New Formulation Method for Reducing the Direct Kinematic Complexity of the 3-6 Stewart-Gough Platform

Se-Kyong Song and Dong-Soo Kwon

Abstract: This paper presents a new formulation to simplify the three resulting constraint equations of the direct kinematics of the 3-6 (Stewart-Gough) Platform. The conventional direct kinematics of the 3-6 Platform has been formulated through complicated steps with trigonometric functions in three angle variables and thus results in the computational burden. In order to reduce the formulation complexity, we replace an angle variable into a length one and express three connecting joints on the moving platform in the same frame. The proposed formulation yields considerable abbreviation of the number of the calculation terms involved in the direct kinematics. It is verified through a series of simulation results.

Keywords: direct kinematics, tetrahedron, Stewart-Gough Platform and parallel manipulator

I. Introduction

The (Stewart-Gough) Platforms have been adopted in a variety of application fields such as simulators, haptic devices, F/T sensors and micro-manipulators because they have good kinematic properties of high rigidity, high local dexterity, low inertia effect and compact size. The Platform consists of two rigid plates (the moving and the base platforms). In the parallel manipulator research fields, there still exist challenging problems. One of major problems is the direct kinematics. In general, the direct kinematics is derived through complicated formulation procedures and requires heavy computation. However, it is necessary to quickly determine the direct kinematic solution.

Many efficient algorithms have been proposed to determine the direct kinematic solution of the Platform [1-4,7-9]. This paper focuses on deriving the direct kinematics of the 3-6 Platform that has known to have sixteen different solutions by Bezout's theorem [1].

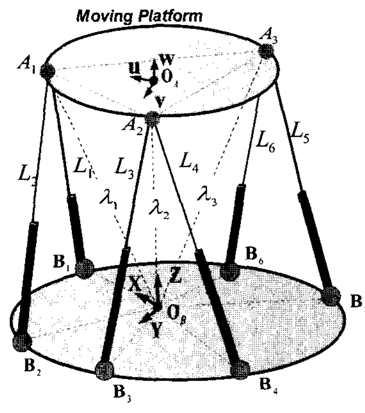

Fig. 1. The 3-6 Stewart-Gough Platform composed of three tetrahedrons.

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The direct kinematics of the 3-6 Platform has been derived in the following three two approaches for different purposes: the *Polynomial-based* and the *Numerical-iterative* Approaches. The *Polynomial-based* Approach is a method to derive a high-order polynomial with a single unknown variable, using a pair of tan-half-angular displacements, by the elimination theories [2-4]. Each real root of the closed-form polynomial corresponds to exact configurations of the Platform. Thus, we can obtain the physical insight of the Platform configurations. Griffis and Duffy [2], Innocenti et. al [3] and Nanua et. al [4] presented a 16th-order polynomial. These direct kinematics has been complicatedly derived through several matrix manipulations with the closed-loop kinematic equations. Since numerical schemes for the polynomial find all solutions, one of them must be always determined as the actual solution. The *Polynomial-based* Approach seems to be much slower than numerical methods based on numerical iteration such as the Newton-Raphson method [5]. Moreover, root finding of a polynomial is very sensitive to the accuracy of its coefficients [6]. Thus the *Polynomial-based* Approach may be appropriate for the design problem determining all the roots of the polynomial rather than an actual solution.

Thus, for real-time calculation of the direct kinematic solutions, numerical iteration methods, called the *Numerical-iterative* Approach, have been rigorously pursued than the *Polynomial-based* Approach. Ku [7] and Liu et. al [8] proposed a numerical procedure using the Newton-Raphson method instead of a 16^{th}-order closed-form polynomial. The direct kinematics in these research works is formulated with three unknown angle variables to determine of each link posture: 1) for the 3-3 Platform, the joint mechanism on the base platform is modeled as a spherical four-bar mechanism with an unknown angle variable [2,9], 2) for the 3-3 or the 3-6 Platforms, the geometric relationship between the base platform and a triangle composed of two links is defined as an unknown angle variable [3,4,7]. In the 3-3 and the 3-6 Platforms, the three resulting equations are yielded from three constraints of the fixed distances between the two adjacent joints on the moving platform. However, these methods lead to complicated formulation procedures and computational burden due to using three unknown angle variables.

In the previous research works of the *Numerical-iterative* Approach, it seems to have escaped notice that the nonlinearity and the number of the calculations of the resulting constraint equations strongly depends on type of unknown variables and coordinates, and this is what leads to the computational burden.

The purpose of this paper is to present an efficient formulation approach that can reduce the complexity of the constraint equations and result in fast computation of the direct kinematic solution by which the three position vectors on the moving platform are expressed with respect to the same frame and their lengths are established as unknown variables. This paper is organized as follows. Section 2 explains a new formulation approach, called *Tetrahedron Approach*, based on the geometric properties of a tetrahedron and presents the direct kinematic analysis for the 3-6 Platform through comparing the conventional approach with the proposed approach. Then, in Section 3 we show numerical results.

II. Direct Kinematic analysis of the 3-6 platform using three tetrahedrons

We proposed the *Tetrahedron Approach* for easy derivation of the direct kinematics of some parallel manipulators with a tetrahedron structure [11,12]. The idea of the *Tetrahedron Approach* arises from that a link between the moving and the base platform is a line constraint that reduces the degree of freedom of a point on the moving platform with respect to the reference (base) frame. In sum, three line constraints are sufficient to identify a point on the moving platform. Geometrically, this geometric structure is a tetrahedron. The proposed *Tetrahedron Approach* will be briefly addressed in the following.

1. Tetrahedron approach

Terminologies and notations used in this paper are defined as follows: firstly, assume that there exist two vectors among the six lines that compose a tetrahedron, as depicted in a tetrahedron of Fig. 1. A *base* is defined as the plane with two vectors that can be expressed with respect to a known reference coordinate. Three lines that lie on the base are called *base lines* and their vectors are called *base vectors*. The three lines that connect the base to a vertex are defined as *space lines* and their vectors are called *space vectors*. The vertex rising above the base from the three space lines is defined as a *top vertex*. A set of three mutual-orthogonal unitary vectors formed from two base vectors is called a *tetrahedron coordinate*. A tetrahedron that satisfies the *Tetrahedron Proposition* is called a *directional tetrahedron*.

