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A Note on a New Two-Parameter Lifetime Distribution with Bathtub-Shaped Failure Rate Function

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Abstract. This paper presents the methodology for obtaining point and interval estimating of the parameters of a new two-parameter distribution with multiple-censored and singly censored data (Type-I censoring or Type-II censoring) as well as complete data, using the maximum likelihood method. The basis is the likelihood expression for multiple-censored data. Furthermore, this model can be extended to a three-parameter distribution that is added a scale parameter. Then, the parameter estimation can be obtained by the graphical estimation on probability plot.

Key Words : *bathtub-shaped failure rate, censoring data, maximum likelihood estimation.*

1. INTRODUCTION

The failure rate function for many mechanical and electronic components may have a bathtub shape. Models in which allow only monotone failure rates might not be appropriate or adequate for modeling the whole bathtub-shaped data. Several models have been proposed to modeling the bathtub-shaped failure rates; see references and they are summarized in Table 1. Among them, a variety of methods for estimation and testing based on general principles such as method of moments, least squares, and maximum likelihood have been examined and discussed for these models. However, most models are not practical to be used by reliability engineers. The parameters of the model by Haupt and Schabe (1992), the additive Weibull model by Xie and Lai (1996), and the additive Burr XII model by Wang (2000) can be estimated by the probability plotting techniques. Recently, Chen (2000) proposes a new two-parameter distribution with bathtub or increasing failure rate function. Compared with other models, this model has some useful properties. First, there are only two parameters to model the bathtub-shaped failure rate function. Second, the confidence intervals and exact joint confidence regions for the parameters have closed form based on Type-II censored data. However, it is important to develop the maximum likelihood estimation of this model under all types of data conditions. Indeed, most laboratory reliability data is either complete or the result of Type-I or Type-II censoring. In fact, the reliability data can be a multiple-censored and for

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this case parameter estimation can be important. It is the intention to develop the methodology for obtaining point and intervals estimates of the parameters of this model with multiple-censored data and then treat Type-I censoring, Type-II censoring and complete as special cases. Thus, there is no need to develop this model parameter estimates from separate likelihood functions.

Table 1. Some models for a bathtub-shaped failure rate

Author	Reliability function & Hazard function	Characteristics
Hjorth (1980)	$R(t) = 1 - F(t) = \frac{e^{-\delta t^2/2}}{(1+\beta t)^{\theta/\beta}}, t \geq 0$ $h(t) = \delta t + \frac{\theta}{1+\beta t}$	$\theta = 0 \to$ Weibull distribution $\delta = \beta = 0 \to$ exponential distribution $\delta = 0 \to$ decreasing failure rate $\delta \geq \theta\beta \to$ increasing failure rate $0 < \delta < \theta\beta \to$ bathtub curve
Haupt & Schabe (1992)	$R(t) = 1 + \beta - \sqrt{\beta^2 + (1+2\beta)t/t_0}$, $0 \leq t \leq t_0$ $h(t) = \frac{1+2\beta}{2t_0 A(1+\beta - A)}, 0 \leq t \leq t_0$ where $A = \sqrt{\beta^2 + (1+2\beta)t/t_0}$	$\beta \leq -1/3 \to$ increasing failure rate $\beta \geq 1 \to$ increasing failure rate $-1/3 < \beta < 1 \to$ bathtub curve
Mudholkar & Srirastava (1993)	$R(t) = 1 - [1 - \exp(-t/\sigma)^{\alpha}]^{\theta}, t \geq 0$ $h(t) = \frac{\alpha\theta(1-A)^{\theta-1} A(t/\sigma)^{\alpha-1}}{\sigma[1-(1-A)^{\theta}]}, t \geq 0$ where $A = \exp(-t/\sigma)^{\alpha}$	$\theta = 1 \to$ Weibull distribution $\alpha = \theta = 0 \to$ exponential distribution $\alpha, \theta < 1 \to$ decreasing failure rate $\alpha, \theta > 1 \to$ increasing failure rate $\alpha > 1, \theta < 1 \to$ bathtub curve or increasing $\alpha < 1, \theta > 1 \to$ unimodal or decreasing
Xie & Lai (1996)	$R(t) = \exp(-(at)^b - (ct)^d)$, $t \geq 0, b > 1, d < 1$ $h(t) = ab(at)^{b-1} + cd(ct)^{d-1}, t \geq 0$	Bathtub curve
Wang (2000)	$R(t) = \exp\{-k_1 \ln[1 + (t/s_1)^{c_1}]$ $- k_2 \ln[1 + (t/s_2)^{c_2}]\}$ $h(t) = \frac{k_1 c_1 (t/s_1)^{c_1 - 1}}{s_1[1+(t/s_1)^{c_1}]} + \frac{k_2 c_2 (t/s_2)^{c_2 - 1}}{s_2[1+(t/s_2)^{c_2}]}$ $t, k_1, k_2, s_1, s_2 \geq 0, 0 < c_1 < 1, c_2 > 2$	Bathtub curve
Chen (2000)	$R(t) = \exp[\lambda(1 - e^{t^{\beta}})], t, \lambda, \beta > 0$ $h(t) = \lambda\beta t^{\beta-1} e^{t^{\beta}}\ t, \lambda, \beta > 0$	$\beta < 1 \to$ bathtub curve $\beta \geq 1 \to$ increasing failure rate

