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Testing Whether New Is Better Than Used of Specified Age Using Moments Inequalities

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Abstract. The class of "new better than used of a specified age" is a large and practical class of life distributions. Its properties, applicability, and testing was discussed by Hollander, Park and Proschan (1986). Their test, while remaining the yardstick for this class, suffers from weak efficiency and weak power, especially for specified ages below the average age. Thus, it is beneficial to have an alternative testing procedure that would work better for early ages and still work well for later ages. This is exactly the subject of the current note. The test developed here is also simpler than that of Hollander, et. al. (1986).

Key Words : $NBU-t_0$ *class of life distributions, asymptotic normality, moment inequality, efficiency, power, simulation methods.*

1. INTRODUCTION

A survival variable X is a nonnegative random variable with distribution function $(df)F(x)=P(X\le x)$ and a corresponding survival function $(sf)\bar{F}=1-F$. Survival distributions are divided into many nonparametric classes based on different aging properties. One large and fairly practical class was introduced by Hollander, Park and Proschan (henceforth written HPP) in 1986. It is called a "new better than used of a specified age "$NBU-t_0$" and is defined as follows:

DEFINITION 1.1. A survival variable X is said to be $NBU-t_0$ if

$$\bar{F}\left(x+t_0\right)\le \bar{F}(x)\bar{F}(t_0),$$

for all $x\ge 0$ find a fixed value $t_0>0$.

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Note that the border class where $\bar{F}(x+t_0)=\bar{F}(x)\bar{F}(t_0)$ includes only the following members:

(i) All exponential distributions.

(ii) All life distributions defined on $[0,t_0]$.

(iii) All life distributions defined freely on $[0,t_0]$ and defined as $\bar{F}^j(t_0)\bar{F}(x-jt_0)$ for $jt_0 \le x < (j+1)t_0, j \ge 1$.

We shall call this border class $\mathbf{A}$. Note that the class $\mathbf{A}$ is a neighborhood of the exponentials.

Basic properties of the above class were discussed by HPP (1986) where they also present a test procedure for testing $H_1 : F \in \mathbf{A}$ against $H_1 : F$ is $NBU-t_0$ where t_0 is a known value. This test was extended to the case when t_0 is unknown by Ahmad (1998). As indicated by HPP, the test they propose does not work well in the sense of efficiency for small values of t_0, hence, we were motivated to look at other alternative tests that would perform better, especially for small t_0, see Table (1), p. 95 of HPP (1986). Our computation of the empirical power of the HPP test confirms the same weakness for small t_0. Note also that small t_0 is important, actually more so than large t_0, from the manufacturer's view point since it is the period the product usually is covered through warranty. Small t_0 is meant to be values smaller than, say, the average or the median life. Hence, there is a need for an alternative approach that remedies the situation.

The approach we take in this investigation is based on developing a moment inequality for the $NBU-t_0$ class and base the testing procedure on a measure of departure from H_0 using this inequality. This approach, which we believe to be new, yields a better (in the sense of efficiency and power) and simpler procedure than that of HPP.

In Section 2, we present a moment inequality for the $NBU-t_0$ class and discuss some of its ramifications. In Section 3, a test procedure for $H_0 : F \in \mathbf{A}$ against $H_1 : F$ is $NBU-t_0$ is presented based on the moment inequality of Section 2. Large sample properties including Pitman asymptotic relative efficiencies and empirical power are given and compared to the test of HPP.

2. A MOMENT INEQUALITY

We let $\mu_{(r)} = E(X^r)$ and assume that it is finite for some positive integer *r*. The following inequality is the main entry of this section.

INEQUALITY 2.1. If *X* is $NBU-t_0$, then for integer $r \ge 0$,

$$E\left[(X-t_0)^{r+1} I\left(X>t_0\right)\right] \leq \mu_{(r+1)} \bar{F}(t_0). \tag{2.1}$$

PROOF. Note that

$$\int_0^\infty x^r \bar{F}(x)dx = E\int_0^\infty x^r I(X>x)dx = E\int_0^X x^r dx = \tfrac{E(X^{r+1})}{r+1} \tag{2.2}$$

Also, we can see that,

$$\begin{aligned}\int_0^\infty x^r \bar{F}(x+t_0)dx &= E\int_0^\infty x^r I(X>x+t_0)dx = E\int_{t_0}^\infty (u-t_0)^r I(X>u)du \\ &= \int_{t_0}^X (u-t_0)^r du = E\int_{t_0}^X (u-t_0)^r I(X>t_0)du \\ &= \tfrac{E[(X-t_0)^{r+1} I(X>t_0)]}{(r+1)}.\end{aligned} \tag{2.3}$$

Now, since $\bar{F}$ is $NBU-t_0, \bar{F}(x+t_0) \leq \bar{F}(x)\bar{F}(t_0)$ for all $x \geq 0$ and a fixed $t_0 \geq 0$, then

$$\int_0^\infty x^r \bar{F}(x+t_0)dx \leq \bar{F}(t_0)\int_0^\infty x^r \bar{F}(x)dx. \tag{2.4}$$

The result follows from (2.2) and (2.3).

