# 탄성평판 굽힘문제를 위한 경계적분 근사해

## A Boundary Integral Approximation for Bending of Elastic Plates

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요 약

본 연구는 굽힘 하중하에서 탄성평판 구조 해석을 위한 경계적분방법 구성을 주목적으로 하고 체계적인 모듈화 시스템 개발의 첫 이론 부분을 확립하였다. 굽힘 문제에서의 4개의 고유변수인 처짐, 기울기, 굽힘모우멘트, 상당 전단력의 항으로 경계적분방정식을 구성하였다. 물리적인 의미를 갖는 두 새로운 핵함수 도입으로 구성된 이들 적분방정식은 경계요소 구성시 나타나는 특이거동의 문제점을 간단한 두 탄성해에 의해 정규화 시켰으며 수치 적분 과정도 Cauchy 주치 적분 수렴성에서의 특별취급과는 달리 효율적으로 일반화시켰다. 경계적분식의 수치해석방법을 서술하였으며 집중하중하의 비대칭문제의 근사수치해를 도시하였다.

주요기술용어 : Boundary Integral Equations(BIEs 경계적분방정식), Fundamental Solution(기본해), Singularity(특이점), Regularization(정규화), Plate Bending(평판 굽힘)

## 1. INTRODUCTION

The formulation of elastic plate bending problem via boundary integral equations(BIEs) or boundary element method furnishes the basis for an alternative to finite difference and finite element approaches to the numerical solution of such problems. Conventional formulation, the so-called direct method is based on a reciprocal work identity and has been used by Forbes and Robinson 1969, Bezine and Gamby 1978, Stern 1979 and others to construct a pair of boundary integral equations involving the natural boundary variables of displacement, normal slope, bending moment, and equivalent shear on the boundary. Central to these developments is the use of special singular auxiliary function called "fundamental solutions" in the reciprocal work identity to generate an analogue of "Green's third identity". The resulting integral equations are generally singular and must be interpreted in a Cauchy principal value sense, which has some unfavorable implications where the equations are discretized for boundary element solution. There

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are several ways to proceed to regularization reducing the strength of the singularities present in BIEs. Du et. al. 1986 suggested an exterior integration technique by placing the source points outside the plate domain such that the field points are never situated within the same element. Tanaka 1991 constructed the discretized slope BIE for higher order quadratic boundary element whose behavior regularized up to the integrable order by using the subtracting and adding-back technique. The obtained BIEs, accordingly are weakly singular and can be integrated accurately by the standard Gaussian quadrature formula.

The main purpose of the present investigation is to formulate boundary integral equations for bending of elastic plates. A new direct formulation is employed to obtain a displacement BIE and a couple of slope BIEs which lead to non-singular BIEs with consequent simplification in the numerical treatment of these equations. While the formulation of the method follows closely in spirit the ideas outlined for thin elastic plate bending analysis by Stern 1979 we are able to introduce the two moment fundamental solutions. We first summarized the reciprocal work identity which is a basis for boundary integral representation. Thereafter we introduce the two fundamental solutions which have the physically meaningful bending solutions, and then three boundary integral equations representing the displacement and normal slope on the plate boundary. Finally, a discretization scheme is outlined for the numerical treatment of the BIEs, and this is illustrated by solving the model problems of concentrated load of square and circular plates.

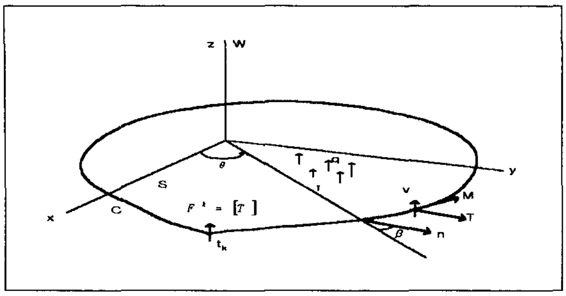
## 2. RECIPROCAL IDENTITY

We suppose that the middle surface of the plate occupies a simply connected bounded region S with total boundary C which is smooth (continuously turning tangent) except possibly for a finite number of corners which we denote $l_1, \ldots, l_k$. This is illustrated in Fig. 1 which also contains some notation that we shall use. Without furnishing any details of the derivation of classical thin plate theory(see e.g. Timoshenko and Woinowsky-Krieger 1959) we note that the deflection of middle surface, denoted $w$, is governed by the equation

$$\nabla^4 w = q/D \text{ in S} \tag{1}$$

where $D = Eh^3/12(1-\nu^2)$ is the plate stiffness and q is the transverse load intensity. To Eq.(1) is appended boundary conditions on C which reflect the manner in which the plate is supported on its boundary.

