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Stability Improved Split-step Parabolic Equation Model

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Abstract

The parabolic equation technique provides an excellent model to describe the wave phenomena when there exists a predominant direction of propagation. The model handles the square root wave number operator in paraxial direction. Realization of the pseudo-differential square root operator is the essential part of the parabolic equation method for its numerical accuracy. The wide-angled approximation of the operator is made based on the Padé series expansion, where the branch line rotation scheme can be combined with the original Padé approximation to stabilize its computational performance for complex modes. The Galerkin integration has been employed to discretize the depth-dependent operator. The benchmark tests involving the half-infinite space, the range independent and dependent environment will validate the implemented numerical model.

Keywords: *Parabolic equation, Padé approximation, Branch line rotation, Galerkin integration*

I. Introduction

Since Tappert's introduction of the parabolic equation (PE) method into underwater acoustics community, a variety of schemes have been applied to solve the range dependent propagation problem. In the beginning, they were often vulnerable to both stability and accuracy problems until highly accurate approximation techniques to the pseudo-differential operator involved in the parabolic equation model were developed.

The wide-angled parabolic equation has been derived to account for the complete angular spectrum of the forward propagation part of the wave equation. Together with the development in accuracy of the square root operator, PE method made further progress by devising a better starting field for more accurate near field solution. But, several authors pointed out the stability problems of the approximated operator in dealing with complex modes[1,2].

Collins developed a range-dependent acoustic model, called RAM[3,4], which has been generally adopted as a benchmark code in the underwater acoustics community. The split-step integrating formulation was taken to propagate the depth-discretized solution. The Padé approximation capable of representing a wide range of angular spectrum makes it possible for the split-step formulation to be a more powerful propagator in accuracy than finite difference schemes. However, its mathematical flaws, i.e. poles located at a physically meaningful region, can cause numerical problems in dealing with the environment in which the complex modes lying near the pole positions of the approximation are excited. Although Collins devised the modified Padé approximation method with stability constraints[2], it does not provide a qualitative

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insight for the operator.

The rotated branch cut rational approximation for the square root operator suggested by Milinazzo et al.[5], one of the remedy to the problem related to the poles, allows the quantification of stability in a clearer way than the others. Moreover, the application is by no means restricted by the order of approximation. The idea, called the branch line controlled Padé approximation, is extended to the approximation of the exponential propagation operator. In this paper, this method of branch line control is applied to the parabolic equation model based on the split-step formulation. Several benchmark tests will validate the numerical accuracy of the implemented code.

II. Derivation of the Split-step Parabolic Equation

Starting with the Helmholtz equation with azimuthal symmetry in cylindrical coordinates (r, φ, z)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\rho(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial p}{\partial z}\right)+k^2 p=0, \qquad (1)$$

the change of variables, $p=\frac{\Psi(r,z)}{\sqrt{r}}$, is used to arrive at the following equation:

$$\frac{\partial^2 \Psi}{\partial r^2}+\rho(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial \Psi}{\partial z}\right)+\left(k^2+\frac{1}{4r^2}\right)\Psi=0. \qquad (2)$$

Additionally, the far-field approximation $(kr \vee 1)$ can be made to give

$$\frac{\partial^2 \Psi}{\partial r^2}+\rho(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial \Psi}{\partial z}\right)+k_0^2 n^2 \Psi=0, \qquad (3)$$

where n ndicates the index of refraction defined by $\frac{c_0}{c(z)}$ and c_0 is the reference sound speed.

Separating paraxial radial dependency of the phase, $e^{ik_0 r}$, the out-going wave equation corresponds to factoring

$$\frac{\partial \phi}{\partial r}=ik_0\left(-1+\sqrt{n^2+\frac{\rho(z)}{k_0^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial}{\partial z}\right)}\right)\phi, \qquad (4)$$

where $\Psi=\phi e^{ik_0 r}$.

It should be noticed that one-way wave equation model is arrived based on the following three assumptions: (1) it is in the far field, (2) no back-scattered field exists or at least it has negligible effect on the full wave solution, and (3) the square root differential operator will have almost no variation in the radial direction. Thus, the parabolic equation technique is incapable of analyzing the wave propagation in the environment where the reflected (back-propagated) field becomes significant. Although the two-way PE can be adopted for this back propagating problems by taking both the forward and backward continuity conditions into consideration, it requires a great amount of computing resources and time.

