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Availability of a Maintained System

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Abstract. In the traditional life testing model, it is assumed that a certain number of identical items are tested under identical condition. This is due to statistical rather than practical considerations. The proportional hazards model can be used to develop a realistic approach to determine the performance of an item. That is also capable of modeling the failure rates of accelerated life testing when the covariates are applied stresses. The proportional hazards model is typically applied for a group of items to assess the importance of factors that may influence the reliability of an item. In this paper we considered the interarrival times of an item rather than the time to first failure for grouped items and provided the availability estimation for the determination of maintenance policy and overhaul time. In order to demonstrate the proposed approach, an example is presented.

Key Words : proportional hazards model, covariates, baseline reliability function, interarrival times, availability

1. INTRODUCTION

This paper concerns the application of proportional hazards modeling to the problem of item performance. In the traditional life testing model, it is assumed that a certain number of identical items are tested under identical condition. This is due to statistical rather than practical considerations as under these assumptions the times to failure of these items are independent and identically distributed and thus are more easily analyzed. Furthemore, in simple hazard modeling, the failure of an item is purely time dependent. In many applications of reliability, however, real interest is in determining the way in which life time depends on other variables, some of which may be under the operator's control.

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The proportional hazards model (PHM) can be used to develop a realistic approach to determine the performance of an item. The performance of an item is influenced not only by the operating time, but by other factors. These influencing factors include operating condition (*e.q.*, vibration levels, temperature, pressure, levels of metal paticles in engine oil, humidity, dust) and operating history (*e.q.*, number of previous overhauls, time since last failure and maintenance). They are generally referred to as covariates or explanatory variables. Given all the possible covariates, PHM can be used to identify the explanatory factors and to predict the performance of a item with those factors.

The PHM is one of the most important statistical regression models. The paper by Cox (1972) aroused much interest in this model. While this model has had a significant impact on the biostatistical field, it has recieved little attention in the reliability literature. At first PHM has been used for the analysis of hardware reliability (Argent *et al.* (1986), Marshall *et al.* (1990)), software reliability (Bendell and Wightman (1986)) and repairable systems (Ascher (1983)). Recently the PHM has been used for a preventive maintenance (PM) scheduling model. Kumar and Westberg (1995) used the PHM to schedule maintenance under age replacement policy. Percy and Kobbacy (1996) considered the analysis of PM policies with exponential times to failure from a Bayesian viewpoint. In the above study, they did not consider the ageing over the lifetime of an item or possible positive reliability growth due to change in design. Kobbacy *et al.* (1997) used the PHM for a simulation framework to schedule the PM interval. In this paper, we will present a way to estimate the availability function with reliability data from an single item using PHM. Using this, optimal PM schedules can be obtained when the availability of an item reaches a predetermined level.

2. ANALYSIS OF THE PROPORTIONAL HAZARDS MODEL

In PHM the failure rate (hazard) function satisfies

$$\lambda(t;\mathbf{z}) = \lambda(t;0)e^{\mathbf{z}^T\beta}, \tag{2.1}$$

where $\lambda(t;\mathbf{z})$ is the failure rate (hazard) function at time t for observations with covariate vector, $\mathbf{z}^T = (z_1, z_2, \cdots, z_p)$, $\beta^T = (\beta_1, \beta_2, \cdots, \beta_p)$ is a vector of unknown regression coefficients and $\lambda(t;0)$ is an unspecified baseline failure rate (hazard) function (*i.e.* the failure rate function when all covariates are zero). The PHM assumes that the covariates act multiplicatively on the failure rate function, so that for different values of the covariates the failure rate functions at each time are proportionl to each other.

The statistical analysis of PHM depends on whether or not we assume a particular functional form for the baseline failure rate function $\lambda(t;0)$. When baseline function is not fitted to a specific model one has a semi-parametric PHM, and when it is assigned to a specific model, *e.g.* Weibull, this leads to a fully parametric model. We suppose that the function $\lambda(t;0)$ is completely unspecified.

