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계층구조의속성을가지는의사결정문제의 선호순위도출을위한수리계획모형 (Mathematical Programming Models for Establishing Dominance with Hierarchically Structured Attribute Tree)

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Abstract

This paper deals with the multiple attribute decision making problem when a decision maker incompletely articulates his/her preferences about the attribute weight and alternative value. Furthermore, we consider the attribute tree which is structured hierarchically. Techniques for establishing dominance with linear partial information are proposed in a hierarchically structured attribute tree. The linear additive value function under certainty is used in the model. The incompletely specified information constructs a feasible region of linear constraints and therefore the pairwise dominance relationship between alternatives leads to intractable non-linear programming. Hence, we propose solution techniques to handle this difficulty. Also, to handle the tree structure, we break down the attribute tree into sub-trees. Due to there cursive structure of the solution technique, the optimization results from sub-trees can be utilized in computing the value interval on the topmost attribute. The value intervals computed by the proposed solution techniques can be used to establishing the pairwise dominance relation between alternatives. In this paper, pairwise dominance relation will be represented as strict dominance and weak dominance, which ware already defined in earlier researches.

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1. Introduction

There are many methods in the field of decision analysis to help a decision maker (DM) come up with a decision, that is, to find an optimal or satisfying solution. However, the gap between theoretical research and practical needs still exists. This gap could be due to the fact that the decision problem or the preferences of the DM are not (yet) structured enough to allow the successful application of most decision analysis methods. For example, the DM may not be willing or able to specify the preferences in the detailed way required by the corresponding method. To narrow the gap between theoretical research and practical application, we need decision models that can be used within decision situations having incomplete information on parameters which describe the decision situation. Initially explored by Fishburn [1], there have been a number of studies which consider linear partial information (LPI) in the field of multi-attribute decision making (MADM), one specific field of decision analysis.

The problem of MADM is defined by selecting or ranking the most preferable alternatives from the alternatives considered. The alternatives are evaluated by a finite discrete set of attributes. This set of attributes can be structured in a hierarchical form. The hierarchically structured attribute tree is used to determine an overall evaluation score of each alternative. The evaluation of alternatives requires that we elicit the value of the consequences on the lowest level attributes and the relative weight of attributes.

The types of multi-attribute value models used have varied from very simple weight linear models to rather complex multiplicative models. In this paper, we use a simple linear additive model for obtaining the overall evaluation score of alternatives, since the linear weighting model is the best known and most widely used. The preferentially independent condition is required for the linear additive model [2]. The preferentially independent condition means that the contributions of an individual attribute to the overall evaluation is independent of other attribute values. It has the advantages of being easily learnedand used by the DM [3].

The aim of this paper is to present tools or techniques for MADM with the DM's

LPI on attribute weights and alternative values. We describe mathematical programming models for establishing pairwise dominance, when the set of attributes is structured hierarchically. To handle the tree structure, we decompose an attribute tree into sub-trees. The optimization results from sub-trees can be utilized in computing the value interval on the upper level attribute. The value interval on the topmost attribute is used for establishing the dominance relation between two competing alternatives. Also, the mathematical model allows the DM to provide any type of LPI on the attribute weight and the value of alternatives, which will become a set of constraints in the model.

2. Problem Statement and Prior Works

2.1 A Multi-Attribute Value Function

Generally, the MADM problem is composed of a finite discrete set A of alternatives, which is valued by a finite discrete set $I=\{1,2,3, . . . ,m\}$ of attributes. Let w_i be an attribute weight which represents the relative importance of the ith attribute and $v_i(x)$ be the value of alternative x on attribute $i \in I$. We will use the following notation to represent a hierarchically structured attribute tree. The hierarchically structured attribute tree can be partitioned into levels, $L1$, $L2$, . . . , Lh such that the kth level attributes are in Lk. By definition $L1$consists of the topmost attribute, $w_1=1$. The set $D(i) \subset I$ consists of the attributes which are structured immediately under attribute i, the set of direct successors of i. The set $DT(i) \subset I$ consists of the lowest level attributes which are structured under attribute i. The set $I_T \in I$ especially consists of the lowest level attributes which are structured under the topmost attribute, $I_T = DT(1)$, $I_T = \{i \in I \mid D(i) = \varnothing\}$. The set $DL(i) \subset I$ consists of the attribute i and all attributes which are structured under attribute i, $D(i), DT(i) \subset DL(i)$.