From the practical point view, the parabolic equation for the impedance-reduced pressure, the energy-conserving model[6,7], is more convenient.

$$\frac{\partial u}{\partial r}=ik_0\left(-1+\sqrt{n^2+\frac{\rho(z)}{\alpha(z)k_0^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial}{\partial z}\alpha(z)\right)}\right)u, \quad (5)$$

where $u(r,z)=\phi(r,z)/\alpha(z)$ and $\alpha(z)=\sqrt{\rho(z)/k(z)}$.

Denoting the differential operator in (5) as $L=ik_0\left(-1+\sqrt{n^2+\frac{\rho(z)}{\alpha(z)k_0^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial}{\partial z}\alpha(z)\right)}\right)$, the general solution is given by $u(r)=e^{L(r-r_0)}u(r_0)$. Therefore, for a range-independent radial computational cell of distance $\vee r$, we obtain the following solution simply by integration.

$$u(r+\vee r)=\exp\left(ik_0\vee r(-1+\sqrt{1+X})\right)u(r), \qquad (6)$$

where $X=n^2-1+\frac{\rho(z)}{\alpha(z)k_0^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial}{\partial z}\alpha(z)\right)$.

III. Propagation Operator

The performance of numerical computation of previous equation (6) via the split-step solution critically depends on two factors: approximation of the differential operator X and realization of the pseudo-differential propagation operator $\exp(ik_0\vee r(-1+\sqrt{1+X}))$. In recent years, the Padé approximation has been in wide spread use for calculating the propagation operator due to its accuracy and numerical implementation ease.

The Padé approximation technique is a rational polynomial approximation using continued fractions. The principal advantage of the Padé approximation over other series expansions is that they provide an extension beyond the bound of convergence of the series[8]. Generally, for serial processing, the approximation will be made in the following multiplicative form,

$$f(x) \approx \prod_{j=1}^{N} \frac{1+\lambda_j x}{1+\mu_j x}. \tag{7}$$

To arrive at the approximation three steps are needed. First, Taylor expansion of order twice that desired is used to equate them with the non-factorized rational polynomial. Then, both the numerator and denominator polynomials are factored into multiples of the 1^{st} order polynomial. Finally, we calculate the reciprocal of the coefficients with the opposite sign to obtain the approximation.

The Padé expansion method shows better performance in operator approximation. However, the poles arising in the approximation inevitably cause the singularity problem. So investigation of whether the pole distribution generates significant degradation in operator approximation needs to be done.