Analysis of the Semi-Parametric PHM

Define, for convenience, $t_{(0)} = 0, t_{(k+1)} = \infty$ and assume, as usual, that $t_{(1)} < t_{(2)} < \cdots < t_{(k)}$ are the observed lifetimes in the random sample of n items and $n-k$ censoring times. Let R_i be the set of items with $t \geq t_{(i)}$, where t may be the either an observed life time or a censored time. Here R_i is the risk set at time $t_{(i)}$; that is those items which were at risk of failing just prior to $t_{(i)}$. To estimate β, we use the partial likelihood function $L(\beta)$ without specifying the failure time distribution:

$$L(\beta) = \prod_{i=1}^{k} \frac{\exp(z_{(i)}^T \beta)}{\sum_{l \in R_i} \exp(z_l^T \beta)} \tag{2.2}$$

where $z_{(i)}$ is the vector of covariates associated with the unit observed to fail at time $t_{(i)}$. Partial likelihood has been discussed Kalbfleisch and Prentice (1973) Cox (1975), Efron (1977), Kalbfleisch and Prentice (1980), and others.

Time-dependent Covariates

In reliability applications it is very often appropriate to relate the current risk of failure to a covariate which is time-dependent, such as an estimate of the current wear or damage sustained by the component. To deal with this it is necessary to replace z by $z(t)$ in PHM (2.1). To estimate covariates the method described above may be adapted when one or more of the covariates is time-dependent.

Estimation of the Baseline Reliability Function

Once the covariates have been estimated by maximization of (2.2), the estimation of the baseline reliability function is often asked. It is clearly of interest to estimate $R(0;\mathbf{z})$, since this would give estimate of $R(t;\mathbf{z})$ for any $\mathbf{z}$. The reliability function for an individual with covariate vector $\mathbf{z}$ is

$$R(t;z) = R(t;0)e^{\mathbf{z}^T \beta},$$

where $R(t;0)$ is the baseline reliability function of individual with $\mathbf{z} = 0$.

Our approach to estimate $R(0;\mathbf{z})$ is to estimate β from the partial likelihood function (2.2) and then to maximize the full likelihood $L(\beta, R(0;\mathbf{z}))$ for $R(0;\mathbf{z})$, assuming that β is equal to the partialy maximum likelihood estimator $\hat{\beta}$ obtained from (2.2). This approach was introduced by Kalbfleisch and Prentice (1973). The following formular is used to estimate $R(0;\mathbf{z})$.

$$R(0;\mathbf{z}) = \prod_{i:t_{(i)}<t} \hat{\alpha}_i,$$

where $\hat{\alpha}_i$ can be obtained by a solution from the following equations.

We assume that there are n_i risk and d_i death at $t_{(i)}$. And let D_i be the set of indivisuals dying at $t_{(i)}$. Defferentiating log likelihood with repect to $\alpha_1, \cdots, \alpha_k$, we can get equations:

$$\sum_{l \in D_i} \frac{\exp(\mathbf{z}_l^T \beta)}{1 - \alpha_i^{\exp(\mathbf{z}_l^T \beta)}} = \sum_{l \in R_i} \exp(\mathbf{z}_l^T \beta), \qquad i = 1, \cdots, k. \tag{2.3}$$

When $d_i = |D_i| = 1$, (2.3) has a solution given by

$$\hat{\alpha}_i^{\exp(\mathbf{z}_{(i)}^T \beta)} = 1 - \frac{\exp(\mathbf{z}_{(i)}^T \beta)}{\sum_{l \in R_i} \exp(\mathbf{z}_l^T \beta)}.$$

When $d_i \geq 1$, (2.3) must be solved iteratively for $\hat{\alpha}_i$.

3. DATA ANALYSIS USING THE PROPORTIONAL HAZARDS MODEL

In order to demonstrate the way to analyse the reliability data using the PHM, an example is presented. This example is based on data for a pump in a plant. The data are shown in Table 1. PM types are different but they consist of minor routine work, *e.g.* adjustment and inspection work. The time of each PM includes the repair work identified in inspection, if relevant. For simplicity of of analysis we assume that all PMs are of the same type. Corrective maintenance (CM) is any maintenance that occurs when the pump is failed.

Selection of Covariates

Before fitting the PHM, it is necessary to identify the potential covariates. The following covariates which were selected are estimated at a point in time just before the event CM or PM:

(a) age (age)
(b) average PM interval (avpm)
(c) total number of failures (nocm)
(d) total number of PMs (nopm)
(e) total down time of all PMs (dtpm)
(f) total man hours of all PMs (mhpm)
(g) time since last CM (tlcm)
(h) time since last PM (tlpm)

All the above variables are obviously time dependent. However, it is decided to correlate the hazard function with the values of these covariates at a point in time just before the event, *i.e.* PM or CM, thus avoiding the need to apply the more complex fitting of the PHM with time-dependent covariates. For example we are interested in the age of equipment just before the event. This is a common way for handing such variables, *e.q.* correlating survival as a function of patient age at the start of treatment.

