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MAPPING PROPERTIES OF THE MARCINKIEWICZ INTEGRALS ON HOMOGENEOUS GROUPS

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Abstract. Under the cancellation property and the Lipschitz condition on kernels, we prove that the Marcinkiewicz integrals defined on a homogeneous group $\mathbb{H}$ are bounded from $H^1(\mathbb{H})$ to $L^1(\mathbb{H})$, from $L_c^\infty(\mathbb{H})$ to $BMO(\mathbb{H})$, and from $L^p(\mathbb{H})$ to $L^p(\mathbb{H})$ for $1 < p < \infty$ assuming the L^q-boundedness for some $q > 1$.

1. Introduction

Stein [8] defined a higher dimensional analogue of the Marcinkiewicz integral by

$$\mu_\Omega f(x) = \left(\int_0^\infty |F_t(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}, \tag{1.1}$$

where

$$F_t(x) = \int_{|y-x|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy,$$

Ω is a homogeneous function of degree zero whose restriction to S^{n-1} belongs to $\Lambda^\alpha(S^{n-1})$ and satisfies the *cancellation* property,

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0.$$

Here, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ and $\Lambda^\alpha(S^{n-1})$ denotes the Lipschitz space of order α on S^{n-1}. The continuity of Marcinkiewicz

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integrals is very useful in harmonic analysis [9, 10, 15]. Stein [8] proved that if Ω is in $\Lambda^\alpha\left(S^{n-1}\right)$ with $0<\alpha\leq 1$, then

$$\left|\{x\in\mathbb{R}^n : \mu_\Omega f(x)>\lambda\}\right| \;\leq\; \frac{C}{\lambda}\,\|f\|_{L^1} \tag{1.2}$$

and

$$\|\mu_\Omega f\|_{L^p} \;\leq\; C_p\,\|f\|_{L^p}\,, \tag{1.3}$$

where $1<p\leq 2$, and if Ω is an integrable odd function, then

$$\|\mu_\Omega f\|_{L^p}\leq C_p\,\|f\|_{L^p} \tag{1.4}$$

for $2<p<\infty$.

The problem most immediately suggested by Marcinkiewicz [7], who conjectured the L^p-boundedness of (1.1) for $n=1$ and for $\Omega(t)=\text{sign }t$ until Zygmund [14] proved that the conjecture holds for $1<p<\infty$, has been extensively studied beginning with the 1958's article of Stein [8]. Benedek, Calderon and Panzone [1] proved that if $\Omega\in C^1\left(\mathbb{R}^n\setminus\{0\}\right)$ is a homogeneous function of degree zero satisfying the cancellation property, then μ_Ω is bounded on $L^p\left(\mathbb{H}\right)$ for $1<p<\infty$. Torchinsky and Wang considered the weighted L^p-boundedness of μ_Ω, and showed that if Ω is in $\Lambda^\alpha\left(S^{Q-1}\right)$ and μ_Ω is bounded on $L^p\left(\mathbb{R}^n\right)$ for $1<p<\infty$, then for ω satisfying an A_p condition, μ_Ω is bounded on $L^p(\omega)$ [13]. Further results on (1.1) were obtained when Ω satisfies some smoothness conditions [1], [8] and [13].

In this paper, we prove the H^1-L^1, L_c^∞-BMO and L^p-L^p $(1<p<\infty)$ boundedness of Marcinkiewicz integrals defined on homogeneous groups under the cancellation property and the Lipschitz condition on Ω and under the L^q-boundedness of μ_Ω.

This paper is organized as follows: in the next section, some preliminary materials are introduced. The main theorem is stated in Section 3. End-point results appear in Sections 4 and 5. Combined with an interpolation argument, the L^p boundedness for $1<p<\infty$ will be shown in Section 6.

2. Preliminaries and notations

In this section, we introduce notations related to homogeneous groups along with some preliminary materials. Mainly, we follow [6].

2.1. Homogeneous groups

A nilpotent Lie group $\mathbb{H}$ with a dilation group $\{\delta_r\}_{r>0}$ is said to be a homogeneous group. The dilation group is given by

$$\delta_r = \exp(A \ln r)$$

with a suitable matrix A having positive eigenvalues.

