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RANDOM FIXED POINT THEOREMS FOR $*$-NONEXPANSIVE OPERATORS IN FRÉCHET SPACES

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Abstract. Some random fixed point theorems for nonexpansive and $*$-nonexpansive random operators defined on convex and starshaped sets in a Fréchet space are proved. Our work extends recent results of Beg and Shahzad and Tan and Yaun to noncontinuous multivalued random operators, sets analogue to an earlier result of Itoh and provides a random version of a deterministic fixed point theorem due to Singh and Chen.

1. Introduction

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. The study of random fixed point theorems was initiated by the Prague school of probabilists in the 1950s. The generalization of these theorems from self maps to nonself maps has gained tremendous importance after the papers by Beg [2], Beg and Shahzad [3, 4], Lin [10], Tan and Yaun [17], and Xu [22]. In particular, Beg [2], Tan and Yaun [17] and Xu [22] have studied continuous multivalued random maps including condensing maps and hemicompact maps. In this paper we study the random fixed point theory of noncontinuous multivalued random maps. A random fixed point theorem for a nonexpansive operator defined on a starshaped subset of a Fréchet space X is established in Theorem 2.3. As an application of this result in Theorem 2.7 a significantly general fixed point theorem for $*$-nonexpansive (noncontinuous) multivalued random operators defined on a suitable subset

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of X is obtained. A random version of [16, Theorem 2] is also derived in the context of a Fréchet space.

Throughout this paper, $(\Omega, \mathcal{A})$ denotes a measurable space with $\mathcal{A}$ a sigma algebra of subset of Ω unless stated otherwise. Let X be a Fréchet space, C a subset of X, 2^X the family of all nonempty subsets of X, $K(X)$ the family of all nonempty compact subsets of X, $CK(X)$ the family of all nonempty compact convex subsets of X, $WK(X)$ the family of all nonempty weakly compact subsets of X and $CB(X)$ the family of all closed bounded subsets of X. A mapping $T : \Omega \to 2^X$ is called measurable if for any open subset B of X, $T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \neq \phi\} \in \mathcal{A}$. A mapping $\xi : \Omega \to X$ is said to be a measurable selector of a measurable mapping $T : \Omega \to 2^X$ if ξ is measurable and for any $\omega \in \Omega$, $\xi(\omega) \in T(\omega)$.

A mapping $T : \Omega \times C \to 2^X$ is a random operator if for any $x \in C, T(\cdot, x)$ is measurable. A mapping $\xi : \Omega \to C$ is said to be

(i) a deterministic fixed point of T if $\xi(\omega) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$

(ii) a random fixed point of T if ξ is a measurable map such that for every $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))$.

A mapping $T : C \to 2^X$ is said to be

(i) upper (lower) semicontinuous if for any closed (open) subset B of $X, T^{-1}(B)$ is closed (open); if T is both upper and lower semicontinuous, then T is called a continuous map;

(ii) demiclosed at 0 if the conditions $x_n \in C$, $x_n \to x$ weakly, $y_n \in T_{x_n}, y_n \to 0$ strongly imply $0 \in T_x$.

A mapping $T : C \to CB(X)$ is called nonexpansive if for all $x, y \in C$,

$$H(Tx, Ty) \leq d(x, y)$$

where H is the Hausdorff metric on $CB(X)$.

A Banach space X satisfies Opial's condition if for each $x \in X$ and each sequence $\{x_n\}$ converging weakly to x, $\liminf_n \|x_n - y\| > \liminf_n \|x_n - x\|$ holds for all $y \neq x$ in X. A mapping $T : C \to 2^X$ is said to

(i) be weakly nonexpansive (cf. [7, 21]) if given $x \in C$ and $u_x \in Tx$, there is a $U_y \in Ty$ for each $y \in C$ such that

$$d(u_x, u_y) \leq d(x, y)$$

(ii) be$*$-nonexpansive (cf. [7, 21]) if for all $x, y \in C$ and $u_x \in Tx$ with $d(x, u_x) = d(x, Tx) = \inf\{d(x, z) : z \in Tx\}$, there exists $u_y \in T_y$ with $d(y, u_y) = d(y, T_y)$ such that

$$d(u_x, u_y) \leq d(x, y)$$

(iii) satisfy boundary condition (α) if for all $x \in C$ and all $y \in Tx, (x,y] \cap C \neq \phi$ where $(x,y] = \{(1-\alpha)x + \alpha y : 0 < \alpha \leq 1\}$ (cf. [16])

(iv) be d-continuous if for any $y \in X$, $x \to d(y, Tx)$ is continuous.

