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Locating the Change Point of Mean Residual Life of Certain Life Distributions †

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Abstract. A class of life distributions, whose mean residual life keeps stable at its earlier phase and then starts to decrease in time, is proposed to model the life of an element having survived its burn-in. A strongly consistent estimator and a nonparametric testing procedure are developed to locate the occurrence of the change-point of the mean residual life. Finally, some numerical simulations are employed to be an illustration as well.

Key Words : *Mean residual life, Change-point, Estimation, Test hypothesis.*

1. INTRODUCTION

As aging properties such as the failure rate and the mean residual life of a random unit are always used to characterize the wear-out phenomenon in reliability and life-testing (*Hollander and Proschan*, 1984); In literature, lots of models concentrate on the monotonicity of the aging notions. In this line of researches, the life distribution of interest has a monotone ageing process, such as IFR and DMRL etc.. Recently, various non-monotone properties of aging notions are frequently discussed by *Guess et al*(1986), *Zacks*(1984), *Deshpande and Suresh*(1990), *Mitra and Basu*(1994) among many others, some existing results focus upon the situation that there exists a turning point such that aging properties on either sides are completely different, for example, BFR(*bathtub failure rate*) and IDMRL(*increasing initially, then decreasing mean residual life*). But in some practical situations, it is also observed that units, especially those having survived their burn-in, exhibit the following aging trend: after functioning steadily for a certain length of time, which may vary at random between similar systems, some complex mechanical systems(cars, airplanes and vessels etc.) enter a wear-out phase during which it becomes less reliable. This is always described as: in the first phase, the system has a constant failure rate, after entering the second one, its failure rate starts to increase in time (*Zacks*, 1984).

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Such a life distribution need not to satisfy the property that the MRL is initially stable and then decreases in time t; Thus, for certain applications, it may be more appropriate to model similar non-monotone aging through the dominance of MRL rather than failure rate. That is, in the earlier failure period, the system appears to have a constant MRL, and after a certain time point, it tends to be smaller than that in the earlier phase and starts to decrease in time. This enable us to have a greater flexibility in modelling the wear-out process. Of course, the epoch of the entrance from the stable phase to the wear-out one is of special interest. Before that, preventive maintenance is unnecessary; while after that point, it can make the system more reliable; So, an effective method to locate the change-point is necessary and beneficial for both engineers and reliability analysts to produce an efficacious maintenance policy.

This paper aims to make some investigations on nonparametric inferences of the change point of the mean residual life of a certain class of life distributions. In section 2, a class of life distributions is proposed to characterize the above wear-out process. Section 3 presents a strongly consistent estimator for the change-point. And in section 4, a nonparametric testing procedure is developed to detect whether the occurrence of the change-point is earlier or latter than a known point t_0. Finally, a simple numerical example is employed to illustrate the procedure.

2. THE E-DMRL CLASS OF LIFE DISTRIBUTIONS

Assume a random life with distribution function F, denote $\bar{F} = 1-F$ the survival function, then its *mean residual life* (MRL) at time $t \geq 0$ is given by

$$m(t) = \int_t^\infty \overline{F}(x)dx/\overline{F}(t). \tag{2.1}$$

MRL is always used to measure the residual life length that X will continue to have when it survives $t \geq 0$.

Definition 1. A life distribution F is said to be *exponential initially, then with decreasing mean residual life* (E-DMRL), if there exists a finite time point $\tau > 0$ before which the MRL keeps a constant, and then it starts to decrease in time $t \geq 0$.

The next theorem presents a characterization result, which gives the general form of the survival function of an EDMRL distribution.

Theorem 2. A component is E-DMRL if and only if it has survival function of the form

$$\overline{F}(x) = \begin{cases} e^{-\lambda x}, & x \leq \tau, \\ e^{-\lambda\tau}\bar{G}(x-\tau), & x > \tau, \end{cases} \tag{2.2}$$

where the value of MRL at the initial point is $1/\lambda$, τ is the change point, and $\bar{G}$ is the survival function of an arbitrary DMRL component with expectation $1/\lambda$.