$\mathbb{H}$ has a natural homogeneous norm $|\cdot|$, and the homogeneous dimension Q. Abusing the notation, the bi-invariant measure on $\mathbb{H}$ will be denoted by $|\cdot|$.

REMARK 2.1. Let $\mathbb{H}$, $\{\delta_r\}$, $|\cdot|$ and Q be as above. The eigenvalues of the matrix defining $\{\delta_r\}$ are listed as $1 = d_1 \leq d_2 \leq \cdots \leq d_n$ and we let $\bar{d} = \max\{d_i : i = 1, \cdots, n\}$.

1. $|\delta_r x| = r|x|$ for each $x \in \mathbb{H}$, $r > 0$.
2. There exist C_1, $C_2 > 0$ such that

$$C_1\|x\| \;\leq\; |x| \;\leq\; C_2\|x\|^{\bar{d}} \quad \text{whenever} \quad |x| \leq 1.$$

Here, $\|\cdot\|$ denotes the euclidean norm.

3. There exists a constant $\gamma > 0$ such that for all $x, y \in \mathbb{H}$,

$$|x \circ y| \;\leq\; \gamma(|x| + |y|) \text{ for all } x \in \mathbb{H} \tag{2.1}$$

$$\big||x \circ y| - |x|\big| \;\leq\; \gamma|y| \text{ for all } x, y \in \mathbb{H} \text{ such that } |y| \leq |x|/2. \tag{2.2}$$

4. $|\delta_r E| = r^Q|E|$.
5. We let

$$S \;=\; \{x \in \mathbb{H} : |x| = 1\}.$$

There is a unique Radon measure σ on S such that for all $f \in L^1(\mathbb{H})$,

$$\int_{\mathbb{H}} f(x)\, dx \;=\; \int_0^\infty \int_S f(\delta_r y) r^{Q-1}\, d\sigma(y)\, dr.$$

2.2. The Hardy space $H^1(\mathbb{H})$

For the definition of the Hardy space $H^1(\mathbb{H})$, we refer the interested readers to [6].

2.2.1. $H^1_{q,0}$*-atoms.* Let $q \in (1, \infty]$. A function $a(x)$ on $\mathbb{H}$ is said to be an $H^1_{q,0}$-atom (associated to a ball B) if it satisfies the following conditions:

(a) $a(x)$ is supported in $\bar{B}$;
(b) $|a(x)| \leq |B|^{\frac{1}{q}-1}$ almost everywhere; and
(c) $\int_{\mathbb{H}} a(x)\, dx = 0$.

REMARK 2.2. Let $a(x)$ be an $H^1_{\infty,0}$-atom. Then we have

$$\|a\|_{L^1(\mathbb{H})} \leq 1. \tag{2.3}$$

2.3. Atomic decomposition

An equivalent way of looking at $H^1(\mathbb{H})$ is the decomposition of elements in $H^1(\mathbb{H})$ into $H^1_{q,0}$-atoms.

THEOREM 2.1 (Decomposition Theorem). *Let $q \in (1, \infty]$. For $f \in H^1(\mathbb{H})$ there exist a collection of $H^1_{q,0}$ atoms $\{a_k\}_{k\in\mathbb{N}}$ and a sequence of nonnegative real numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ with $\sum_{k=1}^{\infty} \lambda_k < \infty$ so that*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k$$

in the sense of distributions, and we have

$$\|f\|_{H^1} \approx \sum_{k=1}^{\infty} \lambda_k.$$

2.4. BMO

DEFINITION 2.3. A locally integrable function $f : \mathbb{H} \to \mathbb{C}$ is said to be in BMO if there exists a constant C such that for each ball B

$$\frac{1}{|B|} \int_B |f(x) - f_B|\, dx \leq C,$$

holds, where

$$f_B = \frac{1}{|B|} \int_B f(x)\, dx.$$

3. Marcinkiewicz integrals

Let Ω be a measurable function on a homogeneous group $\mathbb{H}$, which is homogeneous of degree 0 in the sense that

$$\Omega(\delta_r x) = \Omega(x)$$

holds for a.e. $x \in \mathbb{H} \setminus \{0\}$ and $r > 0$. We define the Marcinkiewicz integral $\mu_\Omega f$ as follows:

$$\mu_\Omega f(x) = \left(\int_0^\infty |F_t(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}, \tag{3.1}$$

where

$$F_t(x) = \int_{B_t(x)} \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} f(y)\, dy.$$