For each $x \in C$, we follow Xu [21] to define the set (possibly empty)

$$P_T(x) = \{u_x \in Tx : d(x, u_x) = d(x, Tx)\}$$

A random operator $T : \Omega \times C \to 2^X$ is said to be continuous (d-continuous, nonexpansive, contraction, $*$-nonexpansive etc.) if for each $\omega \in \Omega$, $T(\omega, \cdot)$ is continuous (d-continuous, nonexpansive, contraction, $*$-nonexpansive etc.)

It is well known that a Hausdorff locally convex topological vector space X is metrizable if and only if X has a countable base of absolutely convex neighbourhoods of zero or, equivalently, X has a countable family of seminorms $\{p_n\}$ that generates the locally convex topology on X. We can always assume that $p_n \leq p_{n+1}, n \geq 1$. A function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ given by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{c_n p_n(x-y)}{1 + p_n(x-y)}$$

for $x, y \in X$ with $c_n > 0$ and $\sum_{n=1}^{\infty} c_n < \infty$, defines a metric on X.

A map $T : C \to X$ is said to be

(i) Contraction if $p_n(T_x - T_y) \leq k_n p_n(x-y)$ for $x, y \in C$, $n \geq 1$ and some $0 \leq k_n < 1$.

(ii) Nonexpansive if $p_n(Tx - Ty) \leq p_n(x-y)$ for $x, y \in C$ and $n \geq 1$.

(iii) Weakly continuous if $Tx_n \to Tx$ weakly whenever $x_n \to x$ weakly.

A ball $B_r(0) = \{z \in X : d(z,0) \leq r\}$ with radius r and centred at 0 is said to be compressible if for every $\lambda > 1$ there is $t > r$ such that $B_t(0) \subset \lambda B_r(0)$. If every ball $B_r(0)$ in (X,d) is compressible (resp. convex), then we say that d is compressible (resp. convex) (see [19]).

The concept of a $*$-nonexpansive multivalued mapping is different from continuity of the map as is clear from the following example.

EXAMPLE 1.1. Let $X = \mathbb{R}^2$ be equipped with Euclidean norm and $C = \left\{(a,0) : \frac{1}{\sqrt{2}} \leq a \leq 1\right\} \cup \{(0,0)\}$. Define $T : C \to 2^X$ by

$$T(a,0) = \begin{cases} (0,1) & \text{if } a \neq 0 \\ L = \text{the line segment } [(0,1),(1,0)] & \text{if } a = 0. \end{cases}$$

Then $P_T(a,0) = \{(0,1)\}$ for all $(a,0) \in C$ with $a \neq 0$ and $P_T(0,0) = \left(\frac{1}{2},\frac{1}{2}\right)$. Clearly T is $*$-nonexpansive but not a continuous multifunction (cf. [14, p. 537]). Moreover for given $x = (0,0)$ and $u_x = (1,0) \in Tx$, there does not exist $y \neq x$ in C and $u_y \in Ty$ such that

$$|u_x - u_y| \leq |x - y|.$$

Recall that for $y \neq x$ in C, $u_y = (0,1)$ and $|u_x - u_y| = |(1,0) - (0,1)| = \sqrt{2} > d(x,y)$. So T is not weakly nonexpansive.

REMARKS 1.2.

(i) A $*$-nonexpansive map may not be weakly nonexpansive (cf. [7, p. 389]).

(ii) $*$-nonexpansiveness and nonexpansiveness are two different concepts for multivalued mappings.

(iii) The concept of an Opial Hausdorff locally convex space may be obtained from a Banach space one by replacing norm with $p \in \{p_n\}$.