The direct development of boundary integral representations rests on a generalized reciprocal work identity associated with plate bending. We



[Fig. 1] Plate region and related notation

recognize from the symmetric bilinear form associated with the first variation of the strain energy that interchanging the roles of u and w leads to the same value for the reciprocal work, and hence the identity.

$$\int_c \left[ V(u)w - M(u)\frac{dw}{dn} + \frac{du}{dn}M(w) - uV(w)\right]ds + \sum_{k=1}^{k}\left[\,[\,|F(u)|\,]w - [\,|F(w)|\,]u\,\right]_{l_k} = \int_S (D\nabla^4 wu - D\nabla^4 uw)dS \qquad (2)$$

The boundary operators V and M represent the equivalent shear, and bending moment on the boundary associated with the deflection function w and the integral along the boundary tractions in the displacement u and normal rotation du/dn. Finally, the double bracket $[\,|F(w)|\,]_{l_k}$ denotes the jump in the twisting couple at the corner $l_k$ which can be interpreted physically as a concentrated force so that the summation accounts for the reciprocal work of these corner forces.

It will be convenient to introduce a polar coordinate system on S with respect to which these operators take the form.

$$M = -\frac{D(1-\nu)}{2}\left[\cos 2\beta\triangle_1 + \sin 2\beta\triangle_2 + \frac{1+\nu}{1-\nu}\nabla^2\right] \qquad (3)$$

$$F = \frac{D(1-\nu)}{2}(\sin 2\beta\triangle_1 - \cos 2\beta\triangle_2) \qquad (4)$$

$$V = -D\frac{d}{dn}\nabla^2 - \left[\sin\beta\frac{\partial}{\partial r} - \frac{\cos\beta}{r}\frac{\partial}{\partial\theta} + \left(\frac{\cos\beta}{r} - \frac{1}{R}\right)\frac{\partial}{\partial\beta}\right]F \qquad (5)$$

where $r,\ \theta,\ \beta$ are considered to be independent variables and

$$\frac{d}{dn} = \cos\beta\frac{\partial}{\partial r} + \frac{\sin\beta}{r}\frac{\partial}{\partial\theta} \qquad (6)$$

$$\triangle_1 = \frac{\partial^2}{\partial r^2} - \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \qquad (7)$$

$$\triangle_2 = 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) \qquad (8)$$

and we note in passing that

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \qquad (9)$$

In Eqs.(3)-(9) the angle $\beta$ is measured from the extension of the radial line to the outward normal as indicated in Fig. 1, and R is the radius of curvature of the boundary, which is negative if the corner of curvature is on the outward normal. The sign conventions associated with the physical quantities defined by these operators are also indicated in Fig. 1.

The region S and the functions w and u must of course be sufficiently well behaved that the terms in Eq.(2) can be evaluated and the integrals make sense. However, the boundary integral representations that we seek are obtained from Eq.(2) when the auxiliary function u is singular in a particular way.

## 3. FUNDAMENTAL SOLUTIONS

A fundamental solution for plate bending is a biharmonic function(and therefore a solution of Eq.(1) for zero load intensity) which is singular at a point P in the interior of S or on the boundary C.

### 3.1 Fundamental Solution for a Boundary Point

If the singular point P is located on the boundary C, then when we delete from S a circular region of radius $\varepsilon$. The boundary of the remaining region $S_\varepsilon$ now consists of the portion of circular boundary $C_\varepsilon$ contained in the interior of S, and another component denoted $C^*$ which consists of the original boundary C minus the two portions of arc within a distance $\varepsilon$ on either side of P. Again we have introduced two new corners near P denoted $l^+$ and $l^-$.

There are two distinct kinds of fundamental solution that we want to consider : a so-called concentrated force solution and a concentrated moment solution. A concentrated force fundamental solution $u_f^*$ for P on the boundary C has the following form

$$u_f^* = \frac{1}{8\pi D} r^2 \ln r \tag{10}$$

for which $V(u_f^*)$ behaves like 1/r near the origin with $N_f^*$, $M_f^*$, $V_f^*$ and $F_f^{*(k)}$ computed by Eqs.(3)-(6).