A new two-parameter distribution with bathtub shape or increasing rate function and the density function and cumulative density function are given by (Chen, 2000)

$$f(t) = \lambda \beta t^{\beta-1} \exp[t^{\beta} + \lambda(1 - e^{t^{\beta}})],\ t, \lambda, \beta > 0 \tag{1}$$

$$F(t) = 1 - \exp[\lambda(1 - e^{t^{\beta}})],\ t, \lambda, \beta > 0 \tag{2}$$

The corresponding failure rate function of this model is $h(t) = \lambda \beta t^{\beta-1} e^{t^{\beta}}$. It can be seen that when $\beta < 1$, the failure rate function is a bathtub type, and when $\beta \geq 1$, the failure rate function is increasing (see, Chen, 2000).

2. MAXIMUM LIKELIHOOD ESTIMATION

2.1 Multiple-censored data

The likelihood function for the multiple-censored data is given by

$$L = f(t_{1,f}, \mathrm{K}, t_{r,f}, t_{1,s}, \mathrm{K}, t_{m,s}) = C \prod_{i=1}^{r} f(t_{i,f}) \prod_{j=1}^{m} [1 - F(t_{j,s})] \tag{3}$$

where C is a constant, $f(.)$ is the density function and $F(.)$ is the distribution function. There are r failure at times $t_{1,f}, \mathrm{K}, t_{r,f}$ and m units with censoring times $t_{1,s}, \mathrm{K}, t_{m,s}$. With $f(t)$ and $F(t)$ given by equations (1) and (2), respectively, the logarithm of the likelihood function becomes

$$\ln L = \ln C + r \ln \lambda + r \ln \beta + \sum_{i=1}^{r} (\beta - 1) \ln t_i + (m + r)\lambda + \sum_{i=1}^{r} t_i^{\beta} - [\sum_{i=1}^{r} \lambda e^{t_i^{\beta}} + \sum_{j=1}^{m} \lambda e^{t_j^{\beta}}] \tag{4}$$

Upon differentiating equation (4) with respect to λ and β, and equating result to zero, two equations must be simultaneously satisfied to obtain the estimates of λ and β These equations are given by

$$\begin{aligned} \frac{\partial \ln L}{\partial \lambda} &= \frac{r}{\lambda} + (m + r) - \sum_{i=1}^{r} e^{t_i^{\beta}} - \sum_{j=1}^{m} e^{t_j^{\beta}} = 0, \\ \frac{\partial \ln L}{\partial \beta} &= \frac{r}{\beta} + \sum_{i=1}^{r} \ln_i + \sum_{i=1}^{r} t_i^{\beta} \ln t_i - \lambda[\sum_{i=1}^{r} e^{t_i^{\beta}} t_i^{\beta} \ln t_i + \sum_{j=1}^{m} e^{t_i^{\beta}} t_i^{\phi} \ln t_j] = 0 \end{aligned} \tag{5}$$