We shall need the following variant of Theorem 2 in [11].

THEOREM 1.3. *Let X be a Hausdorff locally convex space, K a closed subset of X and $T : K \to K(X)$ a contraction satisfying*

$$(x,y] \cap K \neq \phi \text{ for all } x \in K \text{ and } y \in T_x.$$

Then T has a fixed point.

2. Random fixed point theorems

We state single valued version of Theorem 3.1 of Beg and Shahzad [3].

THEOREM 2.1. *Let C be a nonempty separable closed subset of a complete metric space X and $T : \Omega \times C \to X$ a continuous random operator satisfying the following conditions:*

(i) *$F(\omega) = \{x \in C : T(\omega,x) = x\}$ is nonempty for each $\omega \in \Omega$.*
(ii) *Each sequence $\{x_n\}$ in C with $d(x_n, T(\omega,x_n)) \to 0$ as $n \to \infty$ has convergent subsequence.*

Then T has a random fixed point.

The following theorem can be easily deduced from Theorem 2.1.

THEOREM 2.2. *Let C be a nonempty closed separable subset of a Fréchet space X and $T : \Omega \times C \to X$ a random contraction operator. If T satisfies boundary condition (α), then T has a random fixed point.*

Proof. Consider $F : \Omega \to 2^C$ defined by

$$F(\omega) = \{x \in C : T(\omega, x) = x\}.$$

Then by Theorem 1.3 (see also proof of [16, Theorem 1]), $F(\omega)$ is nonempty for each $\omega \in \Omega$. The condition $d(x_n, T(\omega, x_n)) \to 0$ together with the fact that T is a contraction implies that $\{x_n\}$ is a Cauchy sequence. Consequently conditions (i) and (ii) of Theorem 2.1 are satisfied and so by its conclusion, T has a random fixed point. □

As an application of Theorem 2.2 we obtain a random fixed point theorem for nonexpansive random operators satisfying condition (α) in a Fréchet space set up as follows.

THEOREM 2.3. *Let C be a nonempty weakly compact starshaped separable subset of a Fréchet space X and $T : \Omega \times C \to X$ a nonexpansive random operator satisfying boundary condition (α). If for each $\omega \in \Omega, I - T(\omega, \cdot)$ is demiclosed at 0, then T has a random fixed point.*

Proof. Let $z \in C$ be fixed and $r_n = 1 - \frac{1}{n}$ for each $n \geq 1$. Define $T_n : \Omega \times C \to X$ by

$$T_n(\omega, x) = r_n T(\omega, x) + (1 - r_n)z \qquad (n \geq 1).$$

Then for each n, T_n is a random contraction operator with Lipschitz constant r_n and T_n satisfies boundary condition (α) (see [16, Theorem 1]). Thus by Theorem 2.2, T_n has a random fixed point $\xi_n : \Omega \to C$. Define a sequence of mappings $F_n : \Omega \to WK(C)$ by $F_n(\omega) = W\text{-}cl\{\xi_j : j \geq n\}$ where $W - cl$ denotes the weak closure. Let $F : \Omega \to WK(C)$ be a mapping defined by $F(\omega) = \bigcap_{n=1}^{\infty} F_n(\omega)$. Since the weak topology on C is a metric topology (cf. [12, p. 86]) so F is ω-measurable by Theorem 4.1 [6]. By a selection theorem in [9], F has a ω-measurable selector $\xi : \Omega \to C$.

For each $x^* \in X^*$ (dual space of X), the numerically-valued function $x^*(\xi(\cdot))$ is measurable. By Theorem 1 of [20], ξ is measurable. We now show that ξ is the desired random fixed point of T. Fix $\omega \in \Omega$ arbitrarily. Then some subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ converges weakly to $\xi(\omega)$. Since C is bounded, $r_m \to 1$ as $m \to \infty$ and $\xi_m(\omega) - T(\omega, \xi_m(\omega)) = (r_m - 1)\{T(\omega, \xi_m(\omega)) - z\}$ so $\xi_m - T(\omega, \xi_m(\omega))$ converges strongly to zero (cf. [12, p. 22]). As $I - T(\omega, \cdot)$ is demiclosed at 0, it follows that $\xi(\omega) = T(\omega, \xi(\omega))$ as required. □