PHM for Interarrival Times

The PHM is typically applied for a group of items to assess the important of factors with influence the item. For example Jardine *et al.* (1987) used PHM to measure the importance of factors, such as flight hours since last overhaul and the levels of various metal particles in engine oil, on engine condition monitoring and overhaul time. In their analysis they used data for 27 engines.

To apply the PHM here we looked at the interarrival times of each of several items of equipment rather than the time to first failure for grouped items of equipment. Let us consider an item of equipment which has been subjected to time-based PM since the equipment started operation to the present time. When the equipment fails, a CM is taken to restore it to operating condition. Note that the major underlying assumption in this study is that the lives following PM are independent and the lives following CM are independent, conditional on covariates in both case. In this study, the lives following PM and CM are seperately considered using PHM. The failure rate function following PM is

$$\lambda_{pm}(t;\mathbf{z}) = \lambda_{pm}(t;0)e^{\mathbf{z}^T\beta},$$

similarly the failure rate function following CO is

$$\lambda_{cm}(t;\mathbf{x}) = \lambda_{cm}(t;0)e^{\mathbf{x}^T\alpha},$$

where $\mathbf{z}$ and $\mathbf{x}$ are the vectors representing the covariates used in PM and CM proportional hazards model.

All the considered covariates can be calculated at each point of PM and CM from the data shown in Table 1. Table 2 and Table 3, 4 show the prepared data files for PM and CM, respectively. The second column of the tables show the times to failure and the third columns have the variable 'cens' which has a value of 1 for an uncensored life terminated by the failure and 0 for censored life terminated by a PM.

Fitting PHM for Life Times Following PM and CM

The statistical package SPSS was then used to fit the two PHMs for PM and CM. The selection of the best model to fit the data was a learning process. The backward stepwise(Wald) method was used with limit of 20 iterations and entry and exit significance limits of 0.1 and 0.2, respectively. It was essential to rule out models with highly correlated covariates as the resulting coefficients may be misleading.

For the PM model, any two selected covariates of nocm, age, nopm, dtpm and mhpm, have a correlation coeffcient higher than 0.952. The candidate covariates for our PM model was chosen as tlpm, tlcm, avpm and one of the nocm, age, nopm, dtpm, mhpm. The selected model for lives following PM is

$$\lambda_{pm}(t; avpm, age) = \lambda_{pm}(t; 0)e^{-0.45\cdot avpm+0.002\cdot age}.$$

The significant probobility of the significant test for the above model is 0.003. The more significant of the covariates is the age.

For the CM model, adopting the above procedure to identify a suitable model with PHM result in the following model.

$$\lambda_{cm}(t; avpm, nocm) = \lambda_{cm}(t; 0)e^{0.40\cdot avpm-0.049\cdot nocm}.$$

The significant probobility of the significant test for the above model is 0.095. The significants of two covariates are similar. For more significant CM model, we considered a model for lives following CM without covariates. We assume that lives following CM have a Weibull distribution. The reliability function based on Weibull distribution is

$$R(t) = \exp\{-(\frac{t}{\eta})^m\}, \quad m > 0, \eta > 0.$$

From the above equation,

$$\ln\ln\frac{1}{R(t)} = m\ln t - \ln t_0,$$

where $t_0 = \eta^m$. If our assumption of Weibull is correct, when logarithm of the lives following CM and its $\ln\ln\{1/R(t)\}$ are plotted, the relationship appears to be a straight line. In our analysis these appear in Figure 1.

From the Figure 1, we have

$$R_{co}(t) = \exp\{-(\frac{t}{38.063})^{0.9901}\}.$$

In this caes, since shape parameter m is close to 1, life limes following CM are believed to have a exponential distribution. Finally using the procedure discussed in Section 2 baseline reliability function for PM is estimated in Figure 2.

Fitting PHM for Repair Times Following PM and CM

Characteristics of repair times following PM and CM is studied in the similar way as fitting PHM for life times following PM and CM.