We will study the mapping properties of μ_Ω. To be more specific, we will prove:

THEOREM 3.1. *Let $\mathbb{H}$, Ω and μ_Ω be as above. We assume the following:*

- $\Omega|_S \in \Lambda^\alpha(S)$;
- $\int_S \Omega(x')\, d\sigma(x') = 0$; *and*
- μ_Ω *is bounded in* $L^q(\mathbb{H})$ *for some* $q > 1$.

Then the following inequalities hold:

$$\|\mu_\Omega f\|_{L^1} \le C_1 \|f\|_{H^1}, \qquad f \in H^1(\mathbb{H}) \tag{3.2}$$

$$\|\mu_\Omega f\|_{BMO} \le C_\infty \|f\|_{L^\infty}, \qquad f \in L^\infty_c(\mathbb{H}) \tag{3.3}$$

and

$$\|\mu_\Omega f\|_{L^p} \le C_p \|f\|_{L^p}, \qquad f \in L^p(\mathbb{H}) \tag{3.4}$$

for $1 < p < \infty$.

4. H^1-L^1 boundedness

In this section, we establish H^1-L^1 boundedness of the Marcinkiewicz integral. In view of Theorem 2.1 and the sublinearity of μ_Ω, it suffices to verify the inequality (3.2) when f is an arbitrary $H^1_{\infty,0}$-atom. Let $a(x)$

be an $H^1_{\infty,0}$-atom supported in $B_r(x_0)$. We split the integral into two parts,

$$\begin{aligned}\int_{\mathbb{H}} \mu_\Omega a(x)\, dx &= \int_{B_{2\lambda r}(x_0)} \mu_\Omega a(x)\, dx + \int_{\mathbb{H}\setminus B_{2\lambda r}(x_0)} \mu_\Omega a(x)\, dx \\ &\equiv I + II.\end{aligned}$$

4.1. Estimation on I

By hypothesis, we have

$$\int_{B_{2\lambda r}(x_0)} [\mu_\Omega a(x)]^q\, dx \lesssim \int_{B_r(x_0)} |a(x)|^q\, dx \lesssim |B_r(x_0)|^{-q+1}. \tag{4.1}$$

By Hölder's inequality and (4.1), we obtain

$$I \le \left(\int_{B_{2\lambda r}(x_0)} [\mu_\Omega a(x)]^q\, dx\right)^{\frac{1}{q}} |B_{2\lambda r}(x_0)|^{\frac{1}{q'}} \lesssim 1.$$

4.2. Estimation on II

Before we proceed, we introduce some simple facts on balls in $\mathbb{H}$.

Definition 4.1. For $E \subset \mathbb{H}$ and $x \notin E$, we will use the following notation.

$$d(x,E) = \inf\left\{\left|x\circ y^{-1}\right| : y \in E\right\}.$$

Lemma 4.2. *Let $x \notin B_{2\lambda r}(x_0)$. Then we have*

$$d(x, B_r(x_0)) \ge r.$$

Proof. Suppose

$$d(x, B_r(x_0)) < r.$$

Then there exists $y \in B_r(x_0)$ such that $\left|y\circ x^{-1}\right| < r$. So we get

$$\left|x\circ x_0^{-1}\right| \le \lambda\left(\left|x\circ y^{-1}\right| + \left|y\circ x_0^{-1}\right|\right) < 2\lambda r.$$

A contradiction to $x \notin B_{2\lambda r}(x_0)$. □

Lemma 4.3. *Let $x \notin B_{2\lambda r}(x_0)$ and $y \in B_r(x_0)$. Then we have*

$$\left|x\circ y^{-1}\right| \le 2\lambda\left|x\circ x_0^{-1}\right| \le 4\lambda^2\left|x\circ y^{-1}\right|.$$