If $T : \Omega \times C \to C$, then we have:

COROLLARY 2.4. *Let C be a nonempty weakly compact starshaped separable subset of a Fréchet space X and $T : \Omega \times C \to C$ a nonexpansive random operator. If for each $\omega \in \Omega, I - T(\omega, \cdot)$ is demiclosed at 0, then T has a random fixed point.*

REMARK 2.5. Corollary 2.4 generalizes Theorem 3.4 due to Tan and Yaun [17], Theorem 3.2 due to Beg [2], Theorem 2.6 of Itoh [8] and Theorem 1 of Xu [22].

COROLLARY 2.6. *Let C be a nonempty weakly compact separable convex (or starshaped) subset of a Banach space X which is uniformly convex (or satisfies Opial's condition) and $T : \Omega \times C \to X$ a nonexpansive random operator. If T satisfies boundary condition (α), then T has a random fixed point.*

Proof. For each $\omega \in \Omega, T(\omega, \cdot)$ is nonexpansive so $I - T(\omega, \cdot)$ is demiclosed by Theorem 3 of [5] if X is uniformly convex (or by Remark 3.9 (iii) of [24] in case X satisfies Opial's condition). The result now follows from Theorem 2.3. □

The notion of $*$-nonexpansive map is different from continuous multivalued map so the following theorems are new and significantly general in the sense that these can not be implied by corresponding results of Beg and Shahzad [3, 4] and Tan and Yaun [18].

THEOREM 2.7. *Let C be a nonempty closed bounded convex separable subset of a uniformly convex Fréchet space X and $T : \Omega \times C \to 2^X$ closed convex valued $*$-nonexpansive random operator that satisfies the boundary condition (α). If for each $\omega \in \Omega, I - T(\omega, \cdot)$ is demiclosed at 0 and P_T is a random map, then T has a random fixed point.*

Proof. A closed convex subset of a uniformly convex Fréchet space is Chebyshev by Corollary 3 of [1], the set $T(\omega, x)$ is Chebyshev for each $\omega \in \Omega$ and each $x \in C$. Thus $P_T(\omega, x)$ is a singleton and $P_T(\omega, x) \in T(\omega, x)$ for all $\omega \in \Omega$ and all $x \in C$. By definition of $P_T(\omega, x)$ we have for all $\omega \in \Omega$ and all $x, y \in C$

$$d(P_T(\omega, x), P_T(\omega, y)) = d(u_x, u_y) \leq d(x, y).$$

That is $P_T : \Omega \times C \to X$ is a nonexpansive random operator. By hypothesis for fixed $\omega \in \Omega$ and for each $x \in C$ and each $y \in T(\omega, x)$,

$$(x, y] \cap C \neq \phi.$$

Since $P_T(\omega, x) \in T(\omega, x)$, it follows that $(x, P_T(\omega, y)] \cap C \neq \phi$ and hence P_T satisfies boundary condition (α). Suppose $x_n \to x$ weakly and $x_n - y_n = x_n - P_T(\omega, x_n) \to 0$ strongly for any $\omega \in \Omega$. Then

$x_n - y_n = I - P_T(\omega,\cdot)(x_n) \in I - T(\omega,\cdot)(x_n)$ and this gives by the demiclosedness of $I - T(\omega,\cdot)$ at 0 that $0 \in I - T(\omega, x)$. This implies that $x \in T(\omega, x)$ and hence $d(x, T(\omega, x)) = 0$. By definition of P_T, we have for each $y \in C, \omega \in \Omega$, $d(y, P_T(\omega, y)) = d(y, u_y) = d(y, T(\omega, y))$. Thus $d(x, P_T(\omega, x)) = 0$. So $x = P_T(\omega, x)$ gives that $I - P_T(\omega,\cdot)(x) = 0$ for each $\omega \in \Omega$. Thus $I - P_T(\omega,\cdot)$ is demiclosed at 0. Hence by Theorem 2.3, P_T has a random fixed point. That is there is a measurable map $\xi : \Omega \to C$ such that $\xi(\omega) = P_T(\omega, \xi(\omega))$ for all $\omega \in \Omega$. But $P_T(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$. Thus $\xi(\omega) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$ as desired. □

A special case of Theorem 2.7 is the following result which will be useful in applications.