Then, the linear weighting model is

$$v(x) = \sum_{i \in DT(1)} w_i v_i(x) \qquad \text{with}$$

$$v_i(x^0) = 0 \qquad v_i(x^*) = 1 \qquad \forall i \in DT(1)$$

$$\sum_{i \in DT(1)} w_i = 1 \qquad w_i \geq 0 \qquad \forall i \in I$$

where x^0 and x^* are respectively the most and worst preferred value on the ith attribute.

2.2 Linear Partial Information

When the parameter value information , i.e. w_i and $v_i(\cdot)$ for all i, is precisely assessed by the DM, then the most preferred alternative can be easily determined. However, it is no simple matter to measure the DM's precise information on parameters such as attribute weights and alternative values. The reasons may be that (1) a decision should be made under time pressure and lack of knowledge or data [4], (2) many of the attributes are intangible or non-monetary because they reflect social and environmental impacts [5], and (3) the DM has limited attention and information processing capabilities [6], especially on the judgment of numerical values under a complex and uncertain environment. These may take the form of linear partial information such as rankings, interval description, and so on. Examples of the LPI on attribute weights are given by the following forms:

Form 1. $\{w_i \geq w_j\}$
Form 2. $\{w_i - w_j \geq \alpha_i\}$
Form 3. $\{w_i \geq \alpha_i w_j\}$
Form 4. $\{\alpha_i \leq w_i \leq \alpha_i + \varepsilon_i\}$
Form 5. $\{w_i - w_j \geq w_k - w_l\}$ for $j \neq k \neq l$

where $\{\alpha_i\}$ and $\{\varepsilon_i\}$ are non-negative constants [5, 7]. Incomplete information on the value of alternatives can be similarly expressed.

Form 1 is widely used to construct ordinal ranking, because it is one of the most simple forms. Form 5 is a ranking of differences of adjacent parameters obtained by ranking between two parameters, which can be subsequently constructed based on Form 1. A difficulty in taking the information of Forms 2-4 is to precisely justify their constants, since these forms contain numerical values such as α_i and ε_i. However, the assessment of these forms can reduce much of the feasible decision space denoted by LPI.

2.3 Prior Works in MADM with Linear Partial Information

First of all, we discuss the representative methods that consider the incomplete information and multi-attribute value function. The UTA method [8] applies regression analysis to assess linear additive value functions from the ranking of alternatives. Using post-optimality analysis, it calculates all utility functions within a certain range consistent with the DM's information. ARIADNE [9] admits the incomplete information of utilities, attribute weights and probabilities, and allows the DM to induce constraints on the parameters through direct comparison of alternatives. The resulting dominance structure is displayed as a directed graph, called a dominance graph. HOPIE [10]requires the DM to provide holistic judgments on hypothetical alternatives. The alternatives have to be evaluated by the interval description, etc. Based on this incomplete information, it allows the DM to check dominance relation.

In the context of the analytic hierarchy process, Saaty and Vargas [11]suggest the use of pairwise comparison intervals (Form 3 in section 2) which allow the DM to make ratio statements as interval values on a ratio scale, but point out the computational difficulties of analyzing eigenvectors with pairwise comparison interval. Arbel [12]interprets pairwise comparison intervals as linear constraints which at each attribute define a non-empty set of weights called the feasible region.

PAIRS [13] developed an efficient algorithm for synthesizing pairwise comparison intervals into dominance relations among alternatives. These relations resemble those employed in the multi-attribute utility model. The algorithm can analyze even large attribute trees efficiently due to the decomposition of the computations that involve a series of linear programs. Based on the PAIRS, Salo and Hmlinen [14] suggest an interactive decision support process to prevent the feasible region from becoming empty. The interactive decision support process providesthe DM with a consistency interval that guides the DM in the specification of new comparisons.