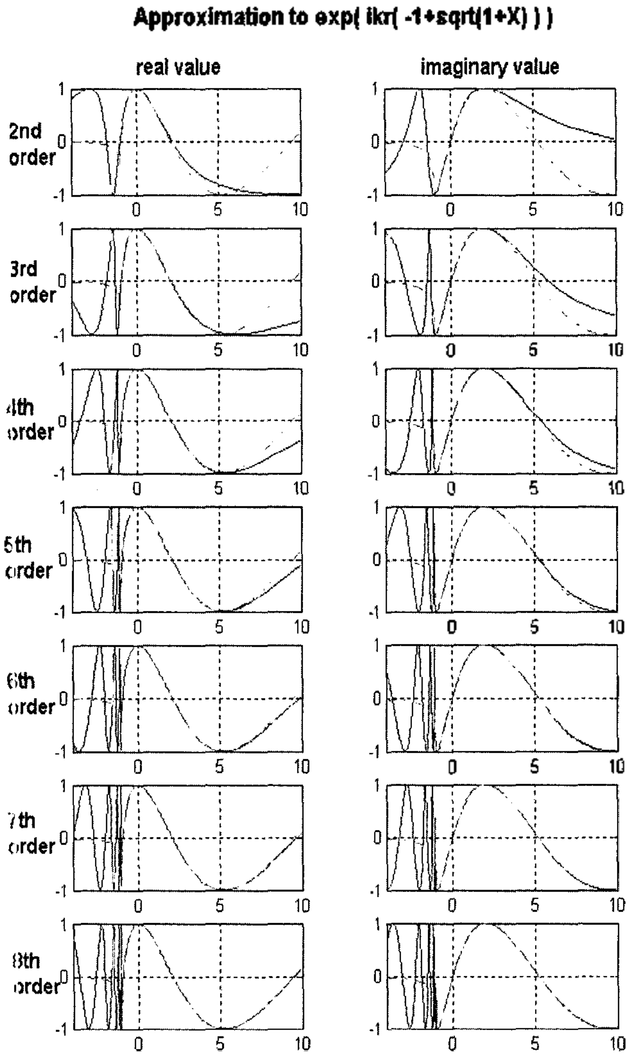

Figure 1. Pade approximation to the exponential propagation operator (with $k_0\sqrt{r}=2.1$). Dash dotted line indicates the analytic function value and the solid line indicates the approximated value for the order of approximation used.

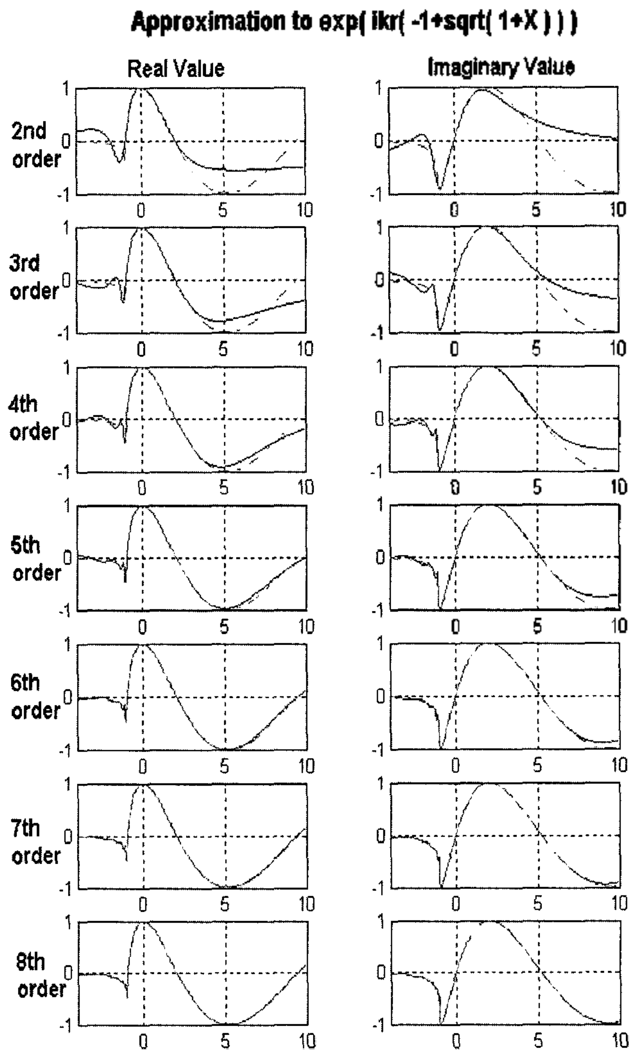

Figure 2. Branch line controlled Padé approximation to the exponential propagation operator (with rotation angle of $\zeta=\pi/6$ and $k_0\sqrt{r}=2.1$). Dash dotted line indicates the analytic function value and the solid line indicates the approximated value for the order of approximation used.

Unfortunately the poles are located at $x \leq -1$ where the significant spectra of the operator exist. Since the limiting distribution of poles of the Padé sequence is congruent with the natural definition of the branch cut[9], the square root function should be appropriately defined into a single-valued function. From the theoretical point of view, the physically acceptable solutions allow only positive imaginary spectrum due to the pre-defined convention of the spatial harmonic term, i.e. e^{ikr}. The singular behavior of the poles lead to deterioration of the performance of the approximated operator thereby motivating PE developers to contrive a different approach to the Padé approximation.

The branch line controlled Padé approximation is a kind of indirect calculation technique, a novel idea to avoid the singularity problem. First we approximate the modified function of which the natural branch cut is rotated by a certain angle. And by changing the form of the argument, the approximation of the original function itself can be reconstructed. It can be restated as the following: the approximations of the true values are done in rotated coordinate and it is transformed into those of the original coordinate. Such a roundabout way of computation makes it possible to avoid the singularity problem at the physically meaningful region[10].