Proof *Sufficiency* Note that,

$$\int_\tau^\infty \bar{F}(x)dx = \int_\tau^\infty e^{-\lambda\tau}\bar{G}(x-\tau)dx,$$

we have, for $0 \le t \le \tau$,

$$m(t) = \left\{\int_t^\tau e^{-\lambda x}dx + \int_\tau^\infty \bar{F}(x)dx\right\}/e^{-\lambda t} = \lambda^{-1};$$

And for $t > \tau$,

$$m(t) = \int_{t-\tau}^\infty \bar{G}(x)dx/\bar{G}(t-\tau);$$

Since G is DMRL, the decreasing property of $m(t)$ then follows immediately.

Necessity For $0 \le t \le \tau$, $m(t) = \lambda^{-1}$ implies that $\lambda^{-1}\bar{F}(t) = \int_t^\infty \bar{F}(x)dx$, then, $\bar{F}(t) = e^{-\lambda t}$. In consideration that $E_F X = \lambda^{-1}$, it holds that

$$\int_0^\infty e^{-\lambda x}dx = \int_0^\infty \bar{F}(x)dx.$$

And hence

$$\int_0^\infty \bar{F}(x+\tau)dx = \int_\tau^\infty \bar{F}(x)dx = \int_\tau^\infty e^{-\lambda x}dx = \lambda^{-1}e^{-\lambda\tau}.$$

This tells that

$$\int_0^\infty e^{\lambda\tau}\bar{F}(x+\tau)dx = \lambda^{-1}.$$

Denote

$$\bar{G}(x) = e^{\lambda\tau}\bar{F}(x+\tau), \qquad \text{for } x \ge 0.$$

Thus, for $t > \tau$, $\bar{F}$ is of the form $e^{-\lambda\tau}\bar{G}(t-\tau)$. Since $m(t)$ is decreasing in $t > \tau$, G is a DMRL life distribution with expectation $1/\lambda$.

In view of the fact that IFR implies DMRL, it follows then from Theorem 2, that a life distribution, which has a constant failure rate up to time τ and then an increasing failure rate since time τ on, is E-DMRL with change piont τ.

3. THE CHANGE-POINT ESTIMATION

A reasonable estimator for the epoch of a E-DMRL life is closely relevant to the maintenance policy in applications, because no one would think of a preventive replacement before the change-point, while after that it may be of special interest. Suppose $X_1, \cdots, X_n$ an independent and identical observations from an E-DMRL life distribution F; In consideration that an E-DMRL life would be exponential if its change-point is infinite, and it is DMRL if the change point is 0, we assume that

(1) The change-point of F, under consideration, is finite, positive and unique.

(2) There exists an upper bound for the change-point, say b, such that $0 < F(b) < 1$.

In most practical situations, the assumption 2 is quite mild because the prior experiences about the population may offer some ideas about the bound b.

For convenience, denote $m(F,t)$ the MRL at time $t \geq 0$, then

$$\tau = \inf\{t : m(F,t) < m(F,0) = \mu\},$$

where $\mu = EX_1$. The *empirical mean residual life* $m(F_n,t)$ and the sample mean $\overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ are reasonable estimators for $m(F,t)$ and μ respectively, the following simple form is convenient.

$$m(F_n,t) = \begin{cases} \left[\sum_{j=1}^{n} I(X_j > t)\right]^{-1} \sum_{i=1}^{n}(X_i - t)I(X_i > t), & X_{(n)} > t, \\ 0, & \text{otherwise}, \end{cases}$$

where $I(A) = 0$ or 1 according as the event A is false or true. For any integer $k > 0$, let

$$\Lambda_n(k) = \{0 < t \leq b : \overline{X}_n - m(F_n,t) = k^{-1}\}, \tag{3.1}$$

and $t_n(k) = \inf \Lambda_n(k)$, it is proven that, with probability 1, $t_n(k)$ converges to $\tau(k)$, the intersect between the curve $m(F,t)$ and the straight line $\mu - k^{-1}$. That is, $m(F,\tau(k)) = \mu - k^{-1}$.