Most of the methods which appear in Table 1 deal mainly with establishing a strict dominance relation: Alternative x strictly dominates y if and only if the value of the worst outcome in $x$is greater that that of the best outcome in y, for a fixed feasible region denoted by LPI. Unfortunately, the strict dominance of two competing alternatives may not be always determined with incomplete information. To cope with the problem, Kmietoicz and Pearman [15] develop a weak dominance relation: alternative x weakly dominates y if and only if selecting x involves less regret, as traditionally defined, than selecting $y$for a fixed feasible region. Furthermore, Park and Kim [5]discovered by introducing a statistical concept that weak dominance is always identified for a fixed feasible region. Thus it may be used as a certain decision rule when the DM is not able to provide additional information.

<Table 1> Earlier methods on MADM with incomplete information

Related Works	(a)Attribute weight	(b)Value of alternative	(c)Multi-leveled attributes
Sarin (1977) [16] Hannan (1981) [17] Kirkwood and Sarin (1985) [18]	Form 1	Cardinal	No
Sage and White (1984) [9]	All Form	All Form	No
Kmietowicz and Pearman (1982) [19]	Form 1,5	Cardinal	No
Kmietowicz and Pearman (1984) [15]	All Form	Cardinal	No
Cook and Kress (1991,1997) [20, 21]	Form 1	Form 1	No
Salo and Hamalainen (1995) [14]	Form 3	Form 3	Yes
Bryson and Mobolurin (1995) [22]	Form 3	Form 3	Yes
Park and Kim (1997) [5]	All Form	All Form	No
Kim and Han (1999) [7]	All Form	All Form	No

In addition to the earlier works mentioned above, there are a number of studies on MADM with incomplete information. We summarize the earlier methods in <Table 1> The characteristics of the studies are divided into three points of view: (a) types of preference information on attribute weights, (b) types of preference information on the value of alternatives'outcome, and (c) consideration of the hierarchically structured attributes tree. In questions (a) and (b), all possible types of LPI in each method are contained in Table 1. The methods which answered "yes"in question (c) consider the attribute tree. When the five forms of LPI on the weights and values are assessed from the DM and the attributes are structured hierarchically, none of the methods in Table 1 is able to provide the dominance relation. In this paper, we will propose a method that is capable of establishing thedominance relation in the situation where the DM articulates all types of LPI on the weights, and values and a hierarchically structured attribute tree is given.

3. Mathematical Programming Model for Establishing Dominance

Given an attribute tree, the DM may provide his LPI on attribute weights, w_i and alternative values, $v_i(\cdot)$. The LPI on attribute weights, w_i can be elicited locally within each set $D(i)$ for $i \notin I_T$. Let $W[D(i)]$ be the set of LPI elicited by the DM, regarding the relative importance of attributes $j \in D(i)$. Let V_i be the set of LPI regarding the alternative value. The LPI on $v_i(\cdot)$ can be elicited by the DM in lowest-level attributes $i \in I_T$. We assume that $W[D(i)]$ and V_i are consistent sets. The sets $W[D(i)]$ and V_i can be composed of any types of LPI mentioned in section 2.

With the DM's LPI, the value interval of alternatives x and y on the attribute k, can be formulated as

$$\varphi_k(x,y) = \min_{w^k, y^k} \text{ or } \max \sum_{i \in DT(k)} w_i \left[v_i(x) - v_i(x) \right] \tag{3.1.a}$$

where

$$W^k = \begin{cases} W[D(i)] & \text{for } i \notin DT(k),\ i \in DL(k) \\ \sum_{j \in D(i)} w_j = w_i & \text{for } i \notin DT(k),\ i \in DL(k) \\ w_k = 1 & \\ w_i \geq 0 & \text{for } i \in DL(k) \end{cases} \quad (3.1.b)$$

$$V^k = \begin{cases} v_i(x), v_i(y) \in V_i & \text{for } i \in DT(k) \\ 0 \leq v_i(x), v_i(y) \leq 1 & \text{for } i \in DT(k) \end{cases} \quad (3.1.c)$$