Fig. 1 shows the comparison of the analytic function value approximated by the usual Padé method while Fig. 2 shows the approximation results by the branch line controlled Padé method. As can be seen in the figures, branch line control method furnishes stable and accurate results even at low order of approximation. Although some work needs to be done in order to find an appropriate rotation angle, it has proven to provide more stable results for most cases.

IV. Depth-dependent Operator

The split-step operator will be represented in multiplicative form of a rational polynomial,

$$X = n^2 - 1 + \frac{\rho(z)}{\alpha(z) k_0^2} \frac{\partial}{\partial z}\left(\frac{1}{\rho(z)} \frac{\partial}{\partial z} \alpha(z)\right). \tag{8}$$

So the discretization of the operator, X, is needed to complete the approximate numerical split-step operator. Following Collins[7], projection scheme of the Galerkin method is adopted.

$$Lf_{z=z_i} \approx \frac{\int (Lf)\, \Psi_i \, dz}{\int \Psi_i \, dz}, \tag{9}$$

where Ψ_i is a linear basis function given at $z = z_i$. The method is closely connected to the FEM solution and since it precludes the projection of the forcing term it cannot guarantee the accuracy of the FEM solver. In spite of this deficiency, it produces a reasonable formulation of the operator for the heterogeneous media.

When the domain is divided into uniform elements, we have the following discretized formula for each term of equation (8).

$$\frac{\partial}{\partial z}\left(\frac{1}{\partial} \frac{\partial \alpha u}{\partial z}\right)_{z=z_i} \approx \frac{1}{2\delta^2}\Big[\left(\frac{1}{\rho_{i-1}} + \frac{1}{\rho_i}\right)\alpha_{i-1}u_{i-1} - \left(\frac{1}{\rho_{i-1}} + \frac{2}{\rho_i} + \frac{1}{\rho_{i-1}}\right)\alpha_i u_i + \left(\frac{1}{\rho_{i+1}} + \frac{1}{\rho_i}\right)\alpha_{i+1}u_{i+1}\Big] \tag{10}$$

$$n^2 u_{z=z_i} \approx \frac{1}{12}\left[(n_{i-1}^2 + n_i^2)u_{i-1} + (n_{i-1}^2 + 6n_i^2 + n_{i+1}^2)u_i + (n_{i+1}^2 + n_i^2)u_{i+1}\right] \tag{11}$$

V. Numerical Validation

In this section, we solve three benchmark problems in order to validate the implemented computer algorithm based on the schemes outlined. They include:

1. Half space problem
2. Range-independent Pekeris wave-guide problem
3. Range-dependent problem of ASA wedge

The first problem, which has an analytic solution using the image method[11], is a simple example to show the validity of the algorithm. The second part of the validation provides the solutions for a common ocean waveguide environment. The results will be compared with the solutions of OASES[12] for range-independent problem and RAM[4] for range-dependent problem. As for the

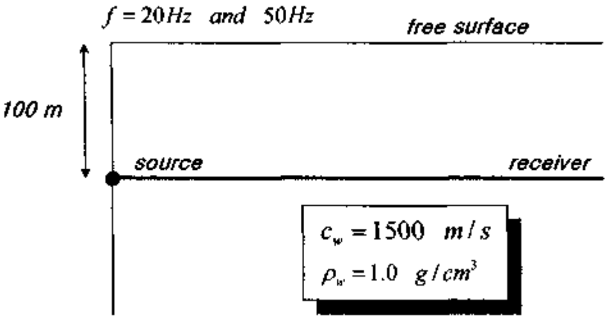

Figure 3. Schematic of the half-space environment.

