

The 3P Gamma distribution provides special properties that are convenient for structural reliability evaluation.

Consider a performance function $G(X)$ of a structural system, where X is the vector of basic random variables. If the first three moments of $G(X)$ can be obtained, the probability of failure, $P(G\le 0)$, can be readily obtained using the 3P Gamma distribution.

For the standardized random variable

$$x_s = \frac{G-\mu_G}{\sigma_G} \quad (13)$$

Since

$$Prob[G\le 0] = Prob\left[x_s \le -\frac{\mu_G}{\sigma_G}\right] = Prob[x_s \le -\beta_{2M}] \quad (14)$$

where, μ_G and σ_G are the mean value and standard deviation, respectively, of the performance function G. Using Eq. (4), the third-moment reliability index can be given as

$$\beta_{3M} = -\Phi^{-1}[F_{g,\lambda^2}[\lambda(\lambda-\beta_{2M})]] \quad (15)$$

where $\beta_{2M}=\mu_G/\sigma_G$ and β_{3M} are the second and third moment reliability indices, respectively, $\lambda = 2/\alpha_{3G}$ is the parameter of the 3P Gamma distribution in which α_{3G} is the skewness of G, and F_{g,λ^2} is the CDF of the standard gamma distribution with parameter λ^2. Although Eq. (15) is not in explicit form, it can be computed quite easily because standard Gamma distribution is a commonly used distribution.

If $G(X)$ is approximated as a second-order surface, the first three moments can be easily obtained and Eq. (15) can be used directly to obtain the second-order third moment reliability index (Zhao and Ono, 2002). In many cases, the performance function is expressed as a linear sum of independent random variables in the original space:

$$G(X) = \sum_{j=1}^{n} a_j x_j \quad (16)$$

In these latter cases, the first three moments of $G(X)$ are as follows:

$$\mu_G = \sum_{j=1}^{n} a_j \mu_j \quad (17a)$$

$$\sigma_G^2 = \sum_{j=1}^{n} a_j^2 \sigma_j^2 \quad (17b)$$

$$\alpha_{3G}\sigma_G^3 = \sum_{j=1}^{n} a_j^3 \sigma_j^3 \alpha_{3j} \quad (17c)$$

Then, substituting μ_G, σ_G, α_{3G} into Eq. (15), the third moment reliability index can be easily obtained.

For a specific example, consider the plastic collapse mechanism of a one-bay frame with the following performance function (after Der Kiureghian *et al.*, 1987):

$$G(X) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_4 - 5x_6 \quad (18)$$

The variables x_i are statistically independent and lognormally distributed with mean values $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 120$, $\mu_5 = 50$ and $\mu_6 = 40$, and standard deviations $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 12$, $\sigma_5 = 15$ and $\sigma_6 = 12$.. All of the random variables in Eq. (18) have known probability distributions, and thus the reliability index can be readily obtained using the First-Order Reliability Method (Ang and Tang, 1984) giving the FORM reliability index of $\beta_F = 2.348$, which corresponds to a failure probability of $P_F = 0.00943$. According to Der Kiureghian *et al.* (1987), the correct failure probability for this problem is $P_F = 0.0121$.

The skewnesses of the variables x_i can be easily obtained as $\alpha_{31} = \alpha_{32} = \alpha_{33} = \alpha_{34} = 0.301$, and $\alpha_{35} = \alpha_{36} = 0.927$. Using Eq. (17), the mean value, standard deviation and skewness of $G(X)$ are $\mu_G = 270$, $\sigma_G = 103.27$ and $\alpha_{3G} = -0.5284$. With these first three moments of $G(\mathbf{X})$, the second- and third-moment reliability indices can be shown to be $\beta_{2M} = 2.6145$ and $\beta_{3M} = 2.2621$, respectively. The probability of failure corresponding to the third-moment reliability index is 0.0118, which is much closer to the correct value of 0.0121 than that of FORM for this example.