When LPI on attribute weights is elicited locally within each set $D(i)$, $i \notin I_T$, the DM may provide LPI in the concept of $\sum_{j \in D(i)} w_j = 1$. Thus, if Form 2 or 4 is included in the set of LPI, $W[D(i)]$ for $i \notin I_T$, then, in order to useForm 2 or 4 in (3.1), the constants $\{\alpha_i\}$ and $\{\varepsilon_i\}$ in Form 2 or 4 should be normalized from $\sum_{j \in D(i)} w_j = 1$ into $\sum_{j \in D(i)} w_j = w_i$. Normalization is not needed for constants $\{\alpha_i\}$ and $\{\varepsilon_i\}$ in Form 3, since the $\{\alpha_i\}$ and $\{\varepsilon_i\}$in Form 3 represent the relative importance of attributes.

If the value of either w_i or $v_i(\cdot)$ is known precisely, then model (3.1) becomes LP, thus it can be easily solved. Assume that the values of w_i and $v_i(\cdot)$ are identified imprecisely. Then the model becomes non-LP, since the objective function is a sum-product type of unknown variables.In this case our first aim is to compute the values, $\varphi_k^+(x,y)$ and $\varphi_k^-(x,y)$, which are respectively maximum and minimum values of the objective function in (3.1). Secondly, we should obtain the value interval on the topmost attribute, w_1, which is necessary for establishing dominance relation between alternatives. We now describe several techniques for this problem.

Technique1. Assume that the DM's LPI on the alternative value for each lowest level attribute are functionally independent, which is formally denoted by a notation,

$V_i \perp V_j$, $\forall i \neq j$, $i,j \in DT(k)$. Then (3.1.a) is "separable" for each lowest level attribute $i \in DT(k)$, thus (3.1) can be solved by the following LPs:

$$\varphi_k(x,y) = \min_{W^k} \ (or \ \max) \sum_{i \in DT(k)} w_i \xi_i(x,y) \tag{3.2.a}$$

with

$$\xi_i(x,y) = \min_{V_i} \ (or \ \max) \ v_i(x) - v_i(y), \qquad i \in DT(k) \tag{3.2.b}$$

If we put k=1 in (3.2), then we can obtain the value interval on the topmost attribute w_1, $[\varphi_1^-(x,y), \varphi_1^+(x,y)]$.

The weight of upper level attributes can be represented bythe sum of attribute weights which are structured immediately under the upper level attribute. Namely, $\sum_{j \in D(i)} w_j = w_i$ for $i \notin I_T$. Also, the weight of upper level attribute can be represented as the sum of the weights of the twig level attributes which are structured under the upper level attribute. Namely, $\sum_{j \in DT(i)} w_j = w_i$ for $i \notin I_T$. Then the constraints, $W[D(i)]$ for $i \notin I_T$ in (3.1), can be represented as $W[DT(i)]$ for $i \notin I_T$. A detailed description of $W[DT(i)]$, with an example, is shown in the next section. Now, the constraints set (3.1.b), Wk , can be changed as follows:

$$W_T^k = \begin{cases} W[DT(i)] & \text{for } i \notin DT(k), \ i \in DL(k) \\ \sum_{i \in DT(k)} w_i = 1 & \\ w_i \geq 0 & \text{for } i \in DT(k) \end{cases} \tag{3.3}$$

If we use the set W_T^k instead of W^k in (3.1), then another solution technique is provided for solving this problem.

