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Joint Structural Importance of two Components

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Abstract. This paper introduces the joint structural importance of two components in a coherent system. Some relationships between joint structural importance and marginal structural importance are presented. It is shown that the sign of joint structural importance can be determined, in advance, without computation in some special structures. The joint structural importance of two components in some series-parallel and parallel-series systems are established. Some practical examples are presented to elucidate some of the derived results.

Key Words : marginal structural importance, coherent system, series-parallel system, parallel-series systems.

1. INTRODUCTION

For a given coherent system, the structural importance (SI) of a component is a quantitative measure of the importance of the individual component in determining whether the system function or not. It is defined as the proportion of the critical path vectors for the component from all possible vectors, see Barlow and Proschan (1981).

This paper introduces the joint structural importance (JSI) for two components of a coherent system. We also establish some relationships between JSI and marginal structural importance (MSI). Based on these relationships, we show that: (i) the value of JSI of two components lies in the interval [-1, 1]; (ii) the sign of JSI of two components can be

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determined in advance for some special cases; and (iii) general forms for JSI in some interesting parallel-series and series-parallel systems can be obtained.

The measures of importance for the components in any system can be utilized in various maintenance, replacement and spare part storage polices. Also JSI and MSI can be similarly exploited.

The discussion in this paper is worded in terms of a system of components. Those readers who prefer graph terminology may mentally substitute network for system, and edge for component. Similarly, those who prefer the most general mathematical interpretation may think in terms of sets instead of systems, and elements instead of components.

The following notations, nomenclature and assumptions are required in this paper.

Notations

N	Number of components in the system
X_i	Indicator for component i; X_i= 1, 0 if component i is functioning or failed
X	$(X_1, X_2, \ldots, X_n)$: state vector
1_i	Component i is in functioning state (state one)
0_i	Component i is in failed state (state zero)
$\Phi(X)$	Structure function of the system
$\Phi(\psi, X)$	$\Phi(X)$ with a known event-set ψ
$JSI(i, j)$	Joint structural importance of components i, j
$MSI(i)$	Marginal structural importance of components i
$MSI(\psi, i)$	$MSI(i)$ with a known event-set ψ

Nomenclature

Min-Path	A minimal set of components whose functioning is sufficient to make the system function
Min-Cut	A minimal set of components whose failure is sufficient to make the system fail
Relevant Component	A component which appears in at least one min-path and/or min-cut
S-Coherent system	Component i is in failed state (state zero)

Assumptions

1. Each component as well as the system has two states: functioning or failed.
2. The system is s-coherent.

3. RELATIONSHIPS BETWEEN JSI AND MSI

Different researchers have discussed the structural importance of a component in a s-coherent system. The following definition for the structural importance of a component in an s-coherent system, see Barlow and Proschan (1981), is based on counting the number of critical paths containing that component.

Definition 1. Given an s-coherent system with structural function $\Phi(X)$, the SI(i) is defined by

$$SI(i) = \frac{N_\Phi(i)}{2^{n-1}} \tag{2.1}$$

where

$$N_\Phi(i) = \sum_{\{X:x_i=1\}} [\Phi(1_i, X) - \Phi(0_i, X)].$$

Next we introduce a definition for marginal structural importance $MSI(\cdot_i, j)$.

Definition 2. Given an s-coherent system with structural function $\Phi(X)$, the $MSI(\cdot_i, j)$ is defined by

$$MSI(\cdot_i, j) = \frac{N_\Phi(\cdot_i, j)}{2^{n-2}} \tag{2.2}$$

where

$$N_\Phi(\cdot_i, j) = \sum_{\{X:x_i=1, x_j=1\}} [\Phi(\cdot_i, 1_j, X) - \Phi(\cdot_i, 0_j, X)]$$

Using the above definition one can define the joint structural importance for two components in an s-coherent system. $MSI(\cdot_i, j)$.