3.4 Reliability Analysis Involving Variables with Unknown Probability Distributions

In the first- or second-order reliability method, the probability distributions of the basic random variables are necessary to perform the normal transformations (the *x-u* transformation and its inverse the *u-x* transformation). Often, in practical applications, the probability distributions of the random variables are unknown, and the probabilistic information may be defined only in terms of the respective first few statistical moments. With the 3P Gamma distribution, first- or second-order reliability analysis can be conveniently performed using the first three moments μ, σ, and α_3 in the *x-u* and *u-x* transformations with the aid of the following equations.

$$u = \Phi^{-1}[F_{g,\lambda^2}[\lambda(\lambda + x_s)]] \quad (19a)$$

$$x_s = \frac{1}{\lambda}F^{-1}_{g,\lambda^2}[\Phi(u)] - \lambda \quad (19b)$$

where x_s is the normalized variable define in Eq. (13), and $F^2_{g,\lambda}$ is the standard Gamma distribution.

Furthermore, random samples of the variables can be easily generated using Eq. (19b) for Monte-Carlo simulations.

For illustration, consider the following performance function of a simple structural column,

$$G(X) = Ax_1x_2 - x_3 \tag{20}$$

where A is the nominal section area, x_1 is a random variable representing the uncertainty in A, x_2 is the yield stress, and x_3 is a compressive load. Assume the column is made of H-shape structural steel with a H300 × 200 (JIS 1977) section having an area $A = 72.38$ cm², and a material of SS41 (JIS1976). The CDFs of x_1 and x_2 are unknown; the respective first three moments are $\mu_1 = 0.990$, $\sigma_1 = 0.051$, $\alpha_{31} = 0.709$, and $\mu_2 = 3.055$ t/cm², $\sigma_2 = 0.364$, $\alpha_{32} = 0.512$. The compressive load x_3 is assumed to be a lognormal variable with a mean value of μ_3=150 t and a standard deviation of $\sigma_3 = 45$t.

Although the CDFs of x_1 and x_2 are unknown, the *x-u* and *u-x* transformations can be obtained with Eqs. (19a) and (19b), without resorting to the Rosenblatt transformations. In the present case, FORM would yield the following after three iterations: the design point in standard normal space is [−0.194, −0.460, 1.161], and the first order reliability index is $\beta_F = 1.2634$ with corresponding failure probability of $P_f = 0.1033$. The average curvature radius of the limit state surface at the design point is 73.22. The performance function may be approximated with the simple parabolic function (Zhao and Ono, 2002)

$$G(U) = \beta_F + 0.0068(u_1^2 + u_2^2) - u_3 \tag{21}$$

The first three moments of G in Eq. (21) can be obtained as $\mu_G = 1.2774$, $\sigma_G = 1.0001$, and $\alpha_{3G} = 0.00001$. With these first three moments of the above second order performance function, the second and third moment reliability indices can be shown to be $\beta_{2M} = \beta_{3M} = 1.277$ with a corresponding probability of failure of $P_f = 0.1008$. Furthermore, using Eq. (19b), the random samples of x_1 and x_2 can be easily generated for Monte-Carlo simulation yielding a probability of failure of $P_f = 0.1013$ with a sample size of 10,000.

4. Conclusions

A three-parameter Gamma distribution is introduced, and its applications are emphasized including its significance in structural reliability assessment. The special attributes of the distribution include the following:

(1) The three independent parameters of the distribution are conveniently related to the mean, variance, and skewness of a random variable.

(2) With three parameters, the distribution has more flexibility for fitting statistical data of basic random variables, and can more effectively fit the histograms of available data than two-parameter distributions.

(3) The 3P Gamma distribution can be used to represent some popular two-parameter distributions, including the standard Gamma, Weibull and lognormal distributions.

(4) For some performance functions, such as second-order performance functions in the standard space, or linear performance functions in the original space, for which the first three moments can be obtained, the 3P Gamma distribution can be conveniently applied to obtain a moment-based reliability index.

(5) For random variables defined only by its first three moments, and otherwise unknown or unspecified probability distributions, first-order or second-order reliability analysis can be conveniently performed by assuming the 3P Gamma distributions for the variables.