[B]=[**X**,**Y**,**Z**] and **[M]**=[**u**,**v**,**w**] are defined as a base frame fixed on the base platform and a moving frame attached on the moving platform, respectively. O_A and O_B are the origins of the moving and the base platform, respectively. **H** is the position vector of the moving platform with respect to the base frame. L_i is i-th link length. B_i and A_i indicate the i-th joints on the moving and the base platform. R and r are the radius of the moving and the base platform. The moving frame **[M]** can be expressed with the three column vectors of the orientation matrix with *roll*(α), *pitch*(β) and *yaw*(γ) angles with respect to the base frame. The three mutual-orthogonal column vectors are denoted by **u**, **v** and **w** in order: **[R]**=[**u**,**v**,**w**]. The orientation matrix **[R]** is identical to the moving frame **[M]**.

From the viewpoint of the procedure in solving the forward kinematics, the *Tetrahedron Approach* differs from the previous works that use tetrahedrons [12,13]. In the *Tetrahedron Approach*, the formulation procedure is simply reduced to a process of first identifying a tetrahedron based on the geometric structure of the linkages, and then using it as a basis to identify and pile up next tetrahedrons. The concepts mentioned above were proposed as the *Tetrahedron Proposition* and the *Tetrahedron Theorem* [11,12]. The *Tetrahedron Proposition* is defined to uniquely identify a tetrahedron based on the geometrical relationship between the moving and the base platform. The *Tetrahedron Theorem* is defined to be a condition that there exists a unique closed-form solution of the direct kinematics and is also used as a formulation guideline to perform the *Tetrahedron Approach* to solve the direct kinematics by using the Tetrahedron Proposition.

2. Deriving the direct kinematics

A new efficient formulation approach using three tetrahedrons is presented to derive a simple form of the direct kinematic solution of the 3-6 Platform. Based on the geometric properties of a tetrahedron, the formulation approach can effectively perform the direct kinematic analysis of the 3-6 Platform when the geometric structure of the 3-6 Platform can be broken down into three tetrahedrons.

In general, the formulation complexity of the direct kinematics greatly depends on the choice of coordinates and unknown variables. In sum, the complexity is mainly caused by the location of the origins of the local frames which are used in the process of the formulation.

The objective of the direct kinematics of the 3-6 Platform is to obtain the position vectors ($\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$) of three connecting joints on the moving platform with respect to the base frame **[B]**. Therefore, most of the existing direct kinematic analyses adopt a formulation approach of utilizing the geometric constraint that the distance ($|\mathbf{A}_i-\mathbf{A}_j| = C_i$) between the two adjacent joints should be the same irrespective to the reference frame **[M]** or **[B]**. Then the three position vectors must satisfy the following three constraint equations simultaneously:

$$\begin{aligned}(\mathbf{A}_i-\mathbf{A}_j)_B\cdot(\mathbf{A}_i-\mathbf{A}_j)_B &= (\mathbf{A}_i\cdot\mathbf{A}_i+\mathbf{A}_j\cdot\mathbf{A}_j-2\mathbf{A}_i\cdot\mathbf{A}_j)_B \\ &= (\mathbf{A}_i-\mathbf{A}_j)_M\cdot(\mathbf{A}_i-\mathbf{A}_j)_M = C_i^2,\ for\ i\neq j, i,j=1,2,3.\end{aligned} \quad (1)$$

In Eq. (1), the dot products yielded between the three position vectors consist of the *Auto-dot* products and *Cross-dot* products. In this paper, the Auto-dot product means the multiplication between the same vectors such as $\mathbf{A}_i\cdot\mathbf{A}_i$ and the *Cross-dot* products mean the multiplication between different vectors such as $\mathbf{A}_i\cdot\mathbf{A}_j$.

- The previous approach

Nanua et. al proposed a formulation approach for the direct kinematics of the 3-6 Platform that includes three passive angles (φ_i) as unknown variables to find the position vectors ($\mathbf{A}_i$) [4,7]. As shown in the left side of Fig. 2, the local frame $[\mathbf{x}_i,\mathbf{y}_i,\mathbf{z}_i]$ is moved on the line between $\mathbf{B}_i$ and $\mathbf{B}_{i+1}$ according to two link lengths (L_1,L_2) and is obtained from the geometric relationship between the base platform and the triangle ($\Delta A_1\mathbf{B}_1\mathbf{B}_2$):

$$\mathbf{y}_i = (\mathbf{B}_i - \mathbf{B}_{i+1})/|\mathbf{B}_i - \mathbf{B}_{i+1}| = y_{i1}\mathbf{X} + y_{i2}\mathbf{Y},$$
$$\mathbf{k}_i = \mathbf{Z},\ \ \mathbf{x}_i = \mathbf{y}_i \times \mathbf{z}_i = x_{i1}\mathbf{X} + x_{i2}\mathbf{Y}. \qquad (2)$$

The vector ($\mathbf{A}_i$) is obtained through the following steps with respect to a local frame $[\mathbf{x}_i,\mathbf{y}_i,\mathbf{z}_i]$:

$$\mathbf{A}_i = \mathbf{L}_i + \mathbf{B}_i = A_{i1}\mathbf{x}_i + A_{i2}\mathbf{y}_i + A_{i3}\mathbf{z}_i = \mathbf{T}_i + \mathbf{Q}_i, \qquad (3a)$$

$$\mathbf{Q}_i = \mathbf{B}_i - M_i\mathbf{y}_i,\ \ \mathbf{T}_i = T_i\cos(\varphi_i)\mathbf{x}_i + T_i\sin(\varphi_i)\mathbf{z}_i, \qquad (3b)$$

where

$$M_i = (|\mathbf{B}_i - \mathbf{B}_{i+1}|^2 + \mathbf{L}_i^2 - \mathbf{L}_{i+1}^2)/2|\mathbf{B}_i - \mathbf{B}_{i+1}|,$$

$$T_i = (L_i^2 - M_i^2)^{1/2},\ \ \mathbf{B}_i = B_{i1}\mathbf{X} + B_{i2}\mathbf{Y} + B_{i3}\mathbf{Z}.$$

The details of the above expressions can be found in [4,7].