Lemma 3. For any fixed k and n sufficiently large, the set $\Lambda_n(k)$ is non-empty.
Proof $m(F,t)$ is strictly decreasing in $t \geq \tau$, there must exist t_1 and t_2 such that $\tau < t_1 < \tau(k) < t_2 < b$ and furthermore,

$$\mu - m(F,t_1) < k^{-1}, \qquad \mu - m(F,t_2) > k^{-1}.$$

By *Yang*(1978), as $n \to \infty$, for any $c > 0$ satisfying $F(c) < 1$,

$$\sup_{0\leq t\leq c} \mid m(F_n,t) - m(F,t) \mid \xrightarrow{a.s} 0. \tag{3.2}$$

Strong Law of Large Number gives that

$$\bar{X}_n - m(F_n,t) \xrightarrow{a.s} \mu - m(F,t), \qquad n \to \infty.$$

Now, for n sufficiently large, with probability 1, it holds that

$$\bar{X}_n - m(F_n,t_1) < k^{-1} \qquad \text{and} \qquad \bar{X}_n - m(F_n,t_2) > k^{-1}.$$

Because $m(F_n,t)$ only has positive jumps, the strict decreasing property of $m(F_n,t)$ in every segment gives the existence of the solution for $\bar{X}_n - m(F_n,t) = k^{-1}$.

Theorem 4. For any fixed integer $k > 0$, $t_n(k)$ is strongly consistent for $\tau(k)$.
Proof By definition, the sequence $\{t_n(k)\}$ is bounded. Assume that $\{t_{n_1}(k)\}$ is an arbitrary convergent subsequence of $\{t_n(k)\}$, and for fixed k, $t_{n_1}(k) \xrightarrow{a.s} \tau^*(k)$,

as $n_1 \to \infty$. Note that the curve $m(F_n, t)$ has only n linear segments with negative slope -1, and each of them lies between two adjacent observations, it is clear that $\Lambda_n(k)$ has not more than n elements. For any $t_{n_1}(k) \in \Lambda_{n_1}(k)$, as n_1 is sufficiently large,

$$0 \leq | \bar{X}_{n_1} - m(F_{n_1}, t_{n_1}(k)) - k^{-1} | \leq | \bar{X}_{n_1} - m(F_{n_1}, \tau(k)) - k^{-1} | .$$

Let $n_1 \to \infty$, it follows that, with probability 1,

$$0 \leq | \mu - m(F, \tau^*(k)) - k^{-1} | \leq | \mu - m(F, \tau(k)) - k^{-1} | = 0.$$

The uniqueness of the intersect $\tau(k)$ guarantees that $\tau^*(k) \overset{a.s.}{=} \tau(k)$.

Thus, every convergent subsequence of $t_n(k)$ converges almost surely to the same intersect $\tau(k)$. This now deduces that, as $n \to \infty$, $t_n(k) \xrightarrow{a.s.} \tau(k)$.

Theorem 5.

$$P\left(\lim_{k\to\infty} \lim_{n\to\infty} t_n(k) = \tau\right) = 1. \tag{3.3}$$

Proof Recall for all $0 \leq t \leq \tau$, $| \bar{X}_n - m(F_n, t) | \leq | \bar{X}_n - \mu | + | m(F_n, t) - \mu |$, as in Lemma 3 there exists an integer $N_1 > 0$ such that, as n exceeds it,

$$| \bar{X}_n - \mu | < (2k)^{-1} \qquad \text{and} \qquad \sup_{0\leq t\leq\tau} | m(F_n, t) - \mu | < (2k)^{-1},$$

and hence $\sup_{0\leq t\leq\tau} | \bar{X}_n - m(F_n, t) | \leq k^{-1}$. That is, for n sufficiently large, $t_n(k) = \inf \Lambda_n(k) > \tau$. So, with probability 1, $\lim_{n\to\infty} t_n(k) = \tau(k) > \tau$. In addition, $\tau(k)$ is strictly decreasing in k because of the same property of $m(F, t)$ in $t > \tau$; Therefore, $\tau(k) \to \tau$, as $k \to \infty$.

4. A TESTING PROCEDURE ON THE CHANGE-POINT

In this section, a testing procedure is developed to detect whether the change point of MRL is larger or smaller than a known point t_0, which is always the primary knowledge about τ drawn from history or experience. To judge whether we are too conservative, that is, $\tau \geq t_0$ may holds in fact, the following hypothesis needs to be tested,

$$H : \tau \geq t_0 \qquad \text{versus} \qquad K : \tau < t_0. \tag{4.1}$$

It is difficult for us to acquire the accurate or asymptotic distribution of the estimator in section 3, for this reason, *empirical mean residual life function* will be used to construct a suitable testing statistic.