For the PM case, The repair rate function for repair times following PM is

$$\mu_{pm}(t; tlpm, age) = \mu_{pm}(t; 0)e^{-0.008\cdot tlpm-0.01\cdot age}.$$

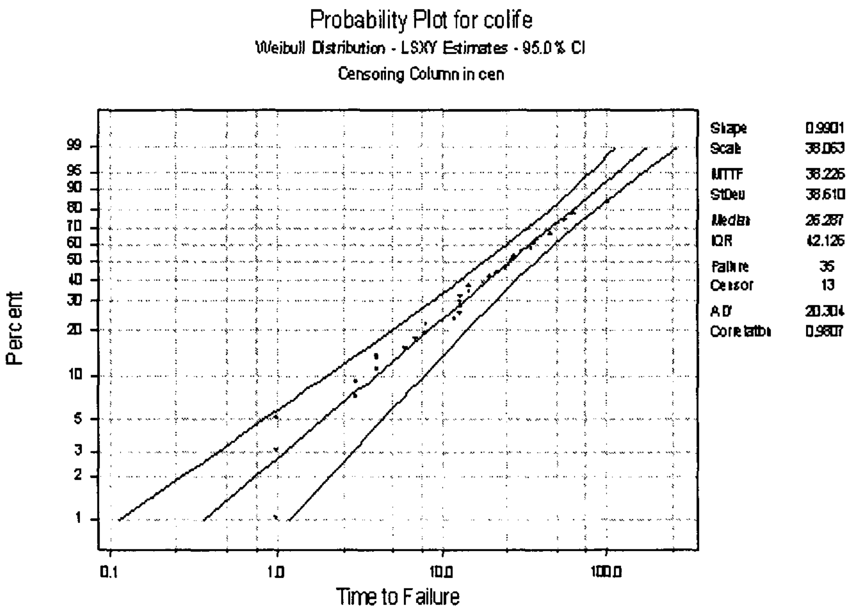

Figure 1. Regression Plot for Life Times Following CM without Covariates.

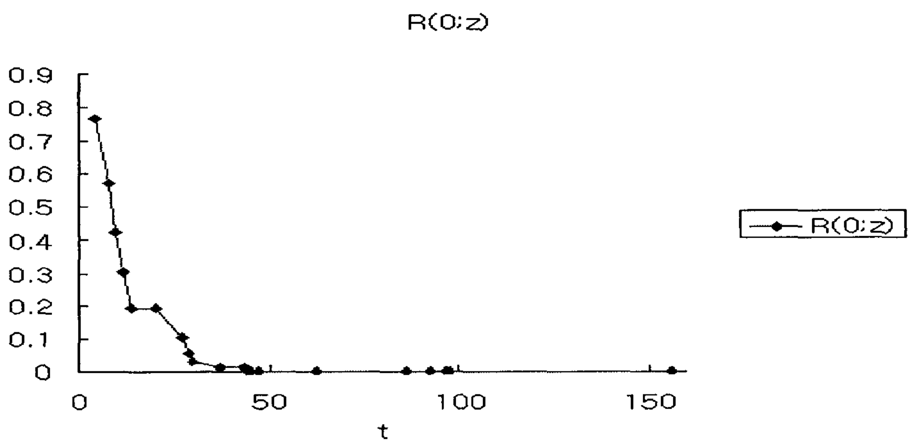

Figure 2. Baseline Reliability Function for Lives Following PM.