In the above form, we have

$$\hat{\lambda} = \frac{r}{(\sum_{i=1}^{r} e^{t_i^{\hat{\beta}}} + \sum_{j=1}^{m} e^{t_j^{\hat{\beta}}}) - m - r} \tag{6}$$

and $\hat{\beta}$ is the solution of

$$\frac{r}{\hat{\beta}} + \sum_{i=1}^{r} \ln t_i + \sum_{i=1}^{r} t_i^{\hat{\beta}} \ln t_i - \frac{r}{(\sum_{i=1}^{r} e^{t_i^{\hat{\beta}}} + \sum_{j=1}^{m} e^{t_j^{\hat{\beta}}}) - m - r} [\sum_{i=1}^{r} e^{t_i^{\hat{\beta}}} t_i^{\hat{\beta}} \ln t_i + \sum_{j=1}^{m} e^{t_j^{\hat{\beta}}} t_j^{\hat{\beta}} \ln t_j] = 0 \tag{7}$$

These equations can be solving by statistical software or IMSL subroutine.

2.2 Type I or Type II censoring data

The likelihood function for the first *r* observations from a sample size *n* drawn from the model in both Type-I and Type-II censoring is given by

$$L = f(t_{1,n}, \ldots, t_{r,n}) = C\prod_{i=1}^{r} f(t_{i,n})[1 - F(t_*)]^{n-r} \tag{8}$$

where $t_* = t_0$, the time of cessation of the test for Type-I censoring and $t_* = t_r$, the time of the *r*th failure for Type-II censoring. Equations (6) and (7) become

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r} e^{t_i^{\hat{\beta}}} + (n-r)e^{t_*^{\hat{\beta}}} - n} \tag{9}$$

$$\frac{r}{\hat{\beta}} + \sum_{i=1}^{r} \ln t_i + \sum_{i=1}^{r} t_i^{\hat{\beta}} \ln t_i - \frac{r}{\sum_{i=1}^{r} e^{t_i^{\hat{\beta}}} + (n-r)e^{t_*^{\hat{\beta}}} - n}\Big[\sum_{i=1}^{r} e^{t_i^{\hat{\beta}}} t_i^{\hat{\beta}} \ln t_i + \sum_{j=1}^{m} e^{t_j^{\hat{\beta}}} t_j^{\hat{\beta}} \ln t_j\Big] = 0 \tag{10}$$

2.3 Complete censored data

Simply replace *r* with *n* in the equations (6) and (7) and ignore the t_j portions. The maximum likelihood equations for the λ and β are given by

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} e^{t_i^{\hat{\beta}}} - n} \tag{11}$$

$$\frac{n}{\hat{\beta}} + \sum_{i=1}^{n} \ln t_i + \sum_{i=1}^{n} t_i^{\hat{\beta}} \ln t_i - \frac{n}{\sum_{i=1}^{n} e^{t_i^{\hat{\beta}}} - n} \times \sum_{i=1}^{n} e^{t_i^{\hat{\beta}}} t_i^{\hat{\beta}} \ln t_i = 0 \tag{12}$$

3. CONFIDENCE INTERVALS OF ESTIMATES

The asymptotic variance-covariance matrix of the parameters (λ and β) is obtained by inverting the Fisher information matrix

$$I_{ij} = E\left[-\frac{\partial^2 L}{\partial\theta_i \partial\theta_j}\right], \quad i, j = 1,2$$

where $\theta_1, \theta_2 = \lambda$ or β (Nelson, 1990). This leads to

$$\begin{bmatrix} Var(\hat{\lambda}) & Cov(\hat{\lambda},\hat{\beta}) \\ Cov(\hat{\lambda},\hat{\beta}) & Var(\hat{\beta}) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial^2 \lambda}\Big|_{\hat{\lambda},\hat{\beta}} & -\frac{\partial^2 \ln L}{\partial\lambda\partial\beta}\Big|_{\hat{\lambda},\hat{\beta}} \\ -\frac{\partial^2 \ln L}{\partial\beta\partial\lambda}\Big|_{\hat{\lambda},\hat{\beta}} & -\frac{\partial^2 \ln L}{\partial^2 \beta}\Big|_{\hat{\lambda},\hat{\beta}} \end{bmatrix}^{-1}$$

Approximate (1-α) 100% confidence intervals on λ and β are obtained from the asymptotic normality of the MLEs (Nelson, 1990). This results in intervals calculated according to:

$$\hat{\lambda} \pm Z_{\alpha/2}\sqrt{Var(\hat{\lambda})} \text{ and } \hat{\beta} \pm Z_{\alpha/2}\sqrt{Var(\hat{\beta})}$$

where $Z_{\alpha/2}$ is upper percentile of standard normal variate.