The reciprocal work identity, Eq.(2), evaluated on $S_\varepsilon$ with $u_f^*$ as the auxiliary function then produces a limiting result as the following boundary integral equation.

$$\begin{aligned} c_f\, w|_{P\ on\ C} &+ \int_C^* (V_f^* w - M_f^* N + N_f^* M \\ &- u_f^* V) ds \\ &+ \sum_{k=1}^{k} [F_f^{*(k)} w^{(k)} - u_f^{*(k)} F^{(k)}] \\ &= \int_S q u_f^* dS \end{aligned} \tag{11}$$

Since the constant $C_f$ depends only on the auxiliary function $u_f^*$ and not on the particular deflection function w, we can evaluate it by considering a special solution, e.g. w defined by

$$\hat{w} \equiv w|_P$$

for which N, M, V and $F^{(k)}$ all vanish identically, as does the corresponding load intensity $\hat{q} = \nabla^4 w$. For this special case, Eq.(11) reduces to

$$C_f\, w\,|_P + \int_C^* V_f^*\, w\,|_P\, ds = 0$$

so that we can rewrite Eq.(11) for the general case in the form

$$\int_c^* [V_f^*(w - w\,|_P) - M_f^* N + N_f^* M - u_f^* V] ds + \sum_{k=1}^{k}\{F_f^{*(k)}[w^{(k)} - w\,|_P] - u_f^{*(k)} F^{(k)}\} = \int_S q u_f^* dS \tag{12}$$

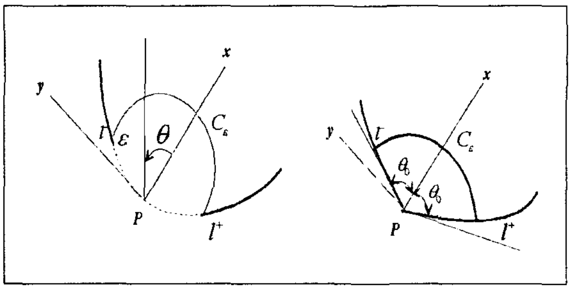
Thus, the single-valued concentrated force fundamental solution is also one that yields a nonsingular boundary integral in Eq.(12) when we disregard the principal value designation.

It remains to identify suitable concentrated moment fundamental solutions. It appears that we want to consider biharmonic functions u(r, $\theta$) for which rV(u) and M(u) behave like 1/r near the origin. We introduce the two additional singular functions

$$u^*_{m_1} = \frac{\partial u_f}{\partial n_p} = \frac{1}{8\pi D} r(1 + 2\ln r)\frac{\partial r}{\partial n_p} \tag{13}$$

$$u^*_{m_2} = \frac{\partial u_f}{\partial t_p} = \frac{1}{8\pi D} r(1 + 2\ln r)\frac{\partial r}{\partial t_p} \tag{14}$$

then we find $M^*_{m_1}$, $V^*_{m_1}$, $M^*_{m_2}$ and $V^*_{m_2}$. Without sacrificing generality we can orient the coordinate system associated with P at any particular boundary point by taking the x axis along the inward normal at a regular boundary point(where the tangent turns continuously) or along the inward bisector of the angle between




[Fig. 2] Orientation of local coordinate system

the tangent lines at a corner with included angle $2\theta_0$ as illustrated in Fig. 2. Now if P is a regular point of the boundary then

$$N\,|_p = -\left.\frac{\partial w}{\partial x}\right|_p \tag{15}$$

and therefore we can get the moment boundary integral equation corresponding to the fundamental solution $u^*_{m_1}$ in the form

$$\int_C \{V^*_{m_1}(w - w\,|_p + N\,|_p\, r\cos\theta) - M^*_{m_1}[N + N\,|_p \cos(\theta+\beta)] + N^*_{m_1} M - u^*_{m_1} V\}ds + \sum_{k=1}^{k}\{F^{*(k)}_{m_1}[w^{(k)} - w\,|_p + N\,|_p\,(r\cos\theta)\,|_{l_k}] - u^{*(k)}_{m_1} F^{(k)}\} = \int_S q u^*_{m_1} dS \tag{16}$$

Note that we have dropped the principal value indication from the boundary integral Eq.(16) since for the particular fundamental solution $u^*_{m_1}$ the integral converges in the ordinary sense.