Proof. Observe that

$$\begin{aligned}\left|x\circ y^{-1}\right| &\leq \lambda\left(\left|x\circ x_0^{-1}\right|+\left|x_0\circ y^{-1}\right|\right)\\ &\leq 2\lambda\left|x\circ x_0^{-1}\right|\\ &\leq 2\lambda^2\left(\left|x\circ y^{-1}\right|+\left|y\circ x_0^{-1}\right|\right)\\ &\leq 4\lambda^2\left|x\circ y^{-1}\right|\end{aligned}$$

from

$$\left|x\circ y^{-1}\right| \geq d\left(x,B_r\left(x_0\right)\right) \geq r \geq \left|y\circ x_0^{-1}\right|. \qquad \square$$

LEMMA 4.4. *Let $x\notin B_{2\lambda r}(x_0)$ and $t<d(x,B_r(x_0))$. Then we have*

$$B_r(x_0)\cap B_t(x) = \emptyset.$$

Proof. If $y\in B_r(x_0)$, then we have

$$\left|y\circ x^{-1}\right| \geq d\left(x,B_r\left(x_0\right)\right) \geq t. \qquad \square$$

Also, observe the following fact.

FACT 4.5. *There exist constants $C>0$, $\varepsilon\in(0,1)$ and $\rho>0$ such that*

$$\left|\delta_s x\circ x^{-1}\right| \leq C|1-s|^{\rho}|x|$$

uniformly in $x\in\mathbb{H}$ and $|1-s|<\varepsilon$.

Fix $x\in\mathbb{H}\setminus B_{2\lambda r}(x_0)$. Then, by Lemma 4.2 we have

$$d\left(x,B_r\left(x_0\right)\right) \leq \left|x\circ x_0^{-1}\right| \leq 2\lambda d\left(x,B_r\left(x_0\right)\right).$$

We have

$$\begin{aligned}\mu_\Omega a(x)^2 &= \int_{d(x,B_r(x_0))}^{\infty}\left|\int_{B_r(x_0)\cap B_t(x)}\frac{\Omega\left(x\circ y^{-1}\right)}{\left|x\circ y^{-1}\right|^{Q-1}}a(y)\,dy\right|^2\frac{dt}{t^3}\\ &= \int_{d(x,B_r(x_0))}^{\infty}\left|\int_{B_r(x_0)\cap B_t(x)}\frac{\Omega\left(x\circ y^{-1}\right)}{t\left|x\circ y^{-1}\right|^{Q-1}}a(y)\,dy\right|\cdot J_t a(x)\,\frac{dt}{t^2},\end{aligned}$$

where

$$J_t a(x) = \left|\int_{B_r(x_0)\cap B_t(x)}\frac{\Omega\left(x\circ y^{-1}\right)}{\left|x\circ y^{-1}\right|^{Q-1}}a(y)\,dy\right|.$$

LEMMA 4.6. *Let Ω, a, and $B_r(x_0)$ be as above. Then we have*

$$\left| \int_{B_r(x_0) \cap B_t(x)} \frac{\Omega\left(x \circ y^{-1}\right)}{t\left|x \circ y^{-1}\right|^{Q-1}} a(y)\, dy \right| \lesssim \mathcal{M}a(x)$$

whenever $t > d(x, B_r(x_0))$, $x \in \mathbb{H} \setminus B_{2\lambda r}(x_0)$.

Proof. There are two cases.
Case 1. $t \leq 2d(x, B_r(x_0))$.

$$\begin{aligned} &\left| \int_{B_r(x_0) \cap B_t(x)} \frac{\Omega\left(x \circ y^{-1}\right)}{t\left|x \circ y^{-1}\right|^{Q-1}} a(y)\, dy \right| \\ \lesssim\ & \frac{\|\Omega\|_\infty}{t \cdot d(x, B_r(x_0))^{Q-1}} \int_{B_t(x)} |a(y)|\, dy \\ \lesssim\ & \frac{\|\Omega\|_\infty}{t^Q} \int_{B_t(x)} |a(y)|\, dy \\ \lesssim\ & \mathcal{M}a(x). \end{aligned}$$