Definition 3. Given an s-coherent system with structural function $\Phi(X)$, the $JSI(i, j)$ is defined by

$$MSI(i, j) = \frac{N_\Phi(i, j)}{2^{n-2}} \tag{2.3}$$

where

$$N_\Phi(i, j) = \sum_{\{X:x_i=1, x_j=1\}} [\Phi(1_i, 1_j, X) + \Phi(0_i, 0_j, X) - \Phi(1_i, 0_j, X) - \Phi(0_i, 1_j, X)].$$

The following theorem describes that $JSI(i, j)$ can be represented in terms of MSI of each component in a modified system in which components are guaranteed to be functioning or failed.

Theorem 1. The $JSI(i, j)$ can be represented by

$$JSI(i,j) = MSI(1_i,j) - MSI(0_i,j) \tag{2.4}$$

Proof. Let us start with the right hand side:

$$MSI(1_i,j) - MSI(0_i,j) = \frac{1}{2^{n-2}}\left[\sum_{\{X:x_i=1,x_j=1\}}[\Phi(1_i,1_j,X) - \Phi(1_i,0_j,X)] + \right.$$
$$\left. - \sum_{\{X:x_i=1,x_j=1\}}[\Phi(0_i,1_j,X) - \Phi(0_i,0_j,X)]\right]$$
$$= \frac{1}{2^{n-2}}\left[\sum_{\{X:x_i=1,x_j=1\}}[\Phi(1_i,1_j,X) + \Phi(0_i,0_j,X) - \Phi(1_i,0_j,X) - \Phi(0_i,1_j,X)]\right]$$
$$= JSI(i,j).$$

Note that one can similarly prove that $JSI(i,j) = MSI(i,1_j) - MSI(i,0_j)$.

The above theorem interprets the relative importance of one of two components when the other is functioning. In particular:

(i) JSI>0 indicates that one of the components becomes more important when the other is functioning.

(ii) JSI<0 indicates that one of the components becomes less important when the other is functioning.

(iii) JSI=0 indicates that one component's important functioning of the other.

The following corollary shows that the value of JSI is bounded.

Corollary 1. For any two components i and j, $JSI(i,j) \in [-1,1]$.

Proof. By using theorem 1, we have $JSI(i,j) = MSI(1_i,j) - MSI(0_i,j)$. But $MSI(1_i,j) \in [0,1]$ and $MSI(0_i,j) \in [0,1]$, which implies that $JSI(i,j) \in [-1,1]$.

Note that, $JSI(i,j) = 1,-1$ respectively in the cases of 2-component series and parallel systems.

3. SIGN OF THE JOINT STRUCTURAL IMPORTANCE

The following corollary shows that it is possible in advance to determine the sign of JSI, in two special cases:

Corollary 2. For any s-coherent system, we have:

a. $JSI(i,j) \le 0$ if the components i, j are connected in parallel;

b. $JSI(i,j) \le 0$ if the components i, j are connected in series.

Proof.
One. Let us firstly assume that the components *i, j* are connected in parallel and component j is functioning (X_j=1), then *i* is irrelevant and then $MSI(1_i, j) = 0$. Using Theorem 1 gives

$$JSI(i,j) = MSI(1_j, i) - MSI(0_j, i) = -MSI(0_j, i) \le 0$$

Two. Now, assume that the components *i, j* are connected in series and component j is failed (X_j=0), then *i* is irrelevant and then $MSI(0_j, i) = 0$. Using theorem 1, we have

$$JSI(i,j) = MSI(1_i, j) \ge 0.$$

In fact that components *i* and *j* are connected in series means that the paths containing *i* contain *j* at the same time. Similarly, components *i* and *j* are connected in parallel means that the cuts containing *i* contain *j* at the same time. In general, there are three situations for the relations of the two components *i* and *j*:

One. There exists no path containing both components *i* and *j*; i.e., there exist some cuts containing both components *i* and *j*.
Two. There exists no cut containing both components *i* and *j* ; i.e., there exist some paths containing both components *i* and *j*.
Three. There exists some paths containing both components *i* and *j* and there exist cuts containing both component *i* and *j*.