Note that it follows from Lemma 1.1 of Ahmad (1998) that X is $NBU-t_0$ if and only if $\bar{F}(x-kt_0) \leq \bar{F}(x)\bar{F}^k(t_0)$ for all integers $k \geq 1$, hence, the following extension of Inequality 2.1 is immediate.

INEQUALITY 2.2. If X is $NBU-t_0$, then for integers $k \geq 1$ and $r \geq 0$,

$$E[(X-kt_0)^{r+1} I(X>kt_0)] \leq \mu_{(r+1)} \bar{F}^k(t_0). \tag{2.5}$$

3. TESTING AGAINST $NBU-t_0$

We want to test $H_0: F \in \mathbf{A}$ where $\mathbf{A} = \{\bar{F}: \bar{F}(x+t_0) = \bar{F}(x)\bar{F}(t_0)$ for all $x \geq 0$ and a fixed $t_0 \geq 0\}$ against $H_1: \bar{F}$ is $NBU-t_0$ and not in $\mathbf{A}$. To this end we use the measure of departure for H_0 given by:

$$\delta_r(t_0) = \bar{F}(t_0)\mu_{(r+1)} - E[(X-t_0)^{r+1} I(X>t_0)], r \geq 0. \tag{3.1}$$

Note that under $H_0, \delta(t_0) = 0$ while it is positive under H_1. However, in order to make the test scale invariant, we divide $\delta_r(t_0)$ by $\sigma_{(r+1)} = \{\mu_{(2r+2)} - \mu_{r+1}^2\}^{\frac{1}{2}}$. Let us call this ratio $\Delta_r(t_0)$ thus

$$\Delta_r(t_0) = \delta_r(t_0)/\sigma_{(r+1)}. \tag{3.2}$$

In order to estimate $\Delta_r(t_0)$, we proceed by estimating both $\delta_r(t_0)$ and $\sigma_{(r+1)}^2$ as follows: Let $X_1, \ldots, X_n$ denote a random sample form F. We estimate $\delta_r(t_0)$ by

$$\begin{aligned}\hat{\delta}_r(t_0) &= \bar{F}_n(t_0)\left\{\tfrac{1}{n}\sum_{i=1}^{n} X_i^{r+1}\right\} - \tfrac{1}{n}\sum_{i=1}^{n}(X_i - t_0)^{r+1} I(X_i > t_0) \\ &= \tfrac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\{X_i^{r+1} I(X_j > t_0) - (X_i - t_0)^{r+1} I(X_i > t_0)\} \\ &= \tfrac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\phi_{t_0,r}(X_i, X_j), \text{ say.}\end{aligned} \tag{3.3}$$

We also estimate $\sigma^2_{(r+1)}$ by:

$$S^2_{(r+1)} = \tfrac{1}{n}\sum_{i=1}^{n} X_i^{2r+2} - \tfrac{1}{n}\left(\sum_{i=1}^{n} X_i^r\right)^2. \tag{3.4}$$

Hence, we estimate $\Delta_r(t_0)$ by $\hat{\Delta}_r(t_0) = \hat{\delta}(t_0)/S_{(r+1)}$. We can now state and prove

THEOREM 2.1. As $n \to \infty, \sqrt{n}(\hat{\Delta}_r(t_0) - \Delta_r(t_0))$ is asymptotically normal with mean 0 and variance $V(\Psi_{r,t_0}(X_1)/\sigma_{r+1})$, where $\Psi_{r,t_0}(X_1)$ is given in (3.6). Under H_0, the variance is equal to $\bar{F}(t_0)F(t_0)$.

PROOF. Note that $\hat{\Delta}_r(t_0)$ and $\hat{\delta}_r(t_0)/\sigma_{r+1}$ have the same limiting distribution, thus, we need only look at $\sqrt{n}(\hat{\delta}_r(t_0) - \delta_r(t_0))$. Using the standard U-statistics theory, cf. Lee (1989), we easily see that,

$$\hat{\delta}_r(t_0) = \tfrac{1}{n}\sum_{i=1}^{n}\Psi_{r,t_0}(X_1) + o_p(n^{-\frac{1}{2}}), \tag{3.5}$$

where

$$\Psi_{r,t_0}(X_1) = X_1^{r+1}\bar{F}(t_0) - (X_1 - t_0)^{r+1} I(X_1 > t_0) + \mu_{(r+1)} I(X_1 > t_0) - \int_{t_0}^{\infty}(x - t_0)^{r+1} dF(x) \tag{3.6}$$