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Appendix A: Derivation of the 3P Gamma distribution

For a standardized variable $x_s=(x-\mu)/\sigma$, consider the Pearson system (Stuart and Ord, 1987).

$$\frac{1}{f}\frac{df}{dx_s} = -\frac{ax_s+b}{c+bx_s+dx_s^2} \tag{A-1}$$

where

$$a = 10\alpha_4 - 12\alpha_3^2 - 18 \tag{A-2}$$

$$b = \alpha_3(\alpha_4+3) \tag{A-3}$$

$$c = 4\alpha_4 - 3\alpha_3^2 \tag{A-4}$$

$$d = 2\alpha_4 - 3\alpha_3^2 - 6 \tag{A-5}$$

When d=0, the distribution is the Pearson's type III distribution. The PDF of x_s is in the form of

$$f(x_s) = K(c+bx_s)^{(ac-b^2)/b^2}\exp\left[-\frac{ax_s}{b}\right] \tag{A-6}$$

The range of x should be limited so that $c+b\,x_s>0$, which means $x>-b/c$ for $b>0$ and $x<-b/c$ for $b<0$.

Substituting $d=0$ into Eqs. (A-2) through (A-4), and then using the results in Eq. (A-6), the PDF can be expressed as,

$$f(x_s) = K(\lambda+x_s)^{\lambda^2-1}\exp(-\lambda x_s) \quad \text{for } \alpha_3>0 \tag{A-7}$$

$$f(x_s) = K(-\lambda-x_s)^{\lambda^2-1}\exp(-\lambda x_s) \quad \text{for } \alpha_3<0 \tag{A-8}$$

where $\lambda = 2/\alpha_3$.

For $\alpha_3>0$, let $t=\lambda(\lambda+x_s)$, then

$$\begin{aligned}\int_{-\lambda}^{+\infty} f(x_s)dx_s &= \int_0^{+\infty}\left(\frac{1}{\lambda}t\right)^{\lambda^2-1}\exp(-t+\lambda^2)\frac{1}{\lambda}dt \\ &= \lambda^{-\lambda^2}\exp(\lambda^2)\Gamma(\lambda^2)\end{aligned} \tag{A-9}$$

that is

$$K = \frac{1}{\lambda^{-\lambda^2}\exp(\lambda^2)\Gamma(\lambda^2)} \tag{A-10}$$

Then the PDF of x_s becomes,

$$f(x_s) = \frac{\lambda^{\lambda^2}}{\exp(\lambda^2)\Gamma(\lambda^2)}(\lambda+x_s)^{\lambda^2-1}\exp(-\lambda x_s) \tag{A-11}$$

Similarly, for $\alpha_3<0$, let $t=\lambda(-\lambda-x_s)$, then the PDF of x_s is obtained

$$f(x_s) = \frac{(-\lambda)^{\lambda^2}}{\exp(\lambda^2)\Gamma(\lambda^2)}(-\lambda-x_s)^{\lambda^2-1}\exp(-\lambda x_s) \tag{A-12}$$

Eqs. (A-11) and (A-12) can be summarized as Eq. (2), and general form is Eq. (1).

Appendix B: Moments of the 3P Gamma Distribution

For $\alpha_3>0$, the rth order moment of x_s is

$$\begin{aligned}M_r &= \int_{-\lambda}^{+\infty} x_s^r f(x_s)dx_s \\ &= \frac{\lambda^{\lambda^2}}{\exp(\lambda^2)\Gamma(\lambda^2)}\int_{-\lambda}^{+\infty} x_s^r(\lambda+x_s)^{\lambda^2-1}\exp(-\lambda x_s)dx_s\end{aligned} \tag{B-1}$$

let $t=\lambda(\lambda+x_s)$, then Eq. (B-1) becomes

$$M_r = \frac{1}{\Gamma(\lambda^2)}\int_0^{+\infty}\left(\frac{1}{\lambda}t-\lambda\right)^r t^{\lambda^2-1}\exp(-t)dt \tag{B-2}$$

Using the expansion formula of $(a+b)^r$, Eq. (B-2) can be summarized as

$$M_r = \frac{1}{\Gamma(\lambda^2)}\sum_{j=0}^{r}\frac{(-1)^j r!}{j!(r-j)!}\lambda^{2j-r}\int_0^{+\infty} t^{\lambda^2+r-j-1}\exp(-t)dt \tag{B-3}$$

which is equivalent to

$$M_r = \frac{1}{\Gamma(\lambda^2)}\sum_{j=0}^{r}\frac{(-1)^j r!}{j!(r-j)!}\lambda^{2j-r}\Gamma(\lambda^2+r-j) \tag{B-4}$$

Using formula $\Gamma(z+1)=z\Gamma(z)$, Eq. (5e) is therefore obtained.

For $\alpha_3<0$, the same results follow.