It is impossible to have no cuts containing both *i* and *j* and no path containing both *i* and *j*. A special case of situation (b) is parallel components whereas a special case of situation (b) is series components. We show that the sign of JSI is non-positive for situation (a), but its sign is non-negative for situation (b).

Theorem 2. If there is no min-path containing both components *i* and *j*, then $JSI(i,j) \le 0$.

Proof. Let us start with a general system, and select any two of its components, say *i* and *j* Create all min-path sets of such system. Using these min-path sets we may represent the system as a parallel-series system, see Figure 1. Each path consists of at least one component.
By the hypothesis, no pat contains both components, but some of these paths contain component i, some contain component j, and some contain neither.
Order of the components. So we can rearrange the components i and j to appear as the last

components on any path as shown in Figure 2.
Combining components into arbitrary modules as follows:

a. All occurrences of component i are recombined into a single i components
b. All occurrences of component j are recombined into a single i components
c. All other components on paths containing i are combined to form module 1
d. All other components on paths containing j are combined to form module 2
e. All other components on paths containing neither i nor j are combined to form module 3

Figure 3 shows the corresponding structure. Let Z_k, k=1,2,3, be the structure function of the module k and $Z=(Z_1,Z_2,Z_3)$. The structure function of this system can be written as:

$$\begin{aligned}\Phi(X_i, X_j, Z) &= (Z_1 X_i) \vee (Z_2 X_j) \vee Z_3 \\ &= X_i(Z_1 - Z_1 Z_3) + X_j(Z_2 - Z_2 Z_3) + X_i X_j (Z_1 Z_2 Z_3 - Z_1 Z_3) + Z_3\end{aligned}$$

That is,

$$\begin{aligned}&\Phi(1_i, 1_j, Z) = (Z_1 - Z_1 Z_3) + (Z_2 - Z_2 Z_3) + (Z_1 Z_2 Z_3 - Z_1 Z_3) + Z_3, \\ &\Phi(0_i, 0_j, Z) = Z_3, \\ &\Phi(1_i, 0_j, Z) = (Z_1 - Z_1 Z_3) + Z_3, \\ &\Phi(0_i, 1_j, Z) = (Z_2 - Z_2 Z_3) + Z_3.\end{aligned}$$

Then $n_\Phi(i,j) = \sum_{\{X: X_i=1, X_j=1\}} (Z_3 - 1) Z_1 Z_2 \le 0$. Using the fact that Z_k=0 or 1 for k=1,2,3, implies $(Z_3 - 1) Z_1 Z_2 \le 0$ and then $n_\Phi(i,j) \le 0$ which completes the proof.

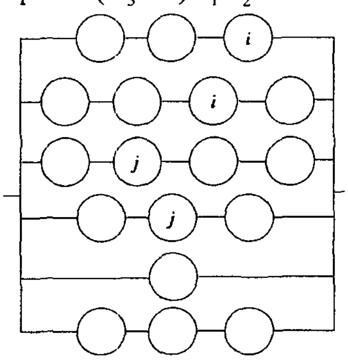

Figure 1. Module Formed.

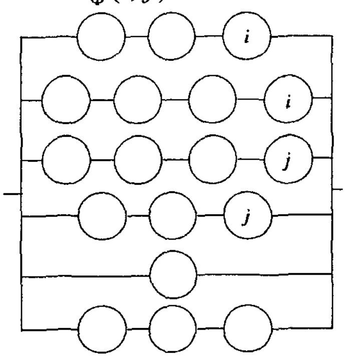

Figure 2. Rearranged components.