Hence, $\sqrt{n}(\hat{\delta}_r(t_0) - \delta_r(t_0))$ is asymptotically normal with mean 0 and variance $\sigma^2 = V(\Psi_{r,t_0}(X_1))$. Under $H_0, \bar{F}(x + t_0) = \bar{F}(x)\bar{F}(t_0)$ for all $x \geq 0$ and a fixed $t_0 \geq 0$. Thus,

$$\Psi_{r,t_0}(X_1) = X_1^{r+1}\bar{F}(t_0) - (X_1 - t_0)^{r+1} I(X_1 > t_0) + \mu_{(r+1)} I(X_1 > t_0) - \mu_{(r+1)}\bar{F}(t_0).$$

Hence, the null variance is equal to (after some algebra):

$$E(\Psi_{r,t_0}(X_1))^2 = \bar{F}(t_0)F(t_0)[\mu_{(2r+2)} - \mu^2_{(r+1)}]. \tag{3.7}$$

The result now follows.

To carry out the test, evaluate $\sqrt{n}\hat{\delta}_r(t_0)/S_{(r+1)}[\bar{F}_n(t_0)F_n(t_0)]^{\frac{1}{2}}$ and reject H_0 if this statistic is larger than Z_α, the $\alpha-th$ percentile of the normal variate. In order to compare the test statistic presented here with that of HPP, we first discuss the Pitman asymptotic efficiency of our test relative to that of HPP and then discuss the simulated powers of the two tests. HPP propose the test statistic

$$\hat{v}(t_0)=\tfrac{1}{2n}\sum_{i=1}^{n}I(X_i>t_0)-\tfrac{1}{n(n-1)}\sum_{i\neq j}I(X_i>X_j+t_0) \quad (3.8)$$

Under H_0, they show that $\sqrt{n}\hat{v}(t_0)/\left[\tfrac{1}{12}(\bar{F}_n(t_0)+\bar{F}_n^{\,2}(t_0)-2\bar{F}_n^{\,3}(t_0)\right]^{\frac{1}{2}}$ is asymptotically standard normal. This statistic is based on the measure of departure from H_0 given by:

$$v(t_0)=\tfrac{1}{2}\bar{F}(t_0)-\int\bar{F}(x+t_0)dF(x). \quad (3.9)$$

To evaluate the efficiency of our test relative to that of HPP, we choose the following alternative distribution:

(i) <u>The linear failure rate distribution</u>:

$$\bar{F}_{1,\theta}(x)=\exp\{-(x+\tfrac{\theta}{2}x^2)\},\theta\geq 0,x\geq 0.$$

(ii) <u>The Makeham distribution</u>:

$$\bar{F}_{2,\theta}(x)=\exp[-\{x+\theta(x+\exp(-x)-1)\}],\theta\geq 0,x\geq 0.$$

The null exponential is at $\theta=0$ for the above two distributions. The asymptotic efficacy of our test works out to be:

$$eff(\hat{\Delta}_r(t_0))=[\bar{F}'_{\theta_0}(t_0)\mu_{\theta_0,(r+1)}-\bar{F}_{\theta_0}(t_0)\mu'_{\theta_r,(r+1)}$$
$$-(r+1)\int_0^\infty x^r\bar{F}'_{\theta_1}(x+t_0)dx]^2/[(2r+2)!-\{(r+1)!\}^2]\bar{F}_{\theta_0}(t_0)F_{\theta_0}(t_0). \quad (3.10)$$

For the above alternatives, we have the following values:

(i) <u>For linear failure rate</u>:

$$[t_0(r+1)(r+1)!]^2/[(2r+2)!-\{(r+1)!\}^2](e^{t_0}-1)$$

(ii) <u>For the Makeham</u>:

$$[(r+1)!]^2[e^{-t_0}(2^r-1)+(r^2-3r+4)/2]^2/[(2r+2)!\{(r+1)!\}^2](e^{t_0-1)})$$

The maximum values are, respectively, at $r=0$ and $r=2$, and they are:

For linear failure rate, $t_0^2/(e^{t_0}-1)$ while for the Makeham, $(3e^{t_0}+1)^2/76e^{2t_0}(e^{t_0}-1)$. The corresponding values for the HPP test are respectively equal to $3t_0^2/4(e^{t_0}+1-2e^{-t_0})$ and $(e^{t_0}-1)^2/3e^{2t_0}(e^{t_0}+1-2e^{-t_0})$. Thus, the asymptotic Pitman efficiency of our test to that of HPP is for the linear failure rate equal to $4(e^{t_0}+1-2e^{-t_0})/3(e^{t_0}-1)>1$ for all t_0. While the relative efficiency for the Makeham is equal to $3(e^{t_0}+3)(3e^{t_0}+1)^2/76(e^{t_0}(e^{t_0}-1))$ which is larger than one for small to moderate values of t_0. Hence, our test remedies the difficulties with that of HPP.