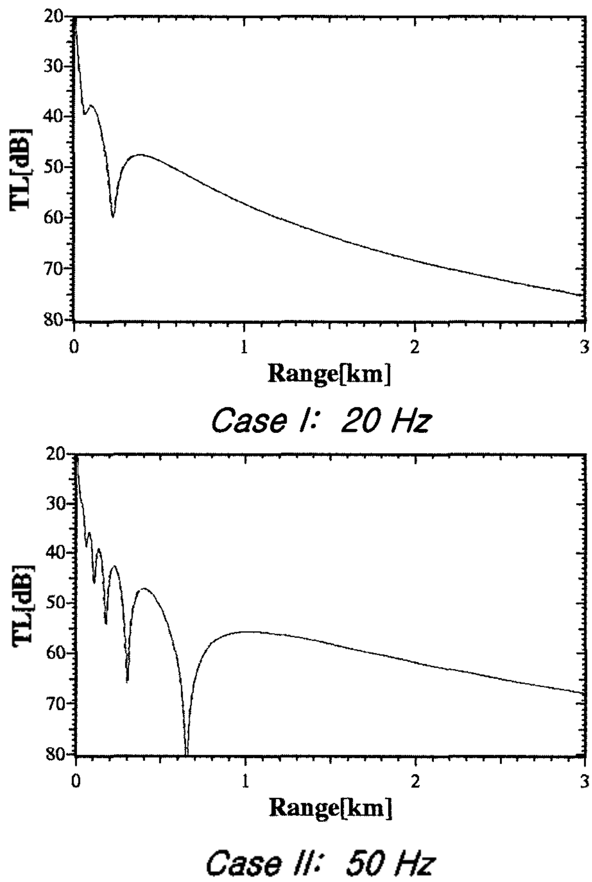

Figure 4. Solutions of the half space problem.

computation time, there is an increase of approximately 20 % compared to RAM due to the rotation manipulation involved.

5.1. Half Space Problem

The environment is given in Fig. 3, which is self-explanatory.

Both the PE and the analytic solutions, shown in Fig. 4 as a single line, coincide exactly and show the Lloyd

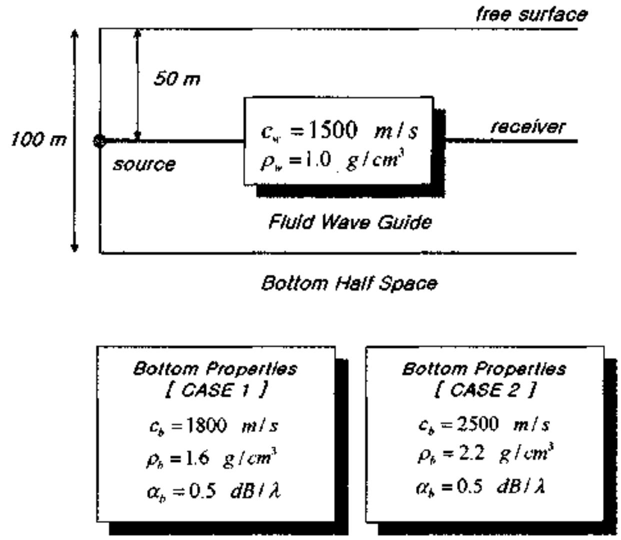

Figure 5. Schematic of the Pekeris waveguide problem.