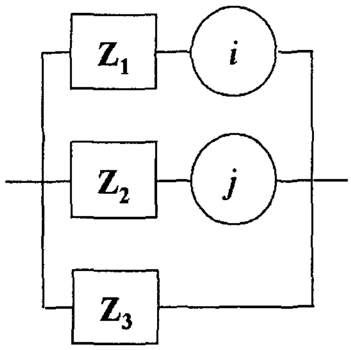

Figure 3. Module Formed.

Theorem 3. If there is no min-cut containing both components i and j, then $JSI(i,j) \ge 0$.

Proof. We can prove this theorem in similar steps to the proof of Theorem 1.

4. PARALLEL-SERIES AND SERIES- PARALLEL SYSTEMS

Note that it is possible to derive the asymptotic reliability functions for series-parallel and parallel-series systems, see Smith (1982) and Kolowrocki (1993, 1994a, 1994b). Other general reliability structures can be expressed in terms of max-min and min-max via the structure function of their corresponding path and cut sets.

In this section we shall introduce the JSI of two components i and j for the series-parallel and parallel-series systems considered by Yamashiro, et al. (1992) and Meng (1993). For this purpose we need the following additional notations. Let $N_1=\{1,2,\ldots,n\}$ and $N_2=\{n+1,n+2,\ldots,2n\}$ be subsets of the index set of the system components.

Theorem 4. For the series-parallel system shown in figure (4.I), we have

$$JSI(i,j)=\begin{cases}\dfrac{2^n-1}{2^{2(n-1)}} & \text{for } i,j\in N_1 \text{ or } i,j\in N_2,\\[2ex] \dfrac{-1}{2^{2(n-1)}} & \text{for } i\in N_1 \text{ and } j\in N_2.\end{cases} \tag{4.1}$$

Proof. Let us firstly start to proof the case when $i,j\in N_1$ or $i,j\in N_2$. Without loss of generality we can let i=1, j=2. Assume that the set of all paths $\{\mathbf{X}: X_1=1, X_2=1\}$ can be divided into the following pairwise disjoint sets:

- P_0: $\{(1,1,\ldots,1)\}$,
- P_1: The set of all vectors having at least one element $X_k=0$ for $k=n+1, n+2, \ldots, 2n$ while $X_l=1$ for $l=3, 4, \ldots, n$.
- P_2: The set of all vectors having at least one element $X_l=0$ for $l=3, 4, \ldots, n$ while $X_l=1$ for $k=n+1, n+2, \ldots, 2n$.
- P_3: The set of all vectors having at least one element $X_l=0$ while $X_l=0$ for $l=3, 4, \ldots, n$ and $k=n+1, n+2, \ldots, 2n$.

Let $\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=\Phi(1_1,1_2,\boldsymbol{X})+\Phi(0_1,0_2,\boldsymbol{X})-\Phi(1_1,0_2,\boldsymbol{X})-\Phi(0_1,1_2,\boldsymbol{X})$, then we can verify that:

$\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=1+1-1-1=0\,\forall\boldsymbol{X}\in P_0$, $\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=1+0-0-0=1\,\forall\boldsymbol{X}\in P_1$,

$\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=1+1-1-1=0\,\forall\boldsymbol{X}\in P_2$ & $\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=0+0-0-0=1\,\forall\boldsymbol{X}\in P_3$, Hence, we have $n_{\Phi}(i,j) = \sum_{\{X: X_1=1, X_2=1\}} \Psi(\cdot_i, \cdot_j, X) = \sum_{X \in P_1} 1 = |P_1| = 2^n - 1$, which completes the proof of the first case. In similar steps one can prove the second case when $i \in N_1$, $j \in N_2$.