Case 2. $t \geq 2(d(x, B_r(x_0)))$. From $B_r(x_0) \cap B_t(x) \subset B_{d(x, B_r(x_0))}(x)$, we have

$$\begin{aligned} &\left| \int_{B_r(x_0) \cap B_t(x)} \frac{\Omega\left(x \circ y^{-1}\right)}{t\left|x \circ y^{-1}\right|^{Q-1}} a(y)\, dy \right| \\ \lesssim\ & \frac{\|\Omega\|_\infty}{d(x, B_r(x_0))^Q} \int_{B_{d(x, B_r(x_0))}(x_0)} |a(y)|\, dy \\ \lesssim\ & \mathcal{M}a(x), \end{aligned}$$

which completes the proof. □

For $J_t a(x)$ we have the following:

LEMMA 4.7. *With Ω, a, and $B_r(x_0)$ as above,*

$$J_t a(x) \lesssim \begin{cases} t\mathcal{M}a(x) \\ \qquad \text{if } \ d(x, B_r(x_0)) \leq t \leq \lambda\left(d(x, B_r(x_0)) + 2\lambda r\right) \\ r^\nu \left|x \circ x_0^{-1}\right|^{-Q+1-\nu} \\ \qquad \text{if } \ t \geq \lambda\left(d(x, B_r(x_0)) + 2\lambda r\right) \end{cases}$$

for any $x \in \mathbb{H} \setminus B_{2\lambda r}(x_0)$, where $\nu = \min\{\alpha, \rho\alpha, 1\}$.

Proof. We have two cases.
Case 1. $t \geq \lambda \left(d\left(x, B_r\left(x_0\right) \right) + 2\lambda r\right)$.
From $B_r\left(x_0\right) \subset B_t(x)$, $\int_{B_r(x_0)} a(y)\, dy = 0$ and the Lipschitz condition on $\Omega|_S$, we get

$$\begin{aligned} J_t a(x) &= \left| \int_{B_r(x_0)\cap B_t(x)} \frac{\Omega\left(x\circ y^{-1}\right)}{\left|x\circ y^{-1}\right|^{Q-1}} a(y)\, dy \right| \\ &= \left| \int_{B_r(x_0)} \left[\frac{\Omega\left(x\circ y^{-1}\right)}{\left|x\circ y^{-1}\right|^{Q-1}} - \frac{\Omega\left(x\circ x_0^{-1}\right)}{\left|x\circ x_0^{-1}\right|^{Q-1}} \right] a(y)\, dy \right|. \end{aligned}$$

Notice the following:

$$\begin{aligned} & \left| \frac{\Omega\left(x\circ y^{-1}\right)}{\left|x\circ y^{-1}\right|^{Q-1}} - \frac{\Omega\left(x\circ x_0^{-1}\right)}{\left|x\circ x_0^{-1}\right|^{Q-1}} \right| \\ \leq\ & \frac{\left|\Omega\left(x\circ y^{-1}\right) - \Omega\left(x\circ x_0^{-1}\right)\right|}{\left|x\circ y^{-1}\right|^{Q-1}} \\ & + \left|\Omega\left(x\circ x_0^{-1}\right)\right| \cdot \left| \frac{1}{\left|x\circ y^{-1}\right|^{Q-1}} - \frac{1}{\left|x\circ x_0^{-1}\right|^{Q-1}} \right| \\ \lesssim\ & \frac{\left| \delta_{|x\circ y^{-1}|^{-1}}\left(x\circ y^{-1}\right) \circ \delta_{|x\circ x_0^{-1}|^{-1}}\left(x\circ x_0^{-1}\right)^{-1} \right|^{\alpha}}{\left|x\circ x_0^{-1}\right|^{Q-1}} + \frac{\left|y\circ x_0^{-1}\right|}{\left|x\circ x_0^{-1}\right|^{Q}}. \end{aligned}$$