Appendix C: The Limit of the 3P Gamma Distribution

The parameter λ of the 3P Gamma distribution is $\lambda=2/\alpha_3$; therefore, as $\alpha_3\to 0$, $\lambda\to\infty$, Hence, as $\alpha_3\to 0$, the limit of the 3P Gamma PDF is,

$$\lim_{\lambda\to\infty} f(x_s) = \lim_{\lambda\to\infty}\frac{\lambda^{\lambda^2}}{\exp(\lambda^2)\Gamma(\lambda^2)}(\lambda+x_s)^{\lambda^2-1}\exp(-\lambda x_s) \tag{C-1}$$

Using Stirling's formula, the limit of the Gamma function in Eq. (C-1) can be expressed as

$$\lim_{\lambda\to\infty}\Gamma(\lambda^2) = \sqrt{2\pi}\lim_{\lambda\to\infty}\frac{1}{\lambda}\lambda^{2\lambda^2}\exp(-\lambda^2) \tag{C-2}$$

Substituting Eq. (C-2) into Eq. (C-1) yields

$$\lim_{\lambda\to\infty}f(x_s) = \frac{1}{\sqrt{2\pi}}\lim_{\lambda\to\infty}\left(1+\frac{x_s}{\lambda}\right)^{\lambda^2-1}\exp(-\lambda x_s) \tag{C-3}$$

Let $y = \left(1+\frac{x_s}{\lambda}\right)^{\lambda^2-1}\exp(-\lambda x_s)$ then

$$\ln(y) = (\lambda^2-1)$$

$$\ln\left(1+\frac{x_s}{\lambda}\right)-\lambda x_s$$

Since

$$\lim_{\lambda\to\infty}[\ln(y)] = \lim_{\lambda\to\infty}\left[\lambda^2\ln\left(1-\frac{x_s}{\lambda}\right)-\lambda x_s\right] = -\frac{1}{2}x_s^2 \tag{C-4}$$

the normal distribution of Eq. (7) is obtained.

Appendix D: Some 3P Distributions

D.1 Three-Parameter Lognormal Distribution

For a standardized variable $x_s=(x-\mu)/\sigma$, The probability density function of the 3P lognormal distribution has been defined (Tichy 1995) as,

$$f(x_s) = \frac{1}{\sqrt{\ln(A)}|x_s-u_b|}\exp\left[\frac{1}{2\ln(A)}\left(\ln(A)\frac{|x_s-u_b|}{|u_b|}\right)^2\right] \tag{D-1}$$

where

$$A = 1+\frac{1}{u_b^2},\quad u_b = (a+b)^{\frac{1}{3}}+(a-b)^{\frac{1}{3}}-\frac{1}{\alpha_3} \tag{D-2}$$

$$a = -\frac{1}{\alpha_3}\left(\frac{1}{\alpha_3^2}+\frac{1}{2}\right),\quad b = -\frac{1}{2\alpha_3^2}\sqrt{\alpha_3^2+4} \tag{D-3}$$

in which μ and σ are the mean and standard deviation of x, respectively, and α_3 is third dimensionless central moment, i.e., the skewness of x.

Since A and u_b in (D-1) only depend only on α_3, the distribution is determined by the three parameters, μ, σ, and α_3.

D.2 Square Normal Distribution

For a standardized variable $x_s=(x-\mu)/\sigma$, the third-moment transformation function is expressed as (Zhao and Ono, 2000)

$$x_s = -\lambda+\sqrt{1-2\lambda^2}u+\lambda u^2 \tag{D-4}$$

and the corresponding probability density function is,

$$f(x_s) = \frac{\phi\left[\frac{1}{2\lambda}(\sqrt{1+2\lambda^2+4\lambda x_s}-\sqrt{1-2\lambda^3})\right]}{\sqrt{1+2\lambda^2+4\lambda x_s}} \tag{D-5}$$

in which

$$\lambda = Sign(\alpha_3)\sqrt{2}\cos\left[\frac{\pi+|\theta|}{3}\right] \tag{D-6}$$

$$\theta = \arctan\left(\frac{\sqrt{8-\alpha_3^2}}{\alpha_3}\right) \tag{D-7}$$

Since Eq. (D-4) is a quadratic form of the standard normal variable, the distribution has been referred to as the "square normal" distribution.