If P is in fact a corner point then there are two distinct limiting normal slopes at P as depicted in Fig. 3 and we have

$$\frac{1}{2}\left( N|_{p^+} + N|_{p^-} \right) = -\left.\frac{\partial w}{\partial x}\right|_p \sin\theta_0 \quad (17)$$

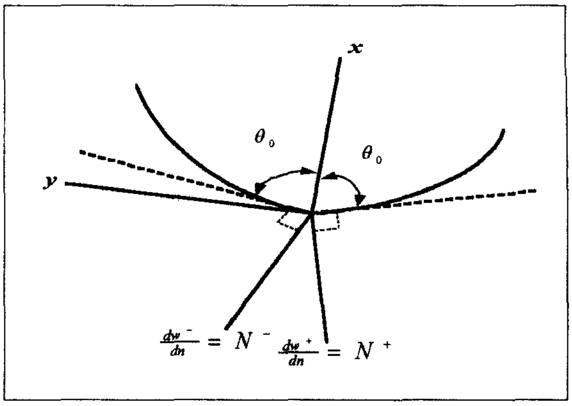
$$\frac{1}{2}\left( N|_{p^+} - N|_{p^-} \right) = -\left.\frac{\partial w}{\partial y}\right|_p \cos\theta_0 \quad (18)$$

For this case we use both $u^*_{m_1}$ and $u^*_{m_2}$ which leads to the independent corner moment boundary integral equations

$$\int_c \Big\{ V^*_{m_1}\Big[ w - w|_p + (N|_{p^+} + N|_{p^-}) \frac{r\cos\theta}{2\sin\theta_0}\Big] - M^*_{m_1}\Big[N + (N|_{p^+} + N|_{p^-}) \frac{\cos(\theta+\beta)}{2\sin\theta_0}\Big] + N^*_{m_1}M - u^*_{m_1}V \Big\} ds + \sum_{k=1}^{k*}\Big\{F^{*(k)}_{m_1}\Big[w^{(k)} - w|_p + (N|_{p^+} + N|_{p^-}) \left.\frac{r\cos\theta}{2\sin\theta_0}\right|_{l_k}\Big] - u^{*(k)}_{m_1}F^{(k)}\Big\} = \int_S q u^*_{m_1} dS \quad (19)$$

$$\int_c \Big\{ V^*_{m_2}\Big[ w - w|_p + (N|_{p^+} - N|_{p^-}) \frac{r\sin\theta}{2\cos\theta_0}\Big] - M^*_{m_2}\Big[N + (N|_{p^+} - N|_{p^-}) \frac{\sin(\theta+\beta)}{2\cos\theta_0}\Big] + N^*_{m_2}M - u^*_{m_2}V \Big\} ds + \sum_{k=1}^{k*}\Big\{F^{*(k)}_{m_2}\Big[w^{(k)} - w|_p + (N|_{p^+} - N|_{p^-}) \left.\frac{r\sin\theta}{2\cos\theta_0}\right|_{l_k}\Big] - u^{*(k)}_{m_2}F^{(k)}\Big\} = \int_S q u^*_{m_2} dS \quad (20)$$

where the asterisk on the summation indicates that the origin corner is not included. We note that if the corner is almost a straight angle(i.e. $\theta_0 \to \pi/2$) then $N|_{p^+}$ and $N|_{p^-}$ are nearly equal and Eq.(19)→Eq.(16) while Eq.(20) becomes indeterminate.




[Fig. 3] Directional derivatives and normal slopes at a corner

## 4. NUMERICAL TREATMENT

We summarise here our formulation of plate bending problems. We suppose that we are given geometrical data defining the region occupied by the plate's middle surface and its boundary. The primary variables of the problem are then the deflection w, the normal slope N, the bending moment M, the equivalent shear V at each point of the boundary and the corner forces $F^{(k)}$ at each boundary corner, produced by a prescribed loading. We also note that at each corner there are two distinct limiting values of N, M and V. Boundary conditions defining the nature of the support(or lack of it) at each point of the plate boundary furnish two relations involving w, N, M and V at that point. The two additional relations needed are the functional equations given by Eq.(12)(but without the principal value designation on the boundary integral) and Eq.(16). In the special case that the point P is a corner point we replace Eq.(16) with two independent relations given by Eqs.(19) and (20). Thus the problem is reduced to finding a solution of the boundary integral equations(in particular, Eqs.(12) and (16) for P a regular boundary point, and Eqs.(12), (19), (20) at each corner) consistent with prescribed boundary conditions. To proceed numerically we replace the continuous problem with a discrete one. Again, there are many different approaches to how this is done; we outline here a fairly direct formulation of the numerical problem which yields good results.