THEOREM 2.8. *Let C be a nonempty weakly compact convex separable subset of a strictly convex Banach space X satisfying Opial's condition (or X is merely uniformly convex Banach space). Suppose that $T : \Omega \times C \to 2^X$ is closed convex valued $*$-nonexpansive random operator such that P_T is a random operator. If T satisfies boundary condition (α), then T has a random fixed point.*

Proof. Clearly the operator $P_T : \Omega \times C \to X$ is nonexpansive in both cases. So $I - P_T(\omega,\cdot)$ is demiclosed in first case by Remark 3.9 (iii) of [24] and in the second case by Theorem 3 of [5]. Hence the result follows from Theorem 2.3. □

We shall now give a general fixed point theorem than Theorem 2.7 under weaker assumptions. For this we require a pair of results stated below.

THEOREM A ([19, Theorem 2.1]). *Let (X, d) be a locally convex metrizable topological vector space with d as convex and compressible metric. Then every weak sequentially compact subset K of X is proximinal.*

THEOREM B ([1, Theorem 2]). *Every convex proximinal set in a strictly convex metric linear space is Chebyshev.*

THEOREM 2.9. *Let C be a nonempty weakly compact convex separable subset of a strictly convex Fréchet space X with convex and compressible metric d and $T : \Omega \times C \to 2^X$ be a weak sequentially compact convex valued $*$-nonexpansive random operator that satisfies boundary condition (α). If for each $\omega \in \Omega, I - T(\omega,\cdot)$ is demiclosed at 0 and P_T is a random operator, then T has a random fixed point.*

Proof. By Theorem A, each set $T(\omega, x)$ is proximinal and hence it becomes Chebyshev by Theorem B. As in the proof of Theorem 2.7, $P_T : \Omega \times C \to X$ is a nonexpansive random operator satisfying boundary condition (α) such that $I - P_T(\omega, \cdot)$ is demiclosed at 0 for each $\omega \in \Omega$. Hence by Theorem 2.3, T has a random fixed point. □

We assume the measure space $(\Omega, \mathcal{A}, \mu)$ to be complete and σ-finite in the rest of this section.

A generalization of Theorem 2.8 to the case of weakly closed sets is established in the following; our result provides a random version of [16, Theorem 2] for *-nonexpansive maps.

THEOREM 2.10. *Let C be a nonempty weakly closed starshaped subset of a separable Fréchet space X satisfying Opial's condition and $T : \Omega \times C \to K(X)$ be a $*$-nonexpansive random operator satisfying boundary condition (α). If for each $\omega \in \Omega$, $T(\omega, C) \subseteq B$ for some weakly compact subset B of X and P_T is a random operator, then T has a random fixed point.*

Proof. Each set $T(\omega, x)$ being compact is proximinal and so $P_T : \Omega \times C \to 2^X$ is well-defined. We can show by definition of T is $*$-nonexpansive that P_T is a nonexpansive map (see also proof of [21, Theorem 2]). The map P_T is compact valued as T is so. Hence $P_T : \Omega \times C \to 2^X$ is compact valued nonexpansive random operator which satisfies the boundary condition (α). Moreover for each $\omega \in \Omega, P_T(\omega, C) \subseteq T(\omega, C) \subseteq B$. Thus by Theorem 2 of [16], $P_T(\omega, \cdot)$ has a deterministic fixed point. We observe that P_T is a d-continuous random mapping. Hence by Lemma 3.1 of [23], P_T and therefore T has a random fixed point. □

In case C is a weakly compact subset of a Banach space X and $T : \Omega \times C \to K(C)$, Theorem 2.10 implies an analogue of Theorem 3.4 due to Itoh [8] for $*$-nonexpansive random operators in the following.