From (2.1), we obtain

$$\begin{aligned} & \left| \delta_{\frac{1}{|x\circ y^{-1}|}}\left(x\circ y^{-1}\right) \circ \delta_{\frac{1}{|x\circ x_0^{-1}|}}\left(x\circ x_0^{-1}\right)^{-1} \right| \\ \lesssim\ & \left| \delta_{\frac{1}{|x\circ y^{-1}|}}\left(\left(x\circ y^{-1}\right) \circ \left(x\circ x_0^{-1}\right)^{-1} \right) \right| \\ & + \left| \delta_{\frac{1}{|x\circ y^{-1}|}}\left(x\circ x_0^{-1}\right) \circ \delta_{\frac{1}{|x\circ x_0^{-1}|}}\left(x\circ x_0^{-1}\right)^{-1} \right| \\ \lesssim\ & \frac{r}{\left|x\circ y^{-1}\right|} + \frac{\left(1 - \frac{|x\circ x_0^{-1}|}{|x\circ y^{-1}|}\right)^{\rho} \left|x\circ x_0^{-1}\right|}{\left|x\circ x_0^{-1}\right|} \\ \lesssim\ & \frac{r}{\left|x\circ x_0^{-1}\right|} + \frac{r^{\rho}}{\left|x\circ x_0^{-1}\right|^{\rho}}, \end{aligned}$$

and so,

$$\begin{aligned}
&\left|\frac{\Omega\left(x\circ y^{-1}\right)}{|x\circ y^{-1}|^{Q-1}}-\frac{\Omega\left(x\circ x_0^{-1}\right)}{\left|x\circ x_0^{-1}\right|^{Q-1}}\right| \\
\lesssim\ & \frac{r^{\alpha}}{\left|x\circ x_0^{-1}\right|^{Q+\alpha-1}}+\frac{r^{\rho\alpha}}{\left|x\circ x_0^{-1}\right|^{Q+\rho\alpha-1}}+\frac{r}{\left|x\circ x_0^{-1}\right|^{Q}} \\
\lesssim\ & \frac{r^{\nu}}{\left|x\circ x_0^{-1}\right|^{Q+\nu-1}}
\end{aligned}$$

since $\left|x\circ x_0^{-1}\right|\geq 2\lambda r$. Therefore we obtain

$$J_t a(x) \ \lesssim\ \frac{r^{\nu}}{\left|x\circ x_0^{-1}\right|^{Q-1+\nu}}.$$

Case 2. $t\leq\lambda\left(d\left(x,B_r\left(x_0\right)\right)+2\lambda r\right)$.
Since $t\sim d\left(x,B_r\left(x_0\right)\right)$, we get

$$\begin{aligned}
J_t a(x) &\leq \|\Omega\|_{\infty}\left|\int_{B_t(x)}\frac{1}{|x\circ y^{-1}|^{Q-1}}\,|a(y)|\,dy\right| \\
&\lesssim \frac{1}{t^{Q-1}}\int_{B_t(x)}|a(y)|\,dy \\
&\lesssim t\mathcal{M}a(x).
\end{aligned}$$

The proof of Lemma 4.7 is completed. □

Thus, for $x\in\mathbb{H}\setminus B_{2\lambda r}\left(x_0\right)$,

$$\begin{aligned}
\mu_{\Omega}a(x)^2 &\lesssim \left(\int_{d(x,B_r(x_0))}^{\infty} J_t a(x)\,\frac{dt}{t^2}\right)\cdot\mathcal{M}a(x) \\
&= \left(\int_{d(x,B_r(x_0))}^{\lambda(d(x,B_r(x_0))+2\lambda r)} J_t a(x)\,\frac{dt}{t^2}\right. \\
&\qquad \left.+\int_{\lambda(d(x,B_r(x_0))+2\lambda r)}^{\infty} J_t a(x)\,\frac{dt}{t^2}\right)\cdot\mathcal{M}a(x)
\end{aligned}$$

$$\lesssim \left(\int_{d(x,B_r(x_0))}^{\lambda(d(x,B_r(x_0))+2\lambda r)} \frac{dt}{t}\right)\mathcal{M}a(x)^2 + \left(\int_{\lambda(d(x,B_r(x_0))+2\lambda r)}^{\infty} \frac{r^\nu}{|x\circ x_0^{-1}|^{Q-1+\nu}}\frac{dt}{t^2}\right)\cdot \mathcal{M}a(x)$$