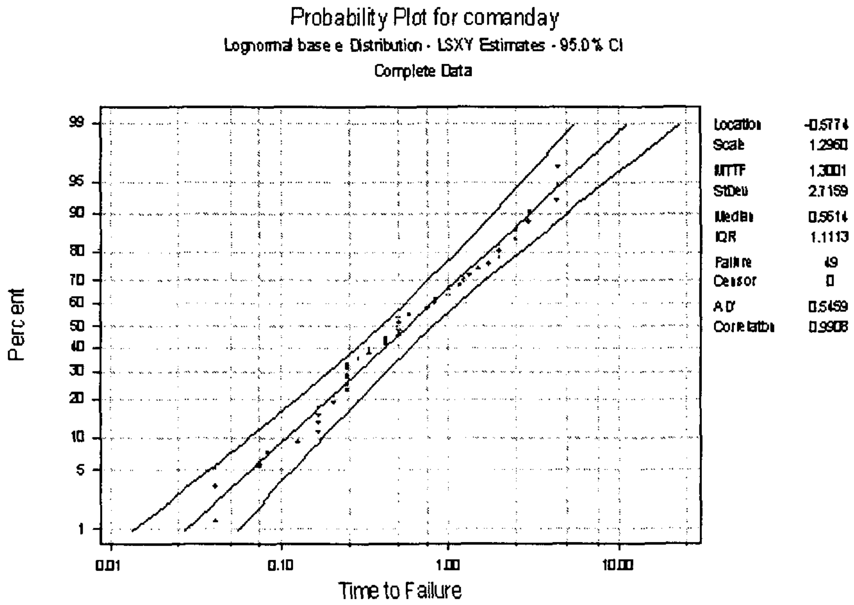

Figure 3. Regression Plot for Repair Times Following CM without Covariates.

The significant probobility of the significant test for the above model is 0.024. For the CM case, a model for repair times following CM without covariates is more significant. We assume that repair times following CM have a Lognormal distribution. The maintainability function based on lognormal distribution is given by

$$M(t) = \Phi(\frac{\ln t - \mu}{\sigma}), \qquad -\infty < \mu < \infty, \sigma > 0.$$

The analysis result appears in Figure 3. From the Figure 3, we have

$$M(t) = \Phi(\frac{\ln t + 0.5774}{1.296}).$$

So far we demonstrated the use of the PHM in estimating reliability and maintainability as well as failure rate and repair rate. This approach can be applied to repairable systems and does not require a group of systems to access the performance of the system.

Availability of Repairable Items

If the life times of the item being renewed are independently and identically distributed (iid), if the nonneggligible repair times are also independently and identically distributed (iid) and if the life times and the repair times are independent, then availability function can be obtianed in the convolution form. Even with the above assumption, availability function can be neither obtianed in the closed form nor calculated easily. In this case, the availability converges rapidly to the steady state availability. In the alternating renewal process, the limiting availability of an item is given by

$$A = \frac{MTBF}{MTBF + MTTR}.$$

In the maintained system, availability function depends on the applied mainteance policy. And the availability function is more complicated than iid case. So I'd like to recommend the following method. If we assume that repair rate is constant, that is constant repair rate $\mu = 1$ / MTTR, the following approximation can be considered (Holcomb (1981)).

$$A(t) = \frac{\mu}{\lambda(t) + \mu}.$$

Using this, optimal PM schedules and overhaul time can be obtained when the availability of an item reaches a predetermined level.

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Table 1. Raw history data for a pump

D	Type	Job time	Man hour	D	Type	Job time	Man hour
-	CM	5	5	64	CM	2	4
34	PM	1	1	8	PM	12	13
14	CM	10	20	62	PM	1	1
81	PM	3	3	8	CM	46	184
86	PM	3	3	46	CM	8	8
156	PM	2	2	22	CM	6	14
20	PM	1	1	51	CM	12	12
96	PM	3	3	51	CM	4	6
47	PM	1	1	15	CM	20	48
45	PM	3	3	18	CM	1	1
97	CM	5	8	1	CM	30	30
88	PM	5	5	26	PM	1	1
30	CM	24	48	37	CM	12	12
4	CM	54	108	36	CM	48	108
1	CM	6	6	2	CM	5	10
4	CM	6	12	12	PM	6	8
13	CM	2	2	27	CM	7	7
27	CM	24	60	102	CM	48	73
8	CM	6	10	3	CM	20	32
148	PM	6	6	13	CM	5	6
92	CM	3	6	8	CM	6	4
13	CM	12	60	6	CM	26	104
13	CM	6	6	26	CM	36	20
67	PM	6	6	15	PM	7	8
29	CM	5	18	10	CM	1	1
12	CM	24	48	25	CM	10	5
1	CM	12	24	44	PM	1	1
37	CM	21	42	30	CM	1	1
28	CM	14	28	31	CM	18	36
38	CM	4	4	12	CM	12	12
20	CM	3	6	69	PM	4	7
28	PM	12	13	12	CM	6	12
44	CM	12	24	65	PM	3	4
3	CM	3	3	43	PM	6	7
56	CM	4	4	4	CM	24	72