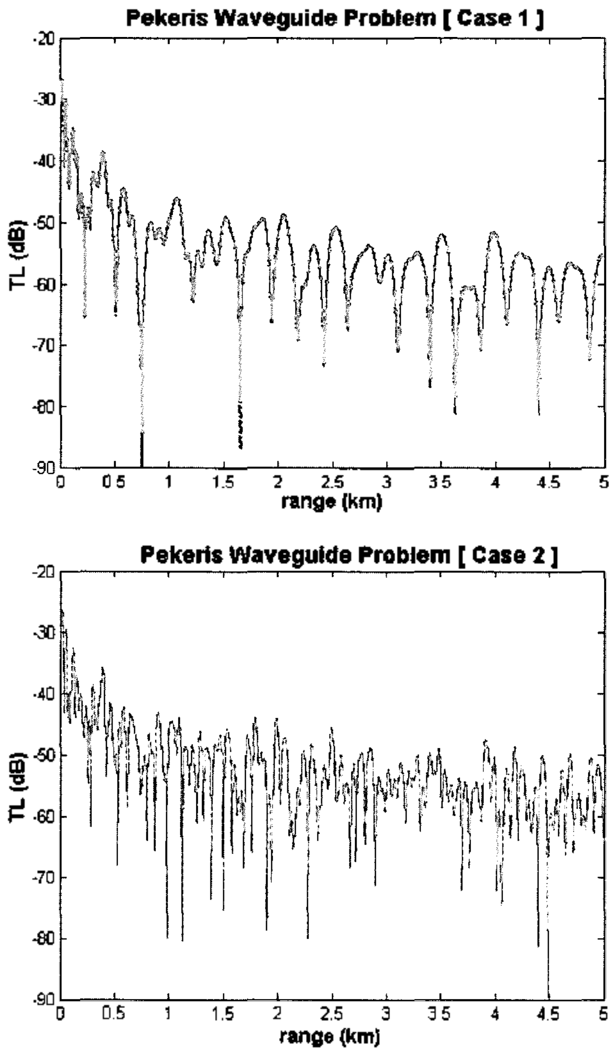

Figure 6. Solutions of the Pekeris waveguide problem.

mirror beam patterns for different frequencies.

5.2. Pekeris Wave-guide Problem

The Pekeris environment is given in Fig. 5. In Fig. 6, the OASES reference solution[12] is given by solid line and present solution is the dash-dotted line. For moderate bottom medium impedance contrast, the numerical specification of step increment dr is less stringent. Half wavelength of the acoustic wave (7.5 m in case 1 and 4 m in case 2) was used avoiding the numerical dispersion successfully, as can be seen from the match of the solutions. As the ocean bottom impedance contrast becomes pronounced, due to the additional propagating modes occurring within the waveguide the propagation pattern becomes more complicated as in Case 2.

5.3. ASA Benchmark: Wedge Problem

Acoustical Society of America (ASA) provides the following range dependent benchmark problem[13]. The ASA wedge problem depicted in Fig. 7 is a representative example to validate the range-dependent numerical solver. The source frequency is 25 Hz located at the midpoint of the 200 m depth furthest away from the apex. The wedge angle is 2.86° which is representative of the sloping ocean. Fig. 8 shows comparison of solutions of the ASA wedge problem with RAM[4] and present algorithm. In Fig. 8, transmission losses along receivers located at depths of 30 m and 150 m are shown. For the receiver located at 30 m, a kink appears at the point of bottom crossover for the RAM solution but has disappeared in the present solution due to increased stability. As can be witnessed from these numerical solutions, the present numerical code of energy conservation model is seen to replicate reliable solutions.

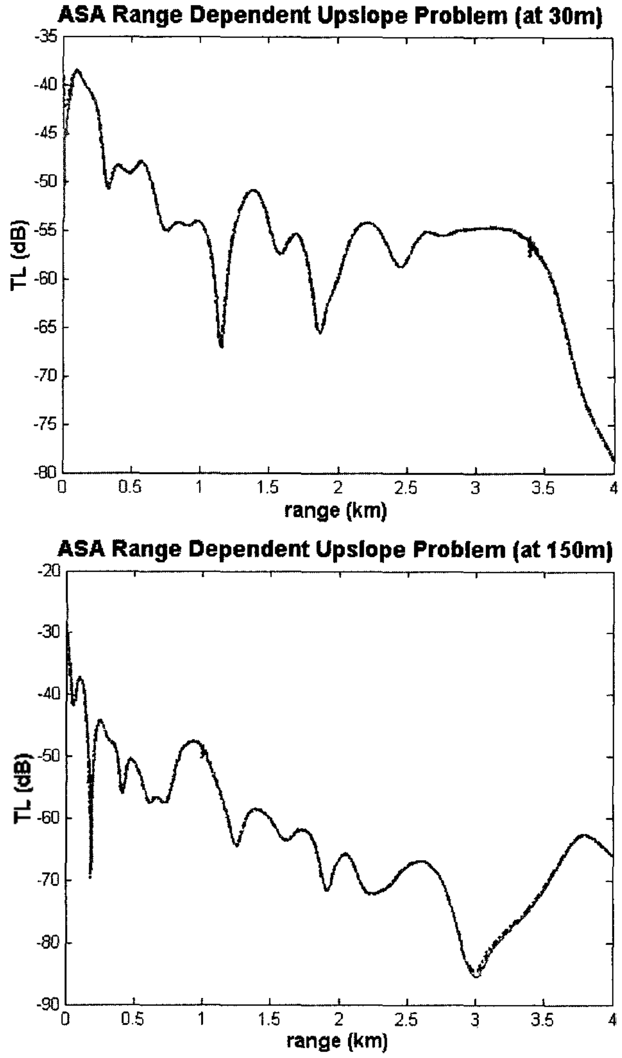

Figure 8. Solutions of the ASA wedge problems (solid: reference by RAM, dashed line: computation).