THEOREM 2.11. *Let C be a nonempty weakly compact starshaped subset of a separable Banach space X satisfying Opial's condition and $T : \Omega \times C \to K(C)$ a $*$-nonexpansive random operator such that P_T is a random map. Then T has a random fixed point.*

For a single valued map the notions of $*$-nonexpansive and nonexpansive coincide. Consequently we obtain a generalization of Theorem 3.4 of Tan and Yaun [17] for starshaped sets in an Opial space as follows.

COROLLARY 2.12. *Let C be a nonempty weakly compact starshaped subset of a separable Banach space X satisfying Opial's condition and*

$T : \Omega \times C \to C$ a nonexpansive random operator. Then T has a random fixed point.

References

[1] G. C. Ahuja, T. D. Narang, and S. Trehan, *Best approximation on convex sets in metric linear spaces*, Math. Nachr. **78** (1977), 125–130.

[2] I. Beg, *Random fixed points of nonself maps and random approximations*, J. Appl. Math. Stoch. Anal. **10** (1997), no. 2, 127–130.

[3] I. Beg and N. Shahzad, *A general fixed point theorem for a class of continuous random operators*, New Zealand J. Math. **26** (1997), 21–24.

[4] ———, *Random fixed point of weakly inward operators in conical shells*, J. Appl. Math. Stoch. Anal. **8** (1995), no. 3, 261–264.

[5] F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.

[6] C. J. Himmelberg, *Measurable relations*, Fund. Math. **87** (1975), 53–72.

[7] T. Husain and A. Latif, *Fixed points of multivalued nonexpansive maps*, Math. Japon. **33** (1988), no. 3, 385–391.

[8] S. Itoh, *Random fixed point theorems with an application to random differential equations in Banach spaces*, J. Math. Anal. Appl. **67** (1979), no. 2, 261–273.

[9] K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 397–403.

[10] T. C. Lin, *Random approximations and random fixed point theorems for continuous 1-set contractive random maps*, Proc. Amer. Math. Soc. **123** (1995), no. 4, 1167–1176.

[11] S. Massa, *Some remarks on Opial's spaces*, Bolletino, U.M.I. **(6) 2-A** (1983), 65–69.

[12] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[13] V. M. Sehgel and S. P. Singh, *A generalization to multifunctions of Fan's best approximation theorem*, Proc. Amer. Math. Soc. **102** (1988), no. 3, 534–537.

[14] ———, *On random approximation and a random fixed point theorem for set-valued mappings*, Proc. Amer. Math. Soc. **95** (1985), 91–94.

[15] N. Shahzad, *Random fixed point theorems for various classes of 1-set contractive maps in Banach spaces*, J. Math. Anal. Appl. **203** (1996), 712–718.

[16] K. L. Singh and Y. Chen, *Fixed points for nonexpansive multivalued mapping in locally convex spaces*, Math. Japon. **36** (1991), no. 3, 423–425.

[17] K. K. Tan and X. Z. Yaun, *Random fixed point theorems and approximation*, Stoch. Anal. Appl. **15** (1997), 103–123.

[18] ———, *Random fixed point theorems and approximation in cones*, J. Math. Anal. Appl. **185** (1994), 378–390.

[19] A. B. Thaheem, *Existence of best approximations*, Port. Math. **42** (1983–84), 435–440.

[20] G. E. F. Thomas, *Integration on functions with values in locally convex Suslin spaces*, Trans. Amer. Math. Soc. **212** (1975), 61–81.

[21] H. K. Xu, *On weakly nonexpansive and $*$-nonexpansive multivalued mappings*, Math. Japon. **36** (1991), no. 3, 441–445.

[22] ———, *Some random fixed point theorems for condensing and nonexpansive operators*, Proc. Amer. Math. Soc. **110** (1990), no. 2, 395–400.

[23] H. W. Yi and Y. C. Zhao, *Fixed point theorems for weakly inward multivalued mappings and their randomizations*, J. Math. Anal. Appl. **183** (1994), no. 3, 613–619.

[24] S. Zhang, *Star-shaped sets and fixed points of multivalued mappings*, Math. Japon. **36** (1991), no. 2, 327–334.

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