No. of days since last action

Table 2. Life time following PM for fitting PHM

	surv	cens	nocm	tlpm	age	nopm	dtpm	mhpm	tlcm	avpm
1	14	1	1	0	34	0	0	0	34	0
2	86	0	2	95	129	1	1	1	81	95
3	156	0	2	86	215	2	4	4	167	91
4	20	0	2	156	371	3	7	7	323	112
5	96	0	2	20	391	4	9	9	343	89
6	47	0	2	96	487	5	10	10	439	91
7	45	0	2	47	534	6	13	13	486	83
8	97	1	2	45	574	7	14	14	531	78
9	30	1	3	185	764	8	17	17	88	78
10	92	1	10	235	999	9	22	22	148	91
11	29	1	13	185	1184	10	28	28	67	107
12	44	1	20	193	1377	11	34	34	28	115
13	62	0	24	175	1552	12	46	47	8	122
14	8	1	24	62	1614	13	58	60	70	127
15	37	1	32	238	1852	14	59	61	26	122
16	27	1	35	87	1939	15	60	62	12	130
17	10	1	42	200	2139	16	66	70	15	127
18	44	1	44	79	2218	17	73	78	44	132
19	12	1	47	145	2363	18	74	79	69	128
20	43	0	48	77	2440	19	78	86	65	129
21	4	1	48	43	2483	20	81	90	108	127

Table 3. Life times following CM for fitting PHM

	surv	cens	nocm	tlpm	age	nopm	dtpm	mhpm	tlcm	avpm
1	34	0	0	0	0	0	0	0	0	0
2	81	0	1	48	14	1	1	1	48	0
3	88	0	2	676	97	8	17	17	628	78
4	4	1	3	794	30	9	22	22	118	91
5	1	1	4	798	34	9	22	22	4	91
6	4	1	5	799	35	9	22	22	1	91
7	13	1	6	803	39	9	22	22	4	91
8	27	1	7	816	52	9	22	22	13	91
9	8	1	8	843	79	9	22	22	27	91
10	148	0	9	851	87	9	22	22	8	91
11	13	1	10	1091	92	10	28	28	240	107
12	13	1	11	1104	105	10	28	28	13	107
13	67	0	12	1117	118	10	28	28	13	107
14	12	1	13	1213	29	11	34	34	96	115
15	1	1	14	1225	141	11	34	34	12	115
16	37	1	15	1226	42	11	34	34	1	115
17	28	1	16	1263	79	11	34	34	37	115
18	38	1	17	1291	107	11	34	34	28	115
19	20	1	18	1329	145	11	34	34	38	115
20	28	0	19	1349	165	11	34	34	20	115
21	3	1	20	1421	44	12	46	47	72	122
22	56	1	21	1424	47	12	46	47	3	122
23	64	1	22	1480	103	12	46	47	56	122
24	8	0	23	1544	167	12	46	47	64	122
25	46	1	24	1622	8	13	59	61	78	122
26	22	1	25	1668	54	14	59	61	46	122
27	51	1	26	1680	76	14	59	61	22	122
28	51	1	27	1741	127	14	59	61	51	122
29	15	1	28	1792	178	14	59	61	51	122
30	18	1	29	1807	193	14	59	61	15	122
31	1	1	30	1825	211	14	59	61	18	122
32	26	0	31	1826	212	14	59	61	1	122
33	36	1	32	1889	37	15	60	62	63	130

Table 4. Life times following CM for fitting PHM (continued)

	surv	cens	nocm	tlpm	age	nopm	dtpm	mhpm	tlcm	avpm
34	7	1	33	1925	73	15	60	62	36	130
35	12	0	34	1927	75	15	60	62	2	130
36	102	1	35	1966	27	16	66	70	39	127
37	3	1	36	2068	129	16	66	70	102	127
38	13	1	37	2071	132	16	66	70	3	127
39	8	1	38	2084	145	16	66	70	13	127
40	6	1	39	2092	153	16	66	70	8	127
41	26	1	40	2098	159	16	66	70	6	127
42	15	0	41	2124	185	16	66	70	26	127
43	25	1	42	2149	10	17	73	78	25	132
44	44	0	43	2174	35	17	73	78	25	132
45	31	1	44	2248	30	18	74	79	74	128
46	15	1	45	2279	61	18	74	79	31	128
47	69	0	46	2294	76	18	74	79	15	128
48	65	0	47	2375	12	19	78	86	81	129