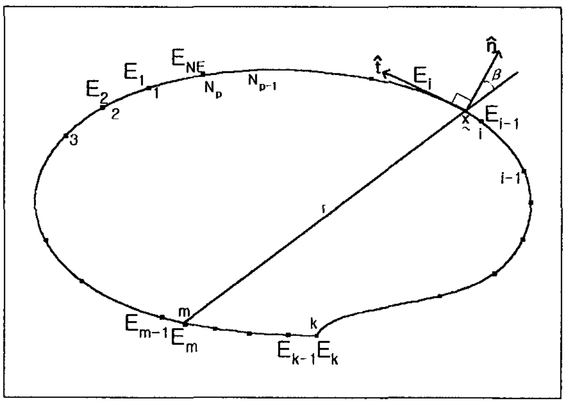
### 4.1 Discretization

There are two different types of variable that are discretized : the geometrical variables defining the boundary of the plate, and the primary variables of the problem. The geometrical data are generally known to any desired degree of accuracy whereas the only a priori data on the primary variables are furnished by boundary conditions. We may therefore choose to approximate the boundary geometry and the primary variables independently. The entire boundary is first decomposed into smooth components between corners. If the component is a straight line or an arc of a circle, then it is easily specified by the end points and, if necessary, a center of curvature. For more complex shapes we would need to furnish additional data, but in any event it is desirable to define each component of the boundary smoothly enough that the normal and curvature are continuous since these enter the calculations directly.

Each component is then partitioned into segments(elements) and on each we identify two nodal points. The segments are now numbered consecutively, say $E_1$ to $E_{NE}$, as are the nodal points from 1 to $N_P$. This is shown in Fig.4 . The nodal number of each corner node must be noted for special treatment as there are twice as many variables at a corner node as at a regular node: On the boundary element, the Gauss rule is used to select integration points and associated weights.

### 4.2 Boundary Conditions

The boundary conditions on each component of the boundary are applied at the nodal points and interpolated linearly on each element. At each



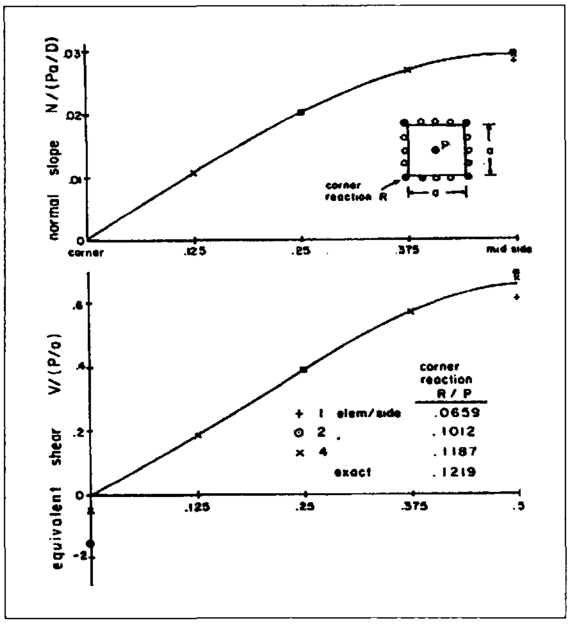

[Fig. 4] Plate geometry and element division regular nodal point there are four nodal variables introduced: deflection, normal slope, moment and shear.

We identify two nodal variables and their values as specified by boundary data, and the remaining two variables join the list of unknowns. Note that associated with each regular node as origin there will be two boundary integral equations.

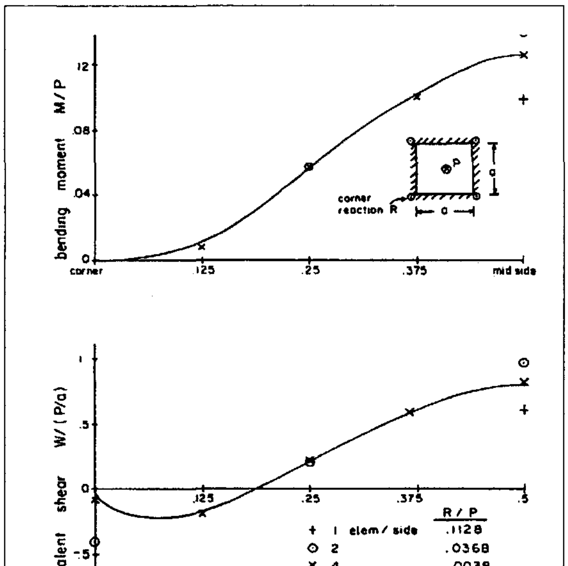
At each of the K corner nodes(common to two adjacent components of the boundary) there are eight nodal variables introduced: deflection and corner force, and two limiting values each of normal slope, moment and shear. There are three independent boundary integral equations associated with each corner so that five boundary conditions are needed. If the corner is free then the corner force, both limiting values of moment and both limiting values of shear vanish. If, however, the deflection is restrained then we must generally appeal to a knowledge of the asymptotic behavior of the solution near the corner. For example, if the corner is clamped along both boundaries then the curvature in every direction must vanishes; so in addition to the vanishing of the deflection and both normal slopes we also prescribe zero limiting bending moments. Similarly, if the corner is simply supported on both sides then shear in every direction must vanish; hence the five boundary conditions for this case are prescribed. In any event, at the conclusion of specifying boundary conditions we know the values of two variables at each regular node and five at each corner, leaving as unknown in the problem two variables at each regular node and three at each corner. We also associate two boundary integral equations with each regular node(as origin) and three boundary integral equations with each corner.