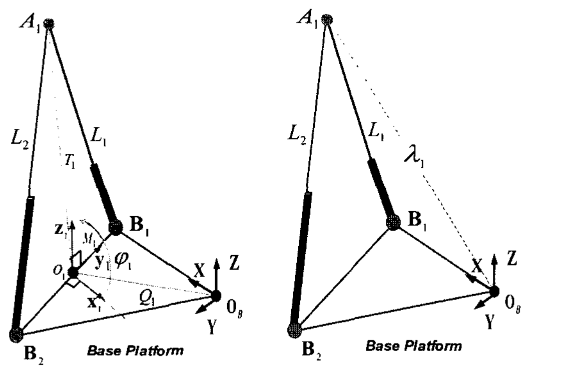
Therefore, the dot products between the three vectors ($\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$) are represented with respect to the local frame as follows:

Auto-dot products:

$$(\mathbf{A}_i \cdot \mathbf{A}_i)_B = A_{i1}A_{i1} + A_{i2}A_{i2} + A_{i3}A_{i3} = B_i^2 + 2\mathbf{B}_i \cdot \mathbf{L}_i + L_i^2, \qquad (4a)$$

Cross-dot products:

$$(\mathbf{A}_i \cdot \mathbf{A}_j)_B = (A_{i1}A_{j1}\mathbf{x}_i \cdot \mathbf{y}_j + A_{i1}A_{j2}\mathbf{x}_i \cdot \mathbf{y}_j) + (A_{i2}A_{j1}\mathbf{y}_i \cdot \mathbf{x}_j + A_{i2}A_{j2}\mathbf{y}_i \cdot \mathbf{y}_j) + (A_{i3}A_{j3}). \qquad (4b)$$



The previous approach The *Tetrahedron Approach*

Fig. 2. Geometric comparison between the *Tetrahedron Approach* with a length variable and the conventional approach with an angle variable.

As known from Eqs. (4a) and (4b), the *Cross-dot* products especially yield a number of algebraic manipulations because the three local frames are all different. The vector position ($\mathbf{A}_i$) must be expressed with an unknown angle variable (φ_i), as shown in Eq. (3b). Therefore, once substituting the dot products in Eqs. (4a) and (4b) into Eq. (1), the three constraint equations involve a number of the complicated dot products that must be calculated in the computational process for the direct kinematics.

- *The Tetrahedron Approach*

Though there is duality between an angle and a length variables, as addressed by Parenti-Castelli and Gregorio [10], introducing length variables has some advantage of easily deriving and simplifying the resulting equations over angle ones.

In order to reduce the formulation complexity and computational burden mentioned above, we introduce a specifically-designed formulation procedure, called the *Tetrahedron Approach*, that can allow easy derivation of the direct kinematics and dramatically reduce the number of the dot products. As depicted in Fig. 1, if the position vector ($\mathbf{A}_i$) is directly expressed in the same frame (the base frame), the Cross-dot products can be eliminated, and if the length of the vector ($\mathbf{A}_i$) is established as an unknown variable (λ_i), the *Auto-dot* products can be reduced to a simplified scalar. Now, assume that the length of the variable (λ_i) is known and its vector ($\boldsymbol{\lambda}_i$) is expressed with respect to the base frame $[\mathbf{X},\mathbf{Y},\mathbf{Z}]$. Under this condition, the dot products of Eq. (1) are dramatically simplified and the number of the calculations included in Eqs. (4a) and (4b) are abbreviated as follows:

$$(\mathbf{A}_i \cdot \mathbf{A}_i)_B = \lambda_i^2,\ \ (\mathbf{A}_i \cdot \mathbf{A}_j)_B = (A_{i1}A_{j1} + A_{i2}A_{j2} + A_{i3}A_{j3}), \qquad (5a)$$

$$(\mathbf{A}_i - \mathbf{A}_j)_B \cdot (\mathbf{A}_i - \mathbf{A}_j)_B = \lambda_i^2 + \lambda_j^2 - 2\mathbf{A}_i \cdot \mathbf{A}_j = C_i^2, \qquad (5b)$$

where

$$\boldsymbol{\lambda}_i = \mathbf{A}_i = A_{i1}\mathbf{X} + A_{i2}\mathbf{Y} + A_{i3}\mathbf{Z}.$$

For the *Tetrahedron Approach*, the Auto-dot products reduce to a single value (λ_i) and the number of the Cross-dot products reduces from 9 to 3. The geometric comparison of the two approaches is depicted in Fig. 2.

Though it may be difficult to directly compare the effectiveness and calculation burden between the two approaches, we carry out a quantitative analysis under the assumption that the number of the calculations involved in three components (A_{i1},A_{i2},A_{i3}) of a vector (A_i) is the same for the both approaches and is symbolized as Θ. The calculation number between the three local frames is expressed in *italics*. As shown in Table 1, the number of calculations is greatly reduced in the *Tetrahedron Approach*. Moreover, Nanua et. al's approach requires calculation of several trigonometric functions where $\sin(\varphi_i)$ and $\cos(\varphi_i)$ may lead to additional computation burden. Consequently, the *Tetrahedron Approach* has an advantage of reducing the computational burden.

As shown in Fig. 1, when three unknown length variables (λ_1,λ_2,λ_3) are added to construct three directional tetrahedrons, three top vertices ($\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$) are obtained as functions of λ_1, λ_2 and λ_3.

