International Journal of Reliability and Applications
Vol. 3, No. 4, pp. 199-202, 2002

Laplace Transform of Forward Recurrence Time in an Alternating Renewal Process

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Abstract. In this paper, we obtain an explicit formula of the Laplace transform of the forward recurrence time at finite time $t > 0$ in an alternating renewal process, by adopting a Markovian approach. As a consequence, we obtain the first two moments of the forward recurrence time.

Key Words : *alternating renewal process, Laplace transform, forward recurrence time*

1. INTRODUCTION

Let $\{(X_n, Y_n), n \geq 1\}$ be an alternating renewal process, where $\{X_n, n \geq 1\}$ denote the sequence of 'up' times and $\{Y_n, n \geq 1\}$ that of 'down' times. we assume that X_n's are independent and identically distributed with distribution function G and density function g, and Y_n's are independent and exponentially distributed with mean $\frac{1}{\lambda}$. For the applications of the exponential 'down' time, see Baxter and Lee(1987).

Baxter(1981) defined the forward recurrence time in the alternating renewal process at time $t > 0$ as the time to transition to 'down', given that the process is then in 'up'. Baxter(1983), later, obtained the moments of the forward recurrence time by making use of renewal arguments.

In an ordinary renewal process, Coleman(1982) obtained the moments of the forward recurrence time by adopting also the renewal arguments. The other transient behaviors of the forward recurrence time, however, have been rarely studied, since the analysis is complicate as Cox(1962) stated. Recently, Lee, Kim and Shim(2002) obtained an explicit formula of the Laplace transform of the forward recurrence time

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by introduing a Markovian approach to the renewal process.

Let $I(t)$ be the state of the alternating renewal process at time $t > 0$, then we define, in this paper, the forward recurrence time $t > 0$ as

$$V(t) = \begin{cases} \text{time to transition to 'down'}, & \text{if } I(t) = \text{'up'} \\ 0, & \text{if } I(t) = \text{'down'} \end{cases}$$

Note that our $V(t)$ is defined on the whole sample space of the process, while Baxter's(1981) on the sample paths where the process is in 'up' at time t. For example, let $V'(t)$ denote the forward recurrence time of Baxter's(1981), then $E[V'(t)] = E[V(t)]/A(t)$, where $A(t) = \Pr\{I(t) = \text{'up'}\}$

In this paper, we obtain an explicit formula of the Laplace transform of $V(t)$ by establishing a Kolmogorov's forward differential equation for the distribution function of $V(t)$. As a consequence, we obtain the moments of $V(t)$ by differentiation.

2. THE LAPLACE TRANSFORM OF THE FORWARD RECURRENCE TIME

First, note that the forward recurrence time $V(t)$ satisfies the Markovian property, that is, once $V(t_0) = x_0$ is given, $V(t)$, $t > t_0$, depends probabilistically only on x_0, since the duration of $V(t)$ being 0 is exponentially distributed. See Figure 1. This fact enables us to deduce a forward differential equation for $F(x,t) = \Pr\{V(t) \le x\}$.

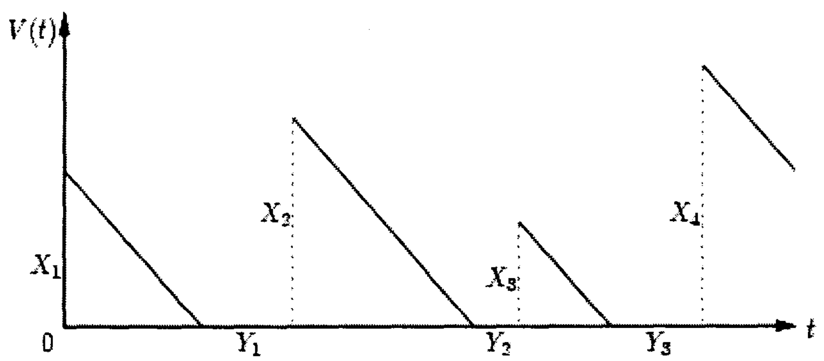

Figure 1. A sample path of forward recurrence time $V(t)$

In a small interval $[t, t+\Delta t]$, $V(t+\Delta t)$ satisfies the followings :

$$V(t+\Delta t) = \begin{cases} V(t) - \Delta t, & \text{almost surely, if } V(t) > \Delta t \\ 0, & \text{almost surely, if } V(t) \le \Delta t \\ & \text{and no transition to 'up' in } [t, t+\Delta t] \\ X - \Delta t', & \text{almost surely, if } V(t) \le \Delta t \\ & \text{and a transition to 'up' in } [t, t+\Delta t], \end{cases}$$

where $0 < \Delta t' \leq \Delta t$ and X is a random variable having distribution function G. These three events are mutually exclusive, and hence