To assess the power of the above test relative to that of HPP, we used simulation. Based on 10,000 replicates of samples of sizes 5(1)25, we calculated the powers corresponding to $\alpha = .90, .95$, and .99 of the two tests for the above mentioned alternatives (the linear failure rate and the Makeham for a number of t_0 values including 0.25, 0.50, and 1.00. We observe the following: our test procedure is more powerful than the HPP test for practical values of t_0, the small to moderate values i.e. t_0 between zero and one, but the HPP test is better for large values of t_0. However, the power of either test reduces considerably as t_0 increases. Thus, for practical t_0 values, our test remedies the problem the HPP test suffers from. In the following tables, we give a glimpse of calculations supporting the conclusions we mentioned above. The first entry is for the HPP test while the second is for our test.

Table 1. Power comparison for the linear failure rate alternative.

$t_0 = 0.25$ θ	2			3			4		
n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
5	.026	.097	.183	0.38	.122	.224	.052	.140	.244
	.052	.188	.285	.069	.219	.322	.079	.234	.344
10	.070	.188	.293	.088	.244	.344	.123	.280	.387
	.115	.274	.399	.154	.338	.459	.186	.377	.505
15	.092	.259	.352	.134	.308	.430	.156	.365	.465
	.173	.373	.502	.228	.457	.583	.277	.513	.634
20	.115	.287	.410	.172	.367	.500	.218	.426	.558
	.236	.461	.567	.316	.544	.654	.380	.608	.707
$t_0 = 0.50$									
5	0.35	.184	.260	.034	.194	.267	.026	.189	.261
	.031	.125	.212	.030	.132	.216	.032	.128	.216
10	.061	.208	.305	.080	.226	.350	.080	.260	.374
	.068	.210	.334	.079	.226	.360	.076	.229	.360
15	.080	.248	.387	.097	.297	.458	.116	.323	.495
	.097	.302	.446	.114	.336	.490	.118	.348	.498
20	.106	.309	.454	.134	.365	.525	.151	.410	.565
	.142	.368	.520	.163	.407	.573	.168	.428	.588
$t_0 = 1.00$									
5	.040	.063	.230	.032	.170	.170	.013	.121	.121
	.011	.056	.113	.010	.050	.099	.008	.043	.094
10	.042	.146	.218	.057	.104	.239	.042	.163	.239
	.017	.093	.170	.018	.075	.139	.014	.062	.129
15	.057	.213	.315	.071	.192	.303	.073	.210	.281
	.028	.124	.228	.025	.101	.182	.021	.081	.149
20	.077	.244	.372	.078	.230	.311	.077	.175	.330
	.038	.161	.294	.027	.129	.240	.026	.106	.185

Table 2. Power comparison for the Makeham alternative.

$t_0 = 0.25$ θ	2			3			4		
n	0.90	.095	.099	0.90	.095	.099	0.90	.095	.099
5	.018	.082	.182	.025	.165	.232	.026	.174	.239
	.039	.148	.242	.051	.175	.276	.063	.195	.298
10	.043	.160	.244	.073	.188	.291	.077	.217	.313
	.072	.204	.312	.106	.252	.373	.126	.288	.413
15	.059	.194	.279	.090	.234	.348	.112	.281	.371
	.104	.267	.386	.151	.341	.463	.190	.395	.528
20	.082	.214	.319	.115	.270	.397	.157	.332	.460
	.142	.318	.423	.206	.409	.518	.261	.482	.594
$t_0 = 0.50$									
5	.026	.159	.225	.025	.165	.232	.026	.171	.239
	.025	.100	.176	.024	.104	.181	.028	.116	.194
10	.043	.147	.228	.048	.175	.278	.054	.143	.243
	.045	.155	.257	.049	.172	.285	.053	.179	.304
15	.052	.177	.297	.066	.228	.373	.080	.261	.380
	.058	.211	.336	.070	.256	.391	.084	.266	.398
20	.067	.216	.334	.083	.266	.401	.104	.301	.445
	.084	.245	.377	.101	.292	.460	.116	.319	.474
$t_0 = 1.00$									
5	.012	.076	.076	.030	.047	.189	.027	.156	.156
	.009	.051	.105	.007	.066	.095			
10	.029	.110	.188	.028	.116	.167	.034	.087	.208
	.016	.079	.151	.012	.069	.135			
15	.042	.141	.257	.043	.170	.231	.043	.169	.275
	.021	.096	.181	.020	.087	.174	.018	.083	.158
20	.046	.164	.290	.054	.186	.303	.057	.187	.265
	.030	.122	.226	.024	.111	.214	.021	.092	.188

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