Four vertices (A_1,B_1,B_2,O_B) yield the first tetrahedron which has three space lines (L_1,L_2,λ_1) and two base vectors ($\mathbf{B}_1$,$\mathbf{B}_2$). By applying the *Tetrahedron Proposition* to this tetrahedron, the first top vertex ($\mathbf{A}_1$) can be obtained with respect to the base frame $[\mathbf{X},\mathbf{Y},\mathbf{Z}]$ as the following function of λ_1:

Table 1. The comparison of the number of calculations included in the conventional approach and the *Tetrahedron Approach*

	The previous approach by Nanua et. al		The *Tetrahedron Approach*	
Auto-dot products $(\mathbf{A}_i\cdot\mathbf{A}_i)_B$	$A_{i1}A_{i1}+A_{i2}A_{i2}+A_{i3}A_{i3}$		λ_i^2	
	Multiplication	Addition	Multiplication	Addition
	$\Theta^2\times 3$	3	1	0
Cross-dot products $(\mathbf{A}_i\cdot\mathbf{A}_j)_B$	$(A_{i1}A_{j1}\mathbf{x}_i\cdot\mathbf{y}_j+A_{i1}A_{j2}\mathbf{x}_i\cdot\mathbf{y}_j)+(A_{i3}A_{j3})$ $+(A_{i2}A_{j1}\mathbf{y}_i\cdot\mathbf{x}_j+A_{i2}A_{j2}\mathbf{y}_i\cdot\mathbf{y}_j)$		$(A_{i1}A_{j1}+A_{i2}A_{j2}+A_{i3}A_{j3})$	
	Multiplication	Addition	Multiplication	Addition
	$\Theta^2\times 5+2\times 4$	$4+1\times 4$	$\Theta^2\times 3$	3
Three constrained equations : f_1, f_2 and f_3	Multiplication	Addition	Multiplication	Addition
	$(\Theta^2\times 3+\Theta^2\times 5+2\times 4)\times 3$	$(3+4+1\times 4)\times 3+2$	$(1+\Theta^2\times 3)\times 3$	$(3)\times 3+2$
Additional calculation	Trigonometric functions: $\sin(\varphi_i)$ and $\cos(\varphi_i)$		None	

$$\boldsymbol{\lambda}_1=\lambda_{11}\mathbf{X}+\lambda_{12}\mathbf{Y}+\lambda_{13}\mathbf{Z}\ ,\quad \mathbf{A}_1=\boldsymbol{\lambda}_1\ , \tag{6}$$

where

$$\lambda_{11}=\frac{\lambda_1^2+B_1^2-L_1^2}{2B_1}\ ,\quad \lambda_{12}=\frac{\lambda_1^2+B_2^2-L_2^2-2\lambda_{11}(\mathbf{B}_2\cdot\mathbf{X})}{2(\mathbf{B}_2\cdot\mathbf{Y})}\ ,$$
$$\lambda_{13}^2=\lambda_1^2-\lambda_{11}^2-\lambda_{12}^2\ .$$

In Eq. (6), λ_{11}, λ_{12} and λ_{13} are functions of λ_1^2. The tetrahedron coordinate is selected to be identical to the base frame in order to eliminate the Cross-dot products.

In a similar manner, four vertices (A_2,B_3,B_4,O_B) and four vertices (A_3,B_5,B_6,O_B) yield the second and the third tetrahedrons which also respectively satisfy the Tetrahedron Proposition. Without loss of generality, assume that the joint positions are symmetrically arranged in each circular plane of the base and the moving platform. Then, the two top vertices ($\mathbf{A}_2$,$\mathbf{A}_3$) can respectively be expressed with respect to the base frame as the following functions of λ_2 and λ_3:

$$\boldsymbol{\lambda}_2=\lambda_{21}(-\mathbf{X}+\sqrt{3}\mathbf{Y})/2+\lambda_{22}(-\sqrt{3}\mathbf{X}-\mathbf{Y})/2+\lambda_{23}\mathbf{Z}\ ,$$
$$\mathbf{A}_2=\boldsymbol{\lambda}_2=(-\lambda_{21}-\sqrt{3}\lambda_{22})\mathbf{X}/2+(\sqrt{3}\lambda_{21}-\lambda_{22})\mathbf{Y}/2+\lambda_{23}\mathbf{Z}\ , \tag{7a}$$
$$\boldsymbol{\lambda}_3=\lambda_{31}(-\mathbf{X}-\sqrt{3}\mathbf{Y})/2+\lambda_{32}(\sqrt{3}\mathbf{X}-\mathbf{Y})/2+\lambda_{33}\mathbf{Z}\ ,$$
$$\mathbf{A}_3=\boldsymbol{\lambda}_3=(-\lambda_{31}+\sqrt{3}\lambda_{32})\mathbf{X}/2+(-\sqrt{3}\lambda_{31}-\lambda_{32})\mathbf{Y}/2+\lambda_{33}\mathbf{Z}\ , \tag{7b}$$

where

$$\lambda_{21}=\frac{\lambda_2^2+B_3^2-L_3^2}{2B_3}\ ,\quad \lambda_{23}^2=\lambda_2^2-\lambda_{21}^2-\lambda_{22}^2\ ,$$
$$\lambda_{22}=\frac{\lambda_2^2+B_4^2-L_4^2-2\lambda_{21}[\mathbf{B}_4\cdot(-\mathbf{X}+\sqrt{3}\mathbf{Y})/2]}{2[\mathbf{B}_4\cdot(-\sqrt{3}\mathbf{X}-\mathbf{Y})/2]}\ ,$$
$$\lambda_{31}=\frac{\lambda_3^2+B_5^2-L_5^2}{2B_5}\ ,\quad \lambda_{33}^2=\lambda_3^2-\lambda_{31}^2-\lambda_{32}^2\ ,$$
$$\lambda_{32}=\frac{\lambda_3^2+B_6^2-L_6^2-2\lambda_{31}[\mathbf{B}_6\cdot(-\mathbf{X}-\sqrt{3}\mathbf{Y})/2]}{2[\mathbf{B}_6\cdot(\sqrt{3}\mathbf{X}-\mathbf{Y})/2]}\ .$$

The components of $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$ are functions of λ_2^2 and λ_3^2, respectively.