Technique2. Assume that the DM's LPI on the alternatives' value for each lowest level attribute is functionally independent. If the constraints set of (3.1.b), W^k, is replaced with the constraints set of (3.3), W_T^k , then (3.1) which has the constraint set W_T^k is "separable" for each lowest level attribute $i \in DT(k)$, thus yielding a set of the following LPs:

$$\varphi_k(x,y) = \min_{W_T^k} \ (or \ \max) \sum_{i \in DT(k)} w_i \xi_i(x,y) \tag{3.3.a}$$

with

$$\xi_i(x,y) = \min_{V_i} \ (or \ \max) \ v_i(x) - v_i(y), \qquad i \in DT(k) \tag{3.3.b}$$

If we put k=1 in (3.3), then we can obtain the value interval on the topmost attribute w_1, $[\varphi_1^-(x,y), \varphi_1^+(x,y)]$.

Now, we will decompose an attribute tree into sub-trees and formulate a mathematical programming model which is capable of computing the value interval. In case that value intervals on one or more attribute $p \in P(k)$, $P(k) \subset DL(k)$, $k \notin I_T$ arealready obtained by Technique 1 or 2, another formulation for obtaining value interval on attribute k, $k \notin I_T$ can be followed as:

$$\varphi_k(x,y) = \min_{W^k, V^k} or \max \sum_{i \in DT^N(k)} w_i [v_i(x) - v_i(y)] \tag{3.4.a}$$

where

$$W^k = \begin{cases} W[D(i)] & \text{for } i \notin DT^N(k), \ i \in DL^N(k) \\ \sum_{j \in D(i)} w_j = w_i & \text{for } i \notin DT^N(k), \ i \in DL^N(k) \\ w_k = 1 & \\ w_i \geq 0 & \text{for } i \in DL^N(k) \end{cases} \tag{3.4.b}$$

$$V^k = \begin{cases} v_i(x), v_i(y) \in V_i & \text{for } i \in DT(k) - DT(p) \\ 0 \le v_i(x), v_i(y) \le 1 & \text{for } i \in DT(k) - DT(p) \\ V_p = \{\varphi_p^-(x,y) \le v_p(x) - v_p(y) \le \varphi_p^+(x,y)\} & \text{for } p \in P(k) \end{cases} \quad (3.4.c)$$

where $DT^N(k) = DT(k) \cup P(k) - DT(p)$ and $DL^N(k) = DL(k) - (DL(p) - P(k))$.

If we want to use the constraints set like (3.3) instead of (3.4.b), then the set (3.3) should also be revised in the formulation of (3.4). The revised set of W_T^k is as follows:

$$W_T^k = \begin{cases} W[DT(i)] & \text{for } i \notin DT^N(k),\ i \in DL^N(k)\ ) \\ \sum_{i \in DT(k)} w_i = 1 & \\ w_i \ge 0 & \text{for } i \in DT^N(k) \end{cases} \quad (3.5)$$

Furthermore, if there exists one or more attributes, $q \in P(p)$, $P(p) \subset DL(p)$, $p \notin I_T$, of which the value interval should be obtained from (3.4), the recursive use of (3.4) is needed to obtain the value intervals on attribute p. For example, the value interval on the attribute p can be obtained by the use of the formulation (3.4) in which p and q are substituted for k and p respectively. Of course, if we encounterthe attribute that is not needed to use (3.4) in the course of using (3.4) recursively, then Technique 1 or 2 will be used for obtaining the value interval for that attribute.

Technique3. Since $Vp, p \in P(k)$ is functionally independent for each attribute $i \in DT^N(k)$, (3.4) can be solved by LPs like (3.2) or (3.3). Using the recursive structure of (3.4), the value interval on the topmost attribute, w_1, can be obtained by putting k=1 in (3.4).

Any combination of the attribute weights and alternative values in the lower level of the tree can not give the value $\varphi_p(x,y)$ that is greater than $\varphi_p^+(x,y)$ and smaller than $\varphi_p^-(x,y)$. In addition, the weights of the upper level attribute are independent of the attribute weight and the alternative value at the lower level of the attribute tree. Based on these facts, the following property of Technique 3 can be derived.

Propertyof Technique3. Using the value interval, $[\varphi_p^-(x,y), \varphi_p^+(x,y)]$, computed at the lower level of the tree in (3.4), guarantees the optimal (minimum or maximum) value interval of the attribute *k*.

Proof. See the Appendix.