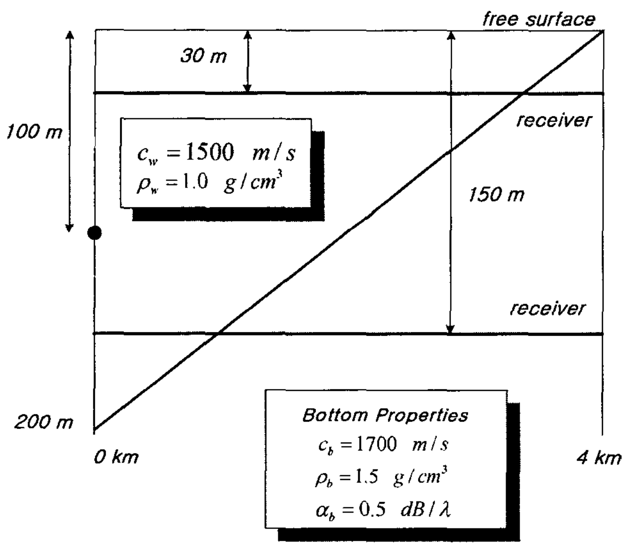

Figure 7. Schematic of the ASA wedge problem.

VI. Concluding Remarks

We have solved the wave propagation problem in a range-dependent ocean environment having negligible back-propagating field by the parabolic equation technique.

The out-going solution was obtained by the split-step integrated formulation. In order to construct the pseudo-differential propagation operator into a feasible operator the Padé approximation method was used but modified for more stability based on the branch line rotation method. As a result of the modification, the stability of the approximated operator can be further improved. Through representative benchmark tests the implemented numerical code was validated.

References

1. B. T. R. Wetton and G. H. Brooke, "One-way wave equations for seismo-acoustic propagation in an ocean overlying an elastic bottom," *J. Acoust. Soc. Am.* **87**, 624-632, 1990.
2. M. D. Collins, "Higher-order Padé approximations for accurate and stable elastic parabolic equations with application to interface wave propagation," *J. Acoust. Soc. Am.* **89**, 1050-1057, 1991.
3. M. D. Collins, "A split-step Padé solution for the parabolic equation," *J. Acoust. Soc. Am.* **93**, 1736-1742, 1993.
4. M. D. Collins, *User's Guide for RAM*, Naval Research Laboratory, Washington D.C., 1995.
5. F. A. Milinazzo, C. A. Zala, and G. H. Brooke, "Rational square-root approximations for parabolic equation algorithms," *J. Acoust. Soc. Am.* **101**, 760-766, 1997.
6. M. B. Porter, F. B. Jensen, and C. M. Ferla, "The problem of energy conservation in one-way models," *J. Acoust. Soc. Am.* **90**, 1058-1067, 1991.
7. M. D. Collins and E. K. Westwood, "A higher-order energy-conserving parabolic equation for range-dependent ocean depth, sound speed, and density," *J. Acoust. Soc. Am.* **89**, 1068-1075, 1991.
8. A. Pozzi, *Applications of Padé Approximation Theory in Fluid Dynamics*, World Scientific, 1994.
9. G. A. Baker, Jr. and P. Graves-Morris, *Padé Approximants Part I: Basic Theory (Encyclopedia of mathematics and Its Applications Volume 13)*, Addison-Wesley Publishing Company, 1981.
10. T. Kim, Numerical Observation of the Statistics of Acoustic Wave propagation through a Shallow Water Environment, Master's thesis, Dept. of Naval Architecture and Ocean Engineering, Seoul National University, 2002.
11. F. B. Jensen, W. A. Kuperman, M. B. Porter, and H. Schmidt, *Computational Ocean Acoustics*, AIP Press, New York, 1994.
12. H. Schmidt, *SAFARI : seismo-acoustic fast field algorithm for range independent environments. User's guide*, SACLANT ASW Research Centre, La Spezia, Italy, 1987.
13. L. B. Felsen, "Benchmarks : An option for quality assessment," *J. Acoust. Soc. Am.* **87**, 1497-1498, 1990.

[Profile]

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