The numerical algorithm of using the proposed *Tetrahedron Approach* needs to include the following if-else statements, because the dot-product signs between the components of $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$ are determined to be either positive or negative according to the angles between two adjacent vectors:

$$\boldsymbol{\lambda}_i\cdot\mathbf{B}_{2i-1}=\begin{cases}+\lambda_{i1}B_{2i-1}, & if\quad \lambda_i^2+B_{2i-1}^2\ge L_{2i-1}^2,\\ -\lambda_{i1}B_{2i-1}, & else\ \lambda_i^2+B_{2i-1}^2<L_{2i-1}^2,\end{cases} \tag{8a}$$

$$\boldsymbol{\lambda}_i\cdot\mathbf{B}_{2i}=\begin{cases}+\lambda_{i1}B_{2i,1}+\lambda_{i2}B_{2i,2}, & if\quad \lambda_i^2+B_{2i}^2\ge L_{2i}^2,\\ -\lambda_{i1}B_{2i,1}-\lambda_{i2}B_{2i,2}, & else\ \lambda_i^2+B_{2i}^2<L_{2i}^2,\end{cases} \tag{8b}$$

$$\lambda_{i3}=\begin{cases}+\sqrt{\lambda_i^2-\lambda_{i1}^2-\lambda_{i2}^2}, & if\ \lambda_i^2\ge\lambda_{i1}^2+\lambda_{i2}^2,\\ +|\lambda_i|, & else\ \lambda_i^2<\lambda_{i1}^2+\lambda_{i2}^2.\end{cases} \tag{8c}$$

Here, $B_{2i,1}$ and $B_{2i,2}$ are components of $\mathbf{B}_{2i}$ and $B_{2i-1}$is a norm of $\mathbf{B}_{2i-1}$.

The three constraint equations of Eq. (5b) can be rewritten as the following three simultaneous functions for i = 1,2,3:

$$\begin{aligned}f_i(\lambda_i,\lambda_j)&=(\mathbf{A}_j-\mathbf{A}_i)\cdot(\mathbf{A}_j-\mathbf{A}_i)-C_i^2\\ &=\lambda_i^2+\lambda_j^2-2\mathbf{A}_i\cdot\mathbf{A}_j-C_i^2=0\end{aligned} \tag{9}$$

where

$$\mathbf{A}_i\cdot\mathbf{A}_j=(A_{i1}A_{j1}+A_{i2}A_{j2}+A_{i3}A_{j3})\ .$$

A_{ij} of $\mathbf{A}_i$ are obtained from the components in Eqs. (6), (7a) and (7b). Once substituting the position vectors ($\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$) with λ_1, λ_2 and λ_3 into Eq. (9), f_1, f_2 and f_3 are represented as simplified forms:

$$F_i(\lambda_i,\lambda_j)=G_{i1}\lambda_i^2+G_{i2}\lambda_j^2+G_{i3}\lambda_i^2\lambda_j^2+G_{i4}-2A_{i3}A_{j3}\ . \tag{10}$$

G_{ik} are constant coefficients determined from the lengths of the six links and the given parameters of the 3-6 Platform for k = 1,2,3,4. The detailed expressions of F_i and G_{ij} are addressed in Appendix A.

From the three nonlinear equations (F_i), three two-variable polynomials (g_i) for i = 1,2,3 are obtained by rearranging Eq. (10) and squaring $2A_{i3}A_{j3}$:

$$\begin{aligned}g_i(\lambda_i,\lambda_j)&=E_{i1}+E_{i2}\lambda_i^2+E_{i3}\lambda_j^2+E_{i4}\lambda_i^2\lambda_j^2\\ &+E_{i5}\lambda_i^2\lambda_j^4+E_{i6}\lambda_i^4\lambda_j^2+E_{i7}\lambda_i^4\lambda_j^4\ .\end{aligned} \tag{11}$$

E_{il} are constant coefficients determined by G_{ik} in Eq. (10) for l = 1,2,...7. It is worth remarking that the three two-variable polynomials could be represented as a single closed-form polynomial with simplified coefficients in one variable (λ_1^2) by introducing well-known eliminant methods which were

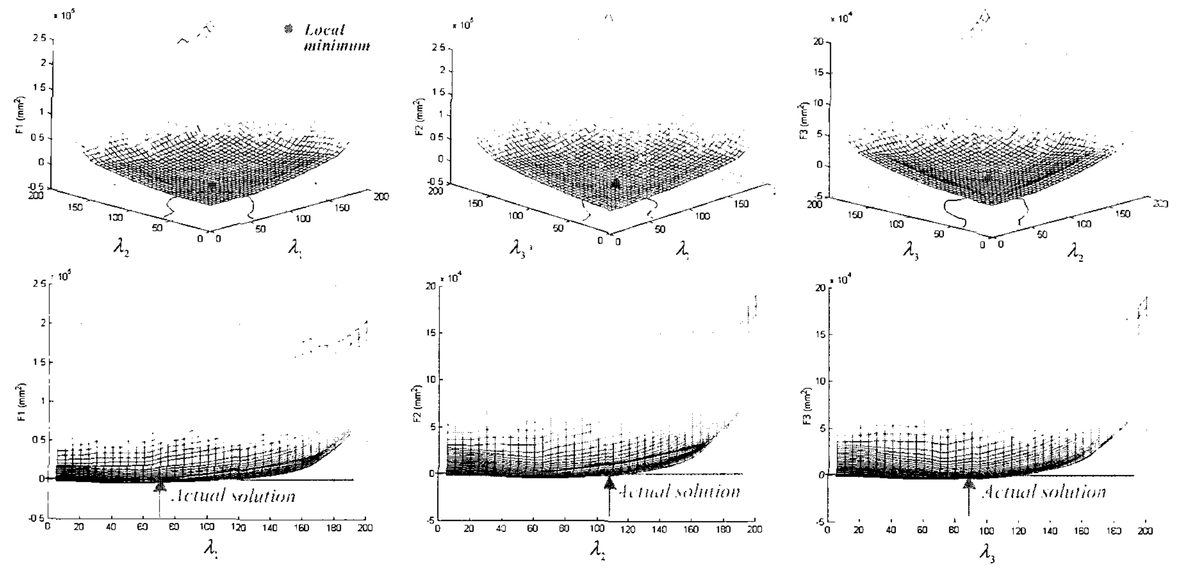


Fig. 3. Contours of the three constraint equations at the *arbitrary* configuration.

applied in much of the literature [2-4,6,8,15]. However, derivation of the closed-form polynomial is beyond the scope of this paper.