In addition, another joint confidence region for the parameters λ and β based on type-II censoring data can be found in Chen (2000).

4. AN EXTENDED MODEL & PROBABILITY PLOTTING PROCEDURE

Here, we may add a scale parameter θ to the model and allow the analyst to manipulate the horizontal and vertical scales to represent many kinds of hazards. The reliability function and hazard function for this modify model are given by

$$R(t) = \exp[\lambda\theta(1 - e^{(t/\theta)^\beta})], t, \lambda, \theta, \beta > 0 \quad (13)$$

$$h(t) = \lambda\beta(t/\theta)^{\beta-1} e^{(t/\theta)^\beta})], t, \lambda, \theta, \beta > 0 \quad (14)$$

When $\theta = 1$, hazards for various β values can be found in Chen (2000). The results show that this model has many different types of bathtub or increasing failure rate function.

Probability plotting is a graphical method used to investigate whether an assumed model adequately fits a set of data. It helps the analyst to assess how well a given theoretical distribution fits the data and to estimate distribution parameters through least squares (LS) procedures. Plotting the data values against the corresponding estimated quantile values, where the scale is adjusted so that the relationship is linear for a given theoretical distribution, produces a probability plot. Thus, a linear pattern of points indicates agreement between the data distribution and the theoretical distribution; a nonlinear pattern indicates that the assumed distribution is not a reasonable representation of the data. If one can take the log-transformation of both sides in equation (13), rewrite the equation, and retake the log-transformation of both sides in equation again, the extended model becomes

$$\ln\ln\left(1 - \frac{\ln R(t_p)}{\lambda\theta}\right) = -\beta\ln(\theta) + \beta\ln(t) \quad (15)$$

Thus, the empirical value of $\ln\ln\left(1 - \frac{\ln R(t_p)}{\lambda\theta}\right)$ is plotted versus the ordered log-transformed data values. Here, $R(t_p)$ is estimated by Herd-Johnson procedure, that is, $\hat{R}(t_p) = \prod_{r=1}^{f} \frac{n-r+1}{n-r+2}$ where *n* is the number of units on test, *r* is the order of each failure after order numbers have been assigned to all units based on their running times, and *f* is the observed number of failures. In the case of complete data, the Herd-Johnson estimate is identical to the maximum likelihood (ML) estimation order-statistic, *i*/(*n*+1). For the simplicity, we may assume $\lambda\theta = 1$, then the estimation of β is the slope of the equation (15), the estimation of θ is equal to $\exp(\frac{\text{intercept}}{\hat{\beta}})$. Thus, the estimation of λ is equal to $\frac{1}{\hat{\theta}}$. For the general case, the Weibull transformation is used to determine the parameter estimation and is given by

$$\ln\left(-\ln R(t_p)\right) = \ln(\lambda\theta) + \ln(e^{(t/\theta)^\beta} - 1)$$

When t is small, we have $e^x - 1 \cong x + O(x)$, where $x = (t/\theta)^\beta$. Then, the parameter β is determined by the slope of the regression line that is $\ln\left(-\ln R(t_p)\right) = \ln(\lambda\theta^{1-\beta}) + \beta \ln t$. When t is large, we have $\ln(e^x - 1) \cong -e^x + x + O(x)$, where $x = (t/\theta)^\beta$. Since $-e^x \to 0$ as x is large, the regression line becomes $\ln\left(-\ln R(t_p)\right) = \ln(\lambda\theta) + (t/\theta)^\beta$. In this case, the parameters λ and θ cannot be easily determined. Therefore, the maximum likelihood estimation method can be used to determine the parameter estimation for the general case.