$$\lesssim \frac{r}{d\left(x, B_r\left(x_0\right)\right)}\cdot \mathcal{M}a(x)^2 + \frac{r^\nu}{|x\circ x_0^{-1}|^{Q+\nu}}\cdot \mathcal{M}a(x),$$

and so we obtain

$$\mu_\Omega a(x) \lesssim \frac{r^{\frac{1}{2}}\cdot \mathcal{M}a(x)}{|x\circ x_0^{-1}|^{\frac{1}{2}}} + \frac{r^{\frac{\nu}{2}}\cdot \mathcal{M}a(x)^{\frac{1}{2}}}{|x\circ x_0^{-1}|^{\frac{Q+\nu}{2}}}.$$

Pick p_1, p_2, q_1, and q_2 with the following properties:

- $2Q < p_1 < \infty$;
- $\frac{2Q}{Q+\nu} < p_2 < 2$;
- $\frac{1}{p_1} + \frac{1}{q_1} = 1$; and
- $\frac{1}{p_2} + \frac{1}{q_2} = 1$.

From Hölder's inequality, the Maximal theorem, and (2.3), we obtain

$$r^{\frac{1}{2}}\int_{\mathbb{H}\setminus B_{2\lambda r}(x_0)} \frac{\mathcal{M}a(x)}{|x\circ x_0^{-1}|^{\frac{1}{2}}}\,dx$$

$$\lesssim r^{\frac{1}{2}}\left(\int_{\mathbb{H}\setminus B_{2\lambda r}(x_0)} |x\circ x_0^{-1}|^{-\frac{p_1}{2}}\,dx\right)^{\frac{1}{p_1}} \|\mathcal{M}a\|_{q_1}$$

$$\lesssim r^{\frac{1}{2}}\left(\int_{2r}^{\infty} \rho^{-\frac{p_1}{2}+Q-1}d\rho\right)^{\frac{1}{p_1}} \|a\|_{q_1}$$

$$\lesssim 1$$

and

$$r^{\frac{\nu}{2}}\int_{\mathbb{H}\setminus B_{2\lambda r}(x_0)} \frac{\mathcal{M}a(x)^{\frac{1}{2}}}{|x\circ x_0^{-1}|^{\frac{Q+\nu}{2}}}\,dx$$

$$\lesssim r^{\frac{\nu}{2}}\left(\int_{\mathbb{H}\setminus B_{2\lambda r}(x_0)} |x-x_0|^{-\frac{(Q+\nu)p_2}{2}}\,dx\right)^{\frac{1}{p_2}} \|\mathcal{M}a\|_{\frac{q_2}{2}}^{\frac{1}{2}}$$

$$\lesssim r^{\frac{\nu}{2}}\left(\int_{2r}^{\infty} \rho^{-\frac{(Q+\nu)p_2}{2}+Q-1}d\rho\right)^{\frac{1}{p_2}} \|a\|_{\frac{q_2}{2}}^{\frac{1}{2}}$$

$$\lesssim 1.$$

This shows

$$II \lesssim 1.$$

Altogether, we obtain

$$\int_{\mathbb{H}} \mu_\Omega a(x)\, dx \lesssim 1$$

and the proof is complete. □

5. L^∞-BMO boundedness

In this section, we study the L^∞-BMO boundedness of the Marcinkiewicz integrals. Let $f \in L^\infty(\mathbb{H})$ be compactly supported and let $B_r(x_0)$ be any ball. We write

$$\begin{aligned} f &= f\chi_{B_{2\lambda r}(x_0)} + f\chi_{\mathbb{H}\setminus B_{2\lambda r}(x_0)} \\ &\equiv f_1 + f_2. \end{aligned}$$

Then $f_1 \in L^2(\mathbb{H})$. By Hölder's inequality and by hypothesis,

$$\begin{aligned} \int_{B_r(x_0)} \mu_\Omega f_1(x)\, dx &\le |B_r(x_0)|^{\frac{1}{q'}} \left(\int_{B_r(x_0)} [\mu_\Omega f_1(x)]^q\, dx \right)^{\frac{1}{q}} \\ &\lesssim |B_r(x_0)|^{\frac{1}{q'}} \|f_1\|_{L^q(\mathbb{H})} \\ &\lesssim |B_r(x_0)|^{\frac{1}{q'}} |B_{2\lambda r}(x_0)|^{\frac{1}{q}} \|f_1\|_{L^\infty} \\ &\lesssim |B_r(x_0)| \cdot \|f\|_{L^\infty}. \end{aligned}$$