Since F_1, F_2 and F_3 are nonlinear, it is not possible to solve them explicitly. The solution of the direct kinematics is simultaneously calculated by the three-dimensional Newton-Raphson method that is expanded in the first order approximation of a Taylor series.

$$F_i(\lambda_1[n+1],\lambda_2[n+1]) = F_i(\lambda_1[n],\lambda_2[n]) + \frac{\partial F_i(\lambda_1[n],\lambda_2[n])}{\partial \lambda_1}(\lambda_1[n+1]-\lambda_1[n]) + \frac{\partial F_i(\lambda_1[n],\lambda_2[n])}{\partial \lambda_2}(\lambda_2[n+1]-\lambda_2[n]) \quad (12)$$

The partial derivatives of the F_i are shown in Appendix B.

Once the initial values are arbitrarily chosen as $\lambda_i[0]$ in a feasible region ($0 < \lambda_i[0] < 2L_i$) where there exist the actual solution, they are repeatedly substituted into the three simultaneous functions (F_1,F_2,F_3). The previous values are updated in the iterative process until the error values of the difference between the updated values ($\boldsymbol{\lambda}$[n+1]) and the previous values ($\boldsymbol{\lambda}$[n]) become smaller than a prescribed convergence tolerance. The iteration procedure is expressed in a matrix form as:

$$[\lambda[n+1]] = [\lambda[n]] - \left[\nabla \mathbf{F}^{-1}\right][\mathbf{F}], \quad (13)$$

$$\| \boldsymbol{\lambda}[n+1] - \boldsymbol{\lambda}[n] \| < \textit{Convergence tolerance}.$$

[$\boldsymbol{\lambda}$] and [**F**] are 3x1 matrix forms of λ_i and F_i, respectively. $[\nabla \mathbf{F}^{-1}]$ is an inverse matrix of the 3x3 partial derivative matrix of [**F**] in terms of [$\boldsymbol{\lambda}$], as expressed in Appendix B.

When the error values exist within the prescribed convergence tolerance, the three calculated values are finally substituted into the three top vertices ($\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$) through the iterative process, respectively. Accordingly, the position (**H**) and the orientation [**u**,**v**,**w**] of the moving platform can be determined from the following geometrical relationships:

$$\textit{Position: } \mathbf{H} = (\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3)/3, \quad (14a)$$

$$\textit{Orientation: } \mathbf{u} = \frac{\mathbf{A}_1 - \mathbf{A}_2}{\sqrt{3}r},\ \mathbf{v} = \frac{-(\mathbf{A}_3 - \mathbf{H})}{r},\ \mathbf{w} = \mathbf{u} \times \mathbf{v}. \quad (14b)$$

When the three connecting joints on the moving platform of the 3-6 Platform are symmetrically arranged, two feasible solutions are mechanically possible, unless a pair of the two adjacent links is coplanar with the moving platform. One of them will be converged to the actual solution according to the initial conditions. It is noted that the *Tetrahedron Approach* does not use the parameterization of the matrix elements of the orientation or the trigonometric equations that are mainly responsible for the complicated formulation and computational burden.

III. Numerical examples

For the proposed *Tetrahedron Approach*, the existence and convergence of the direct kinematic solution is verified through a series of simulations. The Newton-Raphson method will work well, if the contours of the three simultaneous functions [**F**] are moderate in the neighborhood of the actual solution, and if the initial values that are guessed to start the algorithm are not far away from the actual solution. However, if there exist local minima (convex points) in the contours of [**F**], the partial derivatives matrix of [**F**], when crossing the local minima, drops into a numerical singularity which prevents numerical convergence. Therefore, the existence of local minima has been checked within the three contours of [**F**]. Afterward, the convergence is verified through a series of simulations under several initial values.

The initial condition is given by:

Initial parameters and the joint positions

$\mathbf{B}_1 = [R,0,0]^T$, $\mathbf{B}_i = \mathbf{R}(60 \times i)\mathbf{B}_1$, *for* i = 1,2,3,4,5

$\mathbf{A}_1 = [r,0,0]^T$, $\mathbf{A}_j = \mathbf{R}(120 \times j)\mathbf{A}_1$, *for* j = 1,2,3

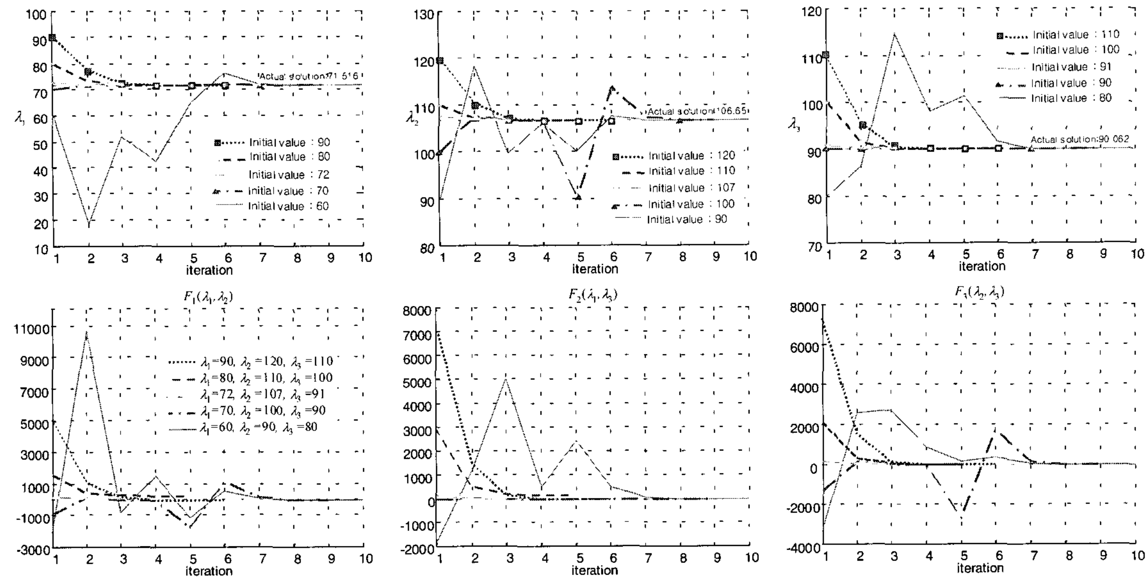


Fig. 4. Iterative converging process of the unknown variables and the constraint equations according to different initial values.