Example: Table 2 contains the times to failure of 50 devices by Aarset [12]; the TTT plot indicates a bathtub-shaped hazard rate (see Wang, 2000). Let us denote the failure times by $t_1, t_2, \ldots, t_{50}$ and assume that $t_1 < t_2 < \ldots < t_{50}$. Muldholkar and Srivastava (1993) analyzed this data using an exponentiated-Weibull model. The parameter estimation by maximum likelihood method is obtained by $\alpha = 4.69, \theta = 0.146, \text{and } \sigma = 91.023$. Using the model by Haupt and Schabe (1992), plotting $i/(n+1)$ versus $(n+1)t_i/i$, we can found the approximately a straight line with intercept $\alpha = 2\beta t_0/(1+2\beta) = 19.889$ and slope $\tan(\phi) = t_0/(1+2\beta) = 108.29$. This gives $t_0 = 128.179$ and $\beta = 0.09$. Using the additive-Weibull model by Xie and Lai (1996), a Weibull plot based on the first fifteen points gives an estimate the slope as 0.4996, which indeed corresponds to a decreasing failure rate at the beginning. The estimated slope for the last ten points is 30.069, which corresponds to an increasing failure rate. The overall parameter estimation is given by $a \approx 0$, $b = 30.069$, $c = 0.0912$ and $d = 0.4996$. Using the additive Burr XII by Wang (2000), a Burr plot based on the first fifteen points gives an estimate the slope as 0.5067 and the intercept as –3.885, which indeed corresponds to a decreasing failure rate at the beginning. The estimated slope for the last ten points is 152.93 and the intercept as –674.39, which corresponds to an increasing failure rate. The overall parameter estimation is given by $c_1 = 0.5067, s_1 = 2137.215, k_1 = 5.5, c_2 = 152.93, s_2 = 85.2526, \text{and } k_2 = 0.5$. Using the extended model, plotting $i/(n+1)$ versus $\ln\ln\left(1 - \ln R(t_p)\right)$ in Figure 1, we can found the approximately a straight line with intercept $2.5039 = \hat{\beta}\ln\hat{\theta}$ and slope $\hat{\beta} = 0.5326$. This gives $\hat{\theta} = 110.09$ and $\hat{\lambda} = 0.0091$. Using the probability plotting procedure estimates as the starting value, the parameter estimation by maximum likelihood method is obtained by $\lambda = 0.0141, \theta = 110.09$ and $\beta = 0.84$.

Furthermore, the estimated parameters and Akaike Information Criterion (AIC) values by several models are listed in Table 3. The result indicates that an extended model by the graphical approach has the third lowest AIC value. The corresponding hazard plots by different models are given in Figure 2. It can be seen that an extended model by the graphical approach (three parameters) is a competitive model for describing the bathtub-shaped failure rate lifetime data.

Table 2. Lifetimes of 50 devices (Aarset, 1987)

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18	18	21	32
36	45	46	47	50	55	60	63	63	67	67	67	67	72	75	79	82	82
83	84	84	84	85	85	85	85	85	86	86							

Table 3. The estimated parameters and AIC values in table 2

Model	**Estimated parameters**	**AIC**	**Rank**
The exponentiated Weibull	$\alpha = 4.69$, $\theta = 0.146$, and $\sigma = 91.023$	748.41	6
Haupt & Schabe	$t_0 = 128.179$ and $\beta = 0.09$	470.52	4
The additive Weibull	$a \approx 0$, $b = 30.069$, $c = 0.0912$ and $d = 0.4996$	532.89	5
The additive Burr XII	$c_1 = 0.5067$, $s_1 = 2137.215$, $k_1 = 5.5$, $c_2 = 152.93$, $s_2 = 85.2526$, and $k_2 = 0.5$	444.63	2
The extended model by the graphical approach	$\lambda = 0.0091, \theta = 110.09, \beta = 0.5326$	489.50	3
The extended model by MLE	$\lambda = 0.0141, \theta = 110.09, \beta = 0.84$	442.78	1