Under H, it holds that

$$m(F, t_0) = m(F, 0) = \mu; \tag{4.2}$$

Intuitively, a larger value of the parameter $m(F,0)-m(F,t_0)$ tends to rejection of H. As a result, $m(F_n,0)-m(F_n,t_0)$ is preferred here to be a reasonable estimation for the above parameter. The following theorem presents its asymptotic distribution.

Theorem 6. Suppose that $VarX_1 = \sigma^2 < \infty$, then, as $n \to \infty$,

$$\sqrt{n}\left[(\bar{X}_n - \mu) - (m(F_n,t_0) - m(F,t_0))\right]/\sqrt{\sigma^2(t_0)} \xrightarrow{L} N(0,1), \tag{4.3}$$

where

$$\sigma^2(t_0) = \Gamma(0,0) - 2\Gamma(0,F(t_0)) + \Gamma(F(t_0),F(t_0)), \tag{4.4}$$

and, for $0 \le s \le t \le 1$,

$$\begin{aligned} \Gamma(s,t) &= [(1-s)(1-t)]^{-1}\sigma^2(t,1) - t(1-s)^{-1}(1-t)^{-2}\theta^2(t,1), \qquad (4.5)\\ \theta(t,u) &= E[I(t < F(X) \le u)X],\\ \sigma^2(t,u) &= Var[I(t < F(X) \le u)X]. \end{aligned}$$

Proof According to Theorem 1 in *Yang* (1978), as $n \to \infty$, the process

$$\sqrt{n}(m(F_n,F^{-1}(t)) - m(F,F^{-1}(t)), \quad \text{for } t \in [0,b], \quad 0 < b < 1,$$

converges in distribution to a *Gaussian process* $U(t)$ with mean zero and covariance function $\Gamma(s,t)$, as in (4.5). Then, as $n \to \infty$, the random vector

$$\sqrt{n}(m(F_n,0) - m(F,0), m(F_n,t_0) - m(F,t_0))'$$

will converge to a bivariate normal distribution with mean vector $(0,0)'$ and covariance metric

$$\begin{pmatrix} \Gamma(0,0) & \Gamma(0,F(t_0)) \\ \Gamma(0,F(t_0)) & \Gamma(F(t_0),F(t_0)) \end{pmatrix}.$$

It follows now from *Cramér-Wold* theorem (*Billingsley*, 1968) that, as $t \to \infty$,

$$\sqrt{n}\left[(m(F_n,0) - m(F,0)) - (m(F_n,t_0) - m(F,t_0))\right] \xrightarrow{L} N(0,\sigma^2(t_0)), \tag{4.6}$$

where $\sigma^2(t_0)$ is determined by (4.4).

Direct evaluation gives that

$$\begin{aligned} \theta(F(t_0),1) &= E[I(F(t_0) < F(X))X] = EXI(X > t_0),\\ \sigma^2(F(t_0),1) &= Var[I(F(t_0) < F(X))X] = EX^2I(X > t_0) - E^2XI(X > t_0). \end{aligned}$$

Furthermore, for $t_0 \le \tau$,

$$\begin{aligned} \Gamma(0,0) &= \sigma^2(0,1) = VarX = \sigma^2,\\ \Gamma(0,F(t_0)) &= e^{\lambda t_0}\sigma^2(F(t_0),1) - \left(e^{2\lambda t_0} - e^{\lambda t_0}\right)\theta^2(F(t_0),1),\\ \Gamma(F(t_0),F(t_0)) &= e^{2\lambda t_0}\sigma^2(F(t_0),1) - \left(e^{3\lambda t_0} - e^{2\lambda t_0}\right)\theta^2(F(t_0),1). \end{aligned}$$