Given the value intervals on the topmost attribute, the strict and weak dominance relation between alternatives *x* and *y*, should be established. If $\varphi_1^-(x,y) \geq 0$ or equivalently $\varphi_1^+(x,y) \leq 0$, then alternative *x* strictly dominates *y*. Simply stated, the worst value for *x* is greater than or equal to the best value for *y*, for a fixed feasible region. If $\varphi_1^-(x,y) \geq \varphi_1^-(y,x)$ or equivalently $\varphi_1^+(x,y) \geq \varphi_1^+(y,x)$, then alternative *x* weakly dominates *y*. Simply, the worst (best) possible value for *x* is greater than or equal to the worst (best) possible value for *y*, for a fixed feasible region.

4. An Illustrative Example

This section illustrates features of the proposed method in the context of an international supplier selection problem. The purchasing department ofa company is considering three competing suppliers which are denoted by *x*: Japanese supplier, *y*: Indonesian supplier and *z*: Korean supplier. The suppliers (alternatives) are evaluated by the attribute tree in Figure 1. A more detailed explanation of the attributes inFig. 1 is provided in the literature of Min [23] and Weber *et al.* [24].

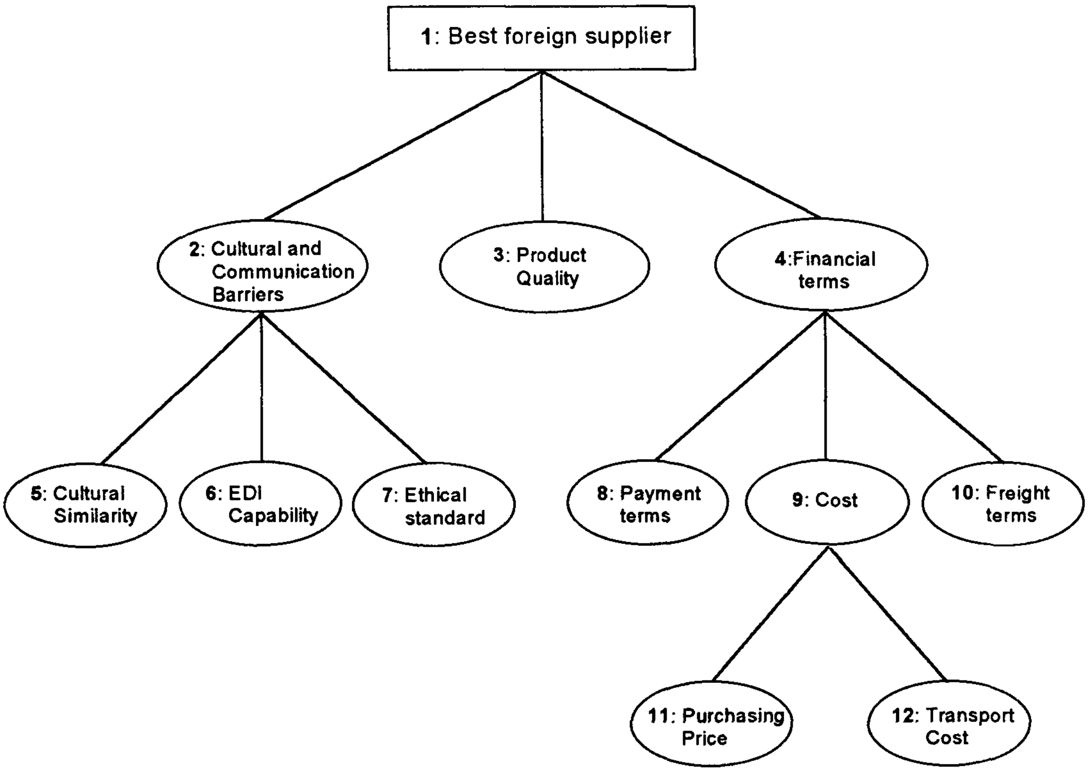


Figure 1. Hierarchically structured attribute tree for the example

Suppose that LPI on attribute weights from the purchasing department are as follows:

$$W