When the two platforms are parallel, the initial lengths of the six links are determined as L_o = 94.114mm through the inverse kinematics.

When the three unknown variables (λ_i) are arbitrary varying in the following wide ranges under considering the mechanical constraints such as spherical joint limitation and link interference:

$$\lambda_{min} = L_o \times 0.1 < \lambda_1,\ \lambda_2,\ \lambda_3 < L_o \times 1.9 = \lambda_{max}. \qquad (15)$$

Three existence ranges of the direct kinematic solution are obtained under the following *arbitrary* configuration:

Arbitrary configuration

$$R = 80,\ r = R/2,\ \alpha = 20°,\ \beta = 30°,\ \gamma = -10°,\ \mathrm{H} = [-10,10,80]^T. \qquad (16)$$

As depicted in Fig 3, three contours are highly nonlinear and have a local minimum where the projective of the contour is of a closed loop. However, since they are very moderate, the numerical solution will be converged well if initial estimate values may be selected in the vicinity of actual solutions (λ_1=71.516,λ_2=106.65,λ_3=90.062).

For the convergence, Figure 4 shows that the three unknown variables (λ_1,λ_2,λ_3) are stably converged to the actual solution within a few iterations in spite of different initial values and a relatively strict given convergence tolerance (10^{-6}). The more initial estimate values stay near the actual solution, the faster becomes the convergence speed. As a result, the feasibility and convergence of the numerical algorithm is verified through a series of simulation results.

IV. Conclusions

The forward kinematics for the 3-6 Platform is presented to be in a simplified form through a new formulation approach of reconfiguring the 3-6 Platform to three tetrahedrons. The formulation simplification has been completed through the proposed *Tetrahedron Approach* in which the three position vectors on the moving platform are expressed with respect to the same frame and their lengths are established as unknown variables. Consequently, the *Tetrahedron Approach* greatly reduces the number of the calculations included in the three resulting constraint equations. The reduction effect has been disclosed in the formulated partial derivative matrix and its inverse one, as shown in Appendix B. The feasibility and convergence of the *Tetrahedron Approach* is verified through a series of simulation results. The *Tetrahedron Approach* has a significant advantage of allowing intuitive derivation and reducing computational burden over the conventional approach that is formulated with trigonometric functions. Moreover, the *Tetrahedron Approach* for the 3-6 Platform can be directly applicable to 6-dof haptic devices with three sets of a serial-parallel linkage because they have three connecting joints on the moving platform.

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Appendix A: Expressions for the coefficients of the F_i

The constraint equations (F_i) are derived under the geometric assumption that the joint positions on the base and moving platform are symmetrically arranged on the circular plane:

$$F_1(\lambda_1,\lambda_2) = -3r^2 + \frac{1}{3}(R^2 - 2L_1^2 + L_2^2 + L_3^2 + \frac{L_1^2L_3^2}{R^2} - \frac{2L_2^2L_3^2}{R^2} - 2L_4^2 + \frac{L_1^2L_4^2}{R^2} + \frac{L_2^2L_4^2}{R^2})$$
$$+ \frac{\lambda_1^2}{3}(4 + \frac{L_3^2}{R^2} - \frac{2L_4^2}{R^2}) + \frac{\lambda_2^2}{3}(4 - \frac{2L_1^2}{R^2} + \frac{L_2^2}{R^2}) + \lambda_1^2\lambda_2^2(\frac{1}{3R^2}) - 2A_{13}A_{23},$$

$$F_2(\lambda_1,\lambda_3) = -3r^2 + \frac{1}{3}(R^2 + L_1^2 - 2L_2^2 - 2L_5^2 + \frac{L_1^2L_5^2}{R^2} + \frac{L_2^2L_5^2}{R^2} + L_6^2 - \frac{2L_1^2L_6^2}{R^2} + \frac{L_2^2L_6^2}{R^2})$$
$$\frac{+\lambda_1^2}{3}(4 - \frac{2L_5^2}{R^2} + \frac{L_6^2}{R^2}) + \frac{\lambda_3^2}{3}(4 + \frac{L_1^2}{R^2} - \frac{2L_2^2}{R^2}) + \lambda_1^2\lambda_3^2(\frac{1}{3R^2}) - 2A_{13}A_{33},$$

$$F_3(\lambda_2,\lambda_3) = -3r^2 + \frac{1}{3}(R^2 - 2L_3^2 + L_4^2 + L_5^2 + \frac{L_3^2L_5^2}{R^2} - \frac{2L_4^2L_5^2}{R^2} - 2L_6^2 + \frac{L_3^2L_6^2}{R^2} + \frac{L_4^2L_6^2}{R^2})$$
$$+ \frac{\lambda_2^2}{3}(4 + \frac{L_5^2}{R^2} - \frac{2L_6^2}{R^2}) + \frac{\lambda_3^2}{3}(4 - \frac{2L_3^2}{R^2} + \frac{L_4^2}{R^2}) + \lambda_2^2\lambda_3^2(\frac{1}{3R^2}) - 2A_{23}A_{33}.$$