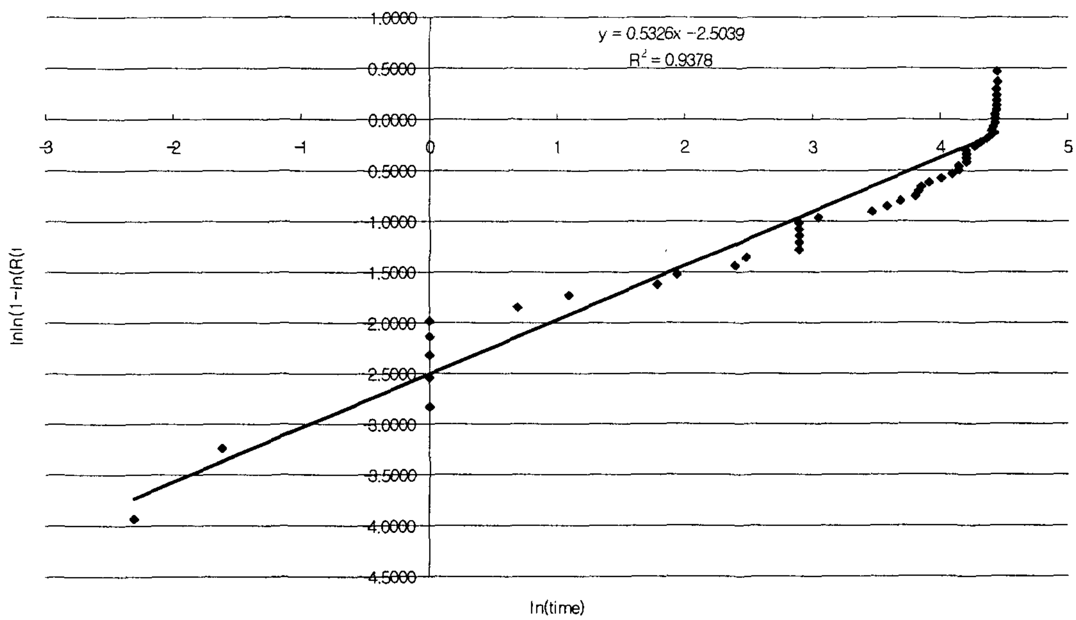

Figure 1. The probability plot on extended model in table 2

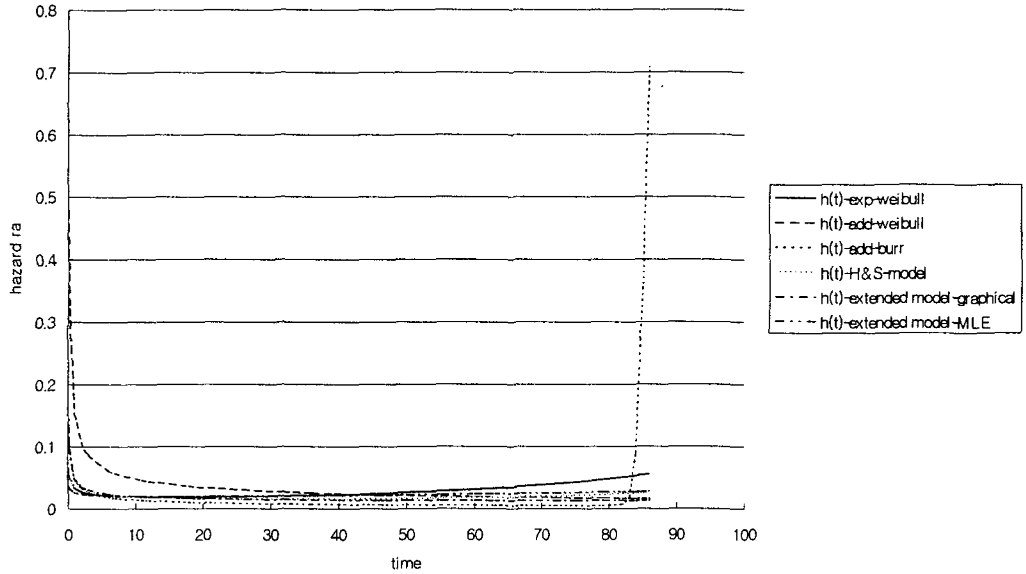

Figure 2. The hazard function plot based on different models in table 2

5. CONCLUSION

In this paper, the maximum likelihood method for obtaining point and interval estimating of the parameters of a new two-parameter distribution with multiple-censored and singly censored data (Type-I censoring or Type-II censoring) as well as complete data is proposed. The basis is the likelihood expression for multiple-censored data. Furthermore, this model can be extended to a three-parameter distribution that is added a scale parameter. Then, the parameter estimation can be obtained by the graphical estimation on probability plot and easily obtained using spreadsheets. The maximum likelihood estimation (MLE) technique has several desirable properties for estimating parameters of models. Using MLE to estimating parameters, the optimization algorithms are often sensitive to the choice of starting values. The graphical approach in this paper can be obtained as the initial guess. The application of the extended model is straightforward. The model can compute further study such as MTTF, burn-in time and replacement time, when the parameters are estimated.

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