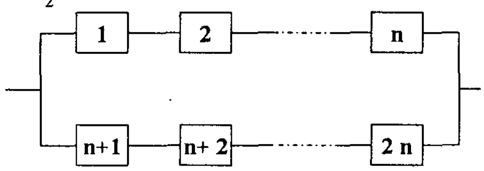

Figure 4.I

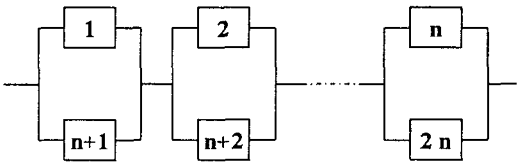

Figure 4.II

Theorem 5. For the parallel-series system shown in Figure 4.II, we have

$$JSI(i,j) = \begin{cases} \dfrac{3^{n-2}}{2^{2(n-1)}} & \text{for } i,j \in N_1 \text{ or } i,j \in N_2, \\ \dfrac{-3^{n-1}}{2^{2(n-1)}} & \text{for } i \in N_1 \text{ and } j \in N_2. \end{cases} \tag{4.2}$$

Proof. Let us firstly prove the first case when $i,j \in N_1$ or $i,j \in N_2$. Without loss of generality one can let i=1, j=2. Let us define the set P as the set of all vectors for which $X_{n+1}=0$, $X_{n+2}=0$ and at least one of X_l and $X_{n+l}=1$ for $l=3,4,\ldots,n$. This means that the set of all paths {**X**: $X_1=1$, $X_2=1$} can be represented as a union of P and its complement.

Let $\Psi(\cdot_i, \cdot_j, X) = \Phi(1_1, 1_2, X) + \Phi(0_1, 0_2, X) - \Phi(1_1, 0_2, X) - \Phi(0_1, 1_2, X)$, then one can verify that:

$$\Psi(\cdot_i, \cdot_j, X) = 1 \,\forall X \in P,\ \Psi(\cdot_i, \cdot_j, X) = 0 \,\forall X \notin P,$$

Hence, we have $n_{\Phi}(i,j) = \sum_{\{X: X_1=1, X_2=1\}} \Psi(\cdot_i, \cdot_j, X) = \sum_{X \in P} 1 = |P| = 3^{n-2}$, which completes the proof of the first case. In a similar approach one can prove the second case when $i \in N_1$, $j \in N_2$.

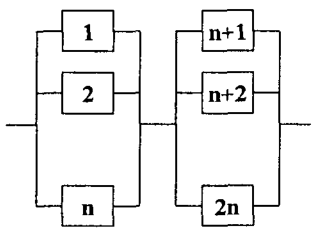

Figure 4.III

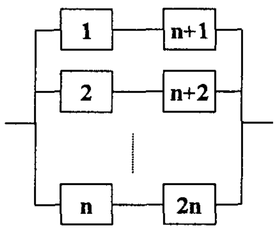

Figure 4.IV

Theorem 6. For the series-parallel system shown in Figure 4.III, we have

$$JSI(i,j)=\begin{cases}\dfrac{-3^{n-2}}{2^{2(n-1)}} & \text{for } i,j\in N_1 \text{ or } i,j\in N_2,\\ \dfrac{3^{n-1}}{2^{2(n-1)}} & \text{for } i\in N_1 \text{ and } j\in N_2.\end{cases} \tag{4.3}$$

Proof. Firstly we can prove the first case when $i,j\in N_1$ or $i,j\in N_2$ as in what follows. Without loss of generality one can let i=1, j=2. Let us define the set P as the set of all vectors for which $X_{n+1}=1$, $X_{n+2}=1$ and at least one of X_l and $X_{n+l}=0$ for $l=3,4,\ldots,n$. That is, the set of all paths {**X**: $X_1=1$, $X_2=1$} can be represented as a union of P and its complement.

Let $\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=\Phi(1_1,1_2,\boldsymbol{X})+\Phi(0_1,0_2,\boldsymbol{X})-\Phi(1_1,0_2,\boldsymbol{X})-\Phi(0_1,1_2,\boldsymbol{X})$, then one can verify that:

$$\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=-1\,\forall\,\boldsymbol{X}\in P,\ \Psi(\cdot_i,\cdot_j,\boldsymbol{X})=0\,\forall\,\boldsymbol{X}\notin P,$$

Hence, we have $n_\Phi(i,j)=\sum_{\{X:X_1=1,X_2=1\}}\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=\sum_{X\in P}-1=-|P|=-3^{n-2}$, which completes the proof of the first case. In similar steps one can prove the second case.