The coefficients of the F_i are obtained as the following simplified forms:

$$G_{11} = \frac{1}{3}(4 + \frac{L_3^2}{R^2} - \frac{2L_4^2}{R^2}),\ G_{12} = \frac{1}{3}(4 - \frac{2L_1^2}{R^2} + \frac{L_2^2}{R^2}),\ G_{13} = \frac{1}{3R^2},$$
$$G_{14} = -3r^2 + \frac{1}{3}(R^2 - 2L_1^2 + L_2^2 + L_3^2 + \frac{L_1^2L_3^2}{R^2} - \frac{2L_2^2L_3^2}{R^2} - 2L_4^2 + \frac{L_1^2L_4^2}{R^2} + \frac{L_2^2L_4^2}{R^2}),$$
$$G_{21} = \frac{1}{3}(4 - \frac{2L_5^2}{R^2} + \frac{L_6^2}{R^2}),\ G_{22} = \frac{1}{3}(4 + \frac{L_1^2}{R^2} - \frac{2L_2^2}{R^2}),\ G_{23} = \frac{1}{3R^2},$$
$$G_{24} = -3r^2 + \frac{1}{3}(R^2 + L_1^2 - 2L_2^2 - 2L_5^2 + \frac{L_1^2L_5^2}{R^2} + \frac{L_2^2L_5^2}{R^2} + L_6^2 - \frac{2L_1^2L_6^2}{R^2} + \frac{L_2^2L_6^2}{R^2}),$$
$$G_{31} = \frac{1}{3}(4 + \frac{L_5^2}{R^2} - \frac{2L_6^2}{R^2}),\ G_{32} = \frac{1}{3}(4 - \frac{2L_3^2}{R^2} + \frac{L_4^2}{R^2}),\ G_{33} = \frac{1}{3R^2},$$
$$G_{34} = -3r^2 + \frac{1}{3}(R^2 - 2L_1^2 + L_4^2 + L_5^2 + \frac{L_3^2L_5^2}{R^2} - \frac{2L_4^2L_5^2}{R^2} - 2L_6^2 + \frac{L_3^2L_6^2}{R^2} + \frac{L_4^2L_6^2}{R^2}).$$

Appendix B: The partial derivatives of the F_i

The partial derivatives of the F_i are readily represented in simplified forms from the resulting equations of Appendix A.

$$\frac{\partial F_1(\lambda_1,\lambda_2)}{\partial\lambda_1} = 2\lambda_1G_{11} + 2\lambda_1\lambda_2^2G_{13} - 2\frac{\partial A_{13}(\lambda_1)}{\partial\lambda_1}A_{23},$$
$$\frac{\partial F_1(\lambda_1,\lambda_2)}{\partial\lambda_2} = 2\lambda_2G_{12} + 2\lambda_2\lambda_1^2G_{13} - 2\frac{\partial A_{23}(\lambda_2)}{\partial\lambda_2}A_{13},$$
$$\frac{\partial F_2(\lambda_1,\lambda_3)}{\partial\lambda_1} = 2\lambda_1G_{21} + 2\lambda_1\lambda_3^2G_{23} - 2\frac{\partial A_{13}(\lambda_1)}{\partial\lambda_1}A_{33},$$
$$\frac{\partial F_2(\lambda_1,\lambda_3)}{\partial\lambda_3} = 2\lambda_3G_{22} + 2\lambda_3\lambda_1^2G_{23} - 2\frac{\partial A_{33}(\lambda_3)}{\partial\lambda_3}A_{13},$$
$$\frac{\partial F_3(\lambda_2,\lambda_3)}{\partial\lambda_2} = 2\lambda_2G_{31} + 2\lambda_2\lambda_3^2G_{33} - 2\frac{\partial A_{23}(\lambda_2)}{\partial\lambda_2}A_{33},$$
$$\frac{\partial F_3(\lambda_2,\lambda_3)}{\partial\lambda_3} = 2\lambda_3G_{32} + 2\lambda_3\lambda_2^2G_{33} - 2\frac{\partial A_{33}(\lambda_3)}{\partial\lambda_3}A_{23},$$
$$\frac{\partial F_1(\lambda_1,\lambda_2)}{\partial\lambda_3} = 0, \frac{\partial F_2(\lambda_1,\lambda_3)}{\partial\lambda_2} = 0, \frac{\partial F_3(\lambda_2,\lambda_3)}{\partial\lambda_1} = 0,$$

where

$$\frac{\partial A_{13}(\lambda_1)}{\partial\lambda_1} = \frac{\lambda_1(1 - \lambda_{11}/R - \lambda_{12}/\sqrt{3}R)}{A_{13}},$$
$$\frac{\partial A_{23}(\lambda_2)}{\partial\lambda_2} = \frac{\lambda_2(1 - \lambda_{21}/R - \lambda_{22}/\sqrt{3}R)}{A_{23}},$$
$$\frac{\partial A_{33}(\lambda_3)}{\partial\lambda_3} = \frac{\lambda_3(1 - \lambda_{31}/R - \lambda_{32}/\sqrt{3}R)}{A_{33}}.$$

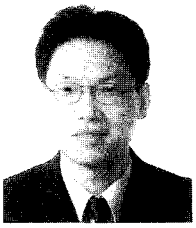
The 3x3 partial derivative matrix $[\nabla\mathbf{F}]$ and its inverse matrix $[\nabla\mathbf{F}^{-1}]$ are expressed as the following simple matrixes:

$$[\nabla\mathbf{F}] = \begin{bmatrix} \frac{\partial F_1}{\partial\lambda_1} & \frac{\partial F_1}{\partial\lambda_2} & 0 \\ \frac{\partial F_2}{\partial\lambda_1} & 0 & \frac{\partial F_2}{\partial\lambda_3} \\ 0 & \frac{\partial F_3}{\partial\lambda_2} & \frac{\partial F_3}{\partial\lambda_3} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & 0 & F_{23} \\ 0 & F_{32} & F_{33} \end{bmatrix},$$

$$[\nabla\mathbf{F}^{-1}] = \frac{1}{Det[\nabla\mathbf{F}]}\begin{bmatrix} -F_{23}F_{32} & -F_{12}F_{33} & F_{12}F_{23} \\ -F_{21}F_{33} & F_{11}F_{33} & -F_{11}F_{23} \\ F_{21}F_{32} & -F_{11}F_{32} & -F_{12}F_{21} \end{bmatrix},$$

where

$$Det[\nabla\mathbf{F}] = -F_{11}F_{23}F_{32} - F_{12}F_{21}F_{33}.$$



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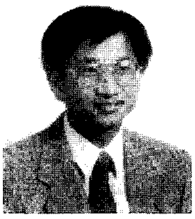

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