So, it follows immediately that, for $t_0 \leq \tau$,

$$\sigma^2(t_0) = \sigma^2 + \left(e^{2\lambda t_0} - 2e^{\lambda t_0}\right)\sigma^2(F(t_0),1) - \left(e^{3\lambda t_0} - 3e^{2\lambda t_0} + 2e^{\lambda t_0}\right)\theta^2(F(t_0),1). \tag{4.7}$$

By *Strong law of large number*, it holds that, with probability 1, as $n \to \infty$,

$$S_n^2 = n^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \to \sigma^2, \qquad \bar{X}_n \to \mu = \lambda^{-1},$$

and

$$A_n(t_0) = n^{-1}\sum_{i=1}^{n} X_i I(X_i > t_0) \to \theta(F(t_0),1),$$

$$B_n(t_0) = n^{-1}\sum_{i=1}^{n} X_i^2 I(X_i > t_0) - \left(n^{-1}\sum_{i=1}^{n} X_i I(X_i > t_0)\right)^2 \to \sigma^2(F(t_0),1).$$

Let

$$\widehat{\sigma^2(t_0)} = S_n^2 + \left(e^{2t_0/\bar{X}_n} - 2e^{t_0/\bar{X}_n}\right)B_n(t_0) - \left(e^{3t_0/\bar{X}_n} - 3e^{2t_0/\bar{X}_n} + 2e^{t_0/\bar{X}_n}\right)A_n^2(t_0), \tag{4.8}$$

from the continuity, we have

Lemma 7. Suppose that $VarX_1 = \sigma^2 < \infty$, under H, it holds that, as $n \to \infty$,

$$\widehat{\sigma^2(t_0)} \xrightarrow{a.s.} \sigma^2(t_0).$$

Theorem 8. Suppose that $VarX_1 = \sigma^2 < \infty$, then, under H, as $n \to \infty$,

$$T_n(t_0) = \sqrt{n}\left(\bar{X}_n - m(F_n,t_0)\right)/\sqrt{\widehat{\sigma^2(t_0)}} \xrightarrow{L} N(0,1), \tag{4.9}$$

where $\widehat{\sigma^2(t_0)}$ is determined by (4.8).

Proof Note that (4.2) holds under H, applying *Slutsky theorem* to Theorem 6 and Lemma 7 will directly give the conclusion.

Now, a testing rule for (4.1) is established as follows: if the observed value of $T_n(t_0)$ is too larger, then it is reasonable for us to reject H.

5. A SIMPLE NUMERICAL EXAMPLE

We restrict our attention in this study to the following model, a special case of (2) with $\alpha = \lambda = 2$, $\tau = 1/2$, to illustrate the previous procedure. It is obvious that

$$m(F,t) = \begin{cases} 1/2, & \text{for } 0 < t \leq 1/2, \\ (4t)^{-1}, & \text{for } 1/2 < t. \end{cases}$$

Table 1. Some numerical simulation results

n	$\bar{X}_n$	$m(F_n, 0.4)$	$m(F_n, 0.6)$	$\widehat{\sigma^2(0.4)}$	$\widehat{\sigma^2(0.6)}$	$T_n(0.4)$	$T_n(0.6)$
30	0.3359	0.3453	0.3037	0.3155	0.6046	-0.0913	0.22697
50	0.3796	0.3551	0.3060	0.2026	0.1727	0.3851	1.2521
60	0.4013	0.3855	0.3210	0.2280	0.2010	0.2561	1.3868

Consequently, the performance of the above testing procedure at the points 0.4 and 0.6 is studied respectively. *Monte Carlo* method is used to produce a sample of size 60. The data in the table 1 are the corresponding numerical results. Two elements are listed in some cells, the upper one corresponds to that of $t_0 = 0.4$, and the lower one, that of $t_0 = 0.6$. For $n = 30$, when $t_0 = 0.6$, $T_n(t_0)$ is too small to reject H, the poor performance is obviously caused by the less observations exceeding t_0; In fact, only 5 ones in 30 is greater than 0.6. For $n = 50$ and $n = 60$, as $t_0 = 0.6$, this testing procedure works well than before, the corresponding powers are 0.8944 and 0.9164, respectively; In the mean time, for $t_0 = 0.4$, $T_n(t_0)$ is not large enough for us to reject H. It can be readily seen that the testing method gradually behaves well as the sample size increases.

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