Theorem 7. For the parallel-series system plotted in Figure 4.IV, we have

$$JSI(i,j)=\begin{cases}-\dfrac{2^{n}-1}{2^{2(n-1)}} & \text{for } i,j\in N_1 \text{ or } i,j\in N_2,\\ \dfrac{1}{2^{2(n-1)}} & \text{for } i\in N_1 \text{ and } j\in N_2.\end{cases} \tag{4.4}$$

Proof. Let us firstly prove the first case when $i,j\in N_1$ or $i,j\in N_2$. Without loss of generality let i=1, j=2. Let us define the set P as the set of all vectors for which at least one element $X_l=1$ for $l=n+1$, $n+2$, $2n$, and $X_k=0$ for $k=3,4,\ldots,n$.

Let $\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=\Phi(1_1,1_2,\boldsymbol{X})+\Phi(0_1,0_2,\boldsymbol{X})-\Phi(1_1,0_2,\boldsymbol{X})-\Phi(0_1,1_2,\boldsymbol{X})$, then one can verify that:

$$\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=-1\,\forall\,\boldsymbol{X}\in P,\ \Psi(\cdot_i,\cdot_j,\boldsymbol{X})=0\,\forall\,\boldsymbol{X}\notin P,$$

Hence, we have $n_\Phi(i,j)=\sum_{\{X:X_1=1,X_2=1\}}\Psi(\cdot_i,\cdot_j,\boldsymbol{X})=\sum_{X\in P}-1=-|P|=-(2^n-1)$, which completes the proof of the first case. In a similar way one can prove the second case.

5. PRACTICAL EXAMPLES

In this section we present some practical examples. In such examples we calculate the MSI and JSI of the system components.

Example 1. Let us assume that we have a coherent system with structure function given by $\Phi(X) = x_1(x_2 \vee x_3)$. For this system we have:

1. MSI(1_1,2)=1/2, MSI(0_1,2)=0 and then JSI(1,2)=1/2. It means that one of the components 1 or 2 becomes more importance when the other is functioning.
2. MSI(1_1,3)=1/2, MSI(0_1,3)=0 and then JSI(1,3)=1/2. It means that one of the components 1 or 3 becomes more importance when the other is functioning.
3. MSI(1_2,3)=0, MSI(0_2,2)=0 and then JSI(1,2)=-1/2. It means that one of the components 2 or 3 becomes less importance when the other is functioning.

Example 2. Consider the bridge system shown in Figure 5. For this system one can deduce that:

1. JSI(1,4)=JSI(2,5)=1/2. It means that one of the components 1 and 4 (or 2 and 5) becomes more importance when the other is functioning.
2. JSI(1,3)=JSI(2,3)=JSI(3,4)=JSI(3,5)=0. It means that the importance of components 3 is unchanged by functioning of any of the components 1, 2, 4 and 4 and vise versa.
3. Similarly one can illustrate JSI(1,5)=JSI(2,4)=0.
4. JSI(1,2)=JSI(4,5)=-1/2. That is, one of the components 1 and 2 (or 4 and 5) becomes less importance when the other is functioning.

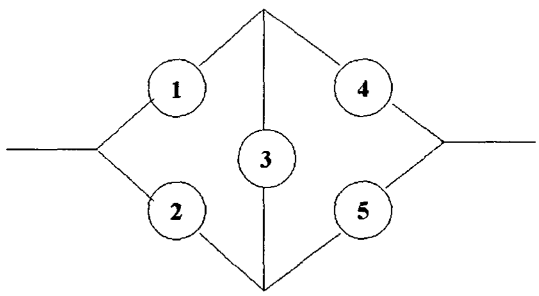

Figure 5. Bridge System

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