Thus we obtain

$$\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \mu_\Omega f_1(x)\, dx \lesssim \|f\|_{L^\infty}. \tag{5.1}$$

For $x \in B_r(x_0)$ we have $\left|x \circ y^{-1}\right| > r$ whenever $y \in \mathbb{H} \setminus B_{2\lambda r}(x_0)$. Let $x \in B_r(x_0)$ and let

$$III = \left(\int_0^\infty |F_t(x) - F_t(x_0)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

Then we have

$$
\begin{aligned}
III &= \left(\int_0^\infty \left| \int_{B_t(x)} \left[\frac{\Omega\left(x \circ y^{-1}\right)}{|x \circ y^{-1}|^{Q-1}} - \frac{\Omega\left(x_0 \circ y^{-1}\right)}{|x_0 \circ y^{-1}|^{Q-1}} \right] f_2(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
&= \left(\int_r^\infty \left| \int_{B_t(x)} \left[\frac{\Omega\left(x \circ y^{-1}\right)}{|x \circ y^{-1}|^{Q-1}} - \frac{\Omega\left(x_0 \circ y^{-1}\right)}{|x_0 \circ y^{-1}|^{Q-1}} \right] f_2(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_r^\infty \left(\int_{B_{2\lambda t}(x_0) \setminus B_{2\lambda r}(x_0)} \frac{r^\nu}{|y - x_0|^{Q-1+\nu}}\, dy \right)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \|f_2\|_{L^\infty} \\
&\lesssim \left(\int_r^\infty \left(\int_{2\lambda r}^{2\lambda t} \frac{r^\nu}{s^\nu}\, ds \right)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \|f_2\|_{L^\infty} \\
&\lesssim \|f\|_{L^\infty} .
\end{aligned}
$$

A triangle inequality provides us

$$
|\mu_\Omega f(x) - \mu_\Omega f_2(x_0)| \ \le\ \mu_\Omega f_1(x) + III,
$$

which verifies

$$
\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |\mu_\Omega f(x) - \mu_\Omega f_2(x_0)|\, dx \ \lesssim\ \|f\|_{L^\infty} .
$$

This proves the L_c^∞-BMO boundedness.

6. L^p-boundedness

From H^1-L^1 boundedness and the L^q-boundedness, it is clear that μ_Ω is bounded in L^p for $1 < p \le q$. To prove the L^p-boundedness for $q < p < \infty$, we define Stein's linearizing function $\varphi(x,t)$ which is a function defined for $x \in \mathbb{H}$, $0 < t < \infty$, so that it satisfies the conditions:

(a) φ vanishes if t is small enough, or if t is large enough and is bounded.

(b) For all x,

$$
\int_0^\infty |\varphi(x,t)|^2 \frac{dt}{t^3} \le 1 \tag{6.1}
$$

holds.

Now we define

$$T_\varphi f(x) = \int_0^\infty \int_{B_t(x)} \frac{\Omega\left(x \circ y^{-1}\right)}{|x \circ y^{-1}|^{Q-1}} f(y)\, dy \varphi(x,t) \frac{dt}{t^3}.$$

By (a), (b) and by Hölder's inequality,

(6.2) $$|T_\varphi f(x)| \le \mu_\Omega f(x) \quad \text{and} \quad \mu_\Omega f(x) = \sup_\varphi |T_\varphi f(x)|$$

for all φ satisfying (a) and (b) of the above.

It can be shown that T_φ is bounded from $L_c^\infty(\mathbb{H})$ to $BMO(\mathbb{H})$ with uniform bounds for those φ with the above properties. An interpolation yields the L^p boundedness for $p \in (q, \infty)$ of T_φ uniformly in φ with above properties, which implies the L^p boundedness of μ_Ω.

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