### 4.3 Concentrated Load Examples

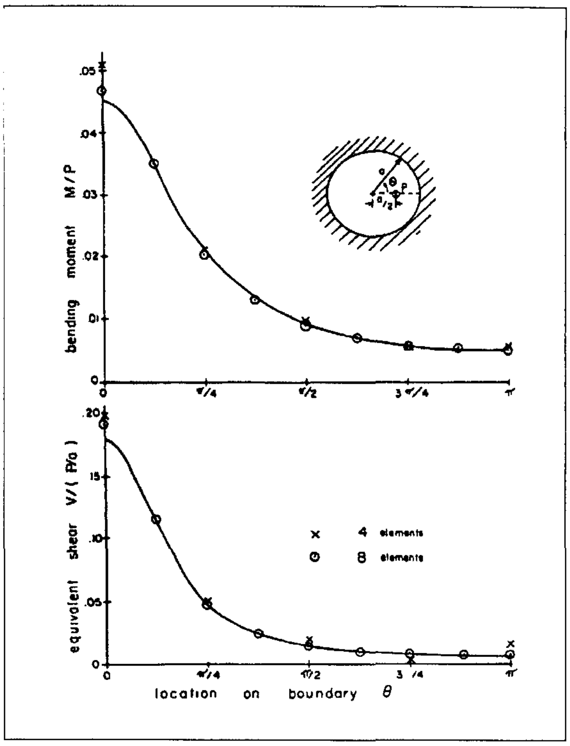
Numerical results for simply supported and clamped square plates subjected to a concentrated load at the center are given in Figs. 5 and 6. The plots are for half of one edge and values are indicated for one, two and four elements per side using six integration points per element. Convergence appears to be fairly rapid and acceptable results are obtained for relatively coarse meshes. All cases are for Poisson's ratio of 0.3. As a final example we look at a clamped circular plate subjected to an off-center concentrated load; the results for uniform meshes are plotted in Fig. 7. As before, convergence appears to be rapid and the results are quite good for such coarse meshes. In many boundary integral equation codes dealing with singular equations, abrupt changes in mesh size should be avoided unless very elaborate procedures are instituted to evaluate principal value integrals over neighboring elements. In the present non singular formulation this is not a problem, as indicated by the results in Fig. 8.



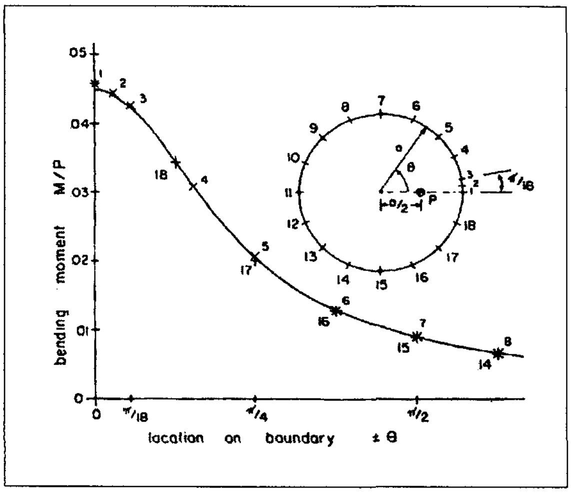

[Fig. 5] Normal slope and equivalent shear for a centrally loaded simply supported square plate




[Fig. 6] Bending moment and equivalent shear for a centrally loaded clamped square plate




[Fig. 7] Bending moment and equivalent shear for an eccentrically loaded clamped circular plate




[Fig. 8] Bending moment for an eccentrically loaded clamped circular plate near an abrupt changes in element size

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