$$\begin{aligned}
\Pr\{V(t+\Delta t) \leq x\} &= \Pr\{V(t) - \Delta t \leq x, V(t) > \Delta t\} \\
&\quad + \Pr\{0 \leq x, V(t) \leq \Delta t, \textit{ no transition}\} \\
&\quad + \Pr\{X - \Delta t' \leq x, V(t) \leq \Delta t, \textit{ a transition}\} \\
&= \Pr\{V(t) - \Delta t \leq x\} \\
&\quad - \Pr\{V(t) - \Delta t \leq x, V(t) \leq \Delta t\} \\
&\quad + \Pr\{V(t) \leq \Delta t\}(1 - \lambda\Delta t + o(\Delta t)) \\
&\quad + \Pr\{X - \Delta t' \leq x, V(t) \leq \Delta t\}(\lambda\Delta t + o(\Delta t)) + o(\Delta t) \\
&= \Pr\{V(t) - \Delta t \leq x\} \\
&\quad - \Pr\{V(t) \leq \Delta t\}(\lambda\Delta t + o(\Delta t)) \\
&\quad + \Pr\{X - \Delta t' \leq x, V(t) \leq \Delta t\}(\lambda\Delta t + o(\Delta t)) \\
&\quad + o(\Delta t). \qquad (2.1)
\end{aligned}$$

Applying the power series expansion to $\Pr\{V(t) - \Delta t \leq x\}$ about x gives

$$\Pr\{V(t) - \Delta t \leq x\} = \Pr\{V(t) \leq x\} + \frac{\partial}{\partial x}\Pr\{V(t) \leq x\}\Delta t + o(\Delta t).$$

Substituting this into (2.1), we have

$$\begin{aligned}
\Pr\{V(t+\Delta t) \leq x\} &= \Pr\{V(t) \leq x\} + \frac{\partial}{\partial x}\Pr\{V(t) \leq x\}\Delta t \\
&\quad - \Pr\{V(t) \leq \Delta t\}(\lambda\Delta t + o(\Delta t)) \\
&\quad + \Pr\{X - \Delta t' \leq x\}\Pr\{V(t) \leq \Delta t\}(\lambda\Delta t + o(\Delta t)) \\
&\quad + o(\Delta t), \qquad (2.2)
\end{aligned}$$

where we use the fact that X does not depend on $V(t)$. Dividing each side by Δt and letting $\Delta t \to 0$ in equation (2.2) give the following differential equation:

$$\frac{\partial}{\partial t}F(x,t) = \frac{\partial}{\partial x}F(x,t) - \lambda F(0,t)(1 - G(x)). \qquad (2.3)$$

Note that $F(x,t) = F(0,t) + \int_0^x f(u,t)du$, since a discrete probability exists when $V(t) = 0$, where $f(x,t) = \frac{\partial}{\partial x}F(x,t)$, $x > 0$.

Taking the Laplace transforms on both sides of equation (2.3) gives

$$\frac{\partial}{\partial t}\frac{1}{s}f^*(s,t) = f^*(s,t) - F(0,t) - \lambda F(0,t)(\frac{1}{s} - \frac{1}{s}g^*(s)), \qquad (2.4)$$

where $f^*(s,t) = \int_0^\infty e^{-sx}dF(x,t)$ and $g^*(s) = \int_0^\infty e^{-sx}g(x)dx$. Solving the above equation (2.4) for $f^*(s,t)$ with boundary condition $f^*(s,0) = g^*(s)$, we have

$$f^*(s,t) = e^{st}\left[\{\lambda(g^*(s) - 1) - s\}\int_0^t e^{-su}F(0,u)du + g^*(s)\right]. \qquad (2.5)$$

Note that $F(0,t)$ can be easily derived by a well known availability theory as follows:

$$F(0,t) = \Pr\{I(t) = 0\} = \int_0^t e^{-\lambda(t-u)} dH_1(u),$$

where $H_1(t) = \sum_{n=0}^{\infty} G^{(n+1)} * K^{(n)}(t)$ with $K(t) = 1 - e^{-\lambda t}$, $t \geq 0$, $*$ being the Stieltjes convolution and (n) the n-fold recursive Stieltjes convolution.

Differentiating equation (2.5) with respect to s gives

$$\begin{aligned} E[V(t)] &= \mu_1 - t + \left[\frac{\lambda\mu_1 + 1}{\lambda}\right] H_2(t) \\ E[V^2(t)] &= t^2 - 2t\mu_1 + \mu_2 + \left[\mu_2 + \frac{2\mu_1}{\lambda} + \frac{2}{\lambda^2}\right] H_2(t) \\ &\quad + \left[\frac{2(\lambda\mu_1 + 1)}{\lambda}\right] \int_0^t (x - t) dH_1(x), \end{aligned}$$

where $\mu_1 = E[X]$, $\mu_2 = E[X^2]$ and $H_2(t) = K * H_1(t)$. These moments are consistent with Baxter's(1983).

ACKNOWLEDGEMENT

This paper was supported by Sookmyung Women's University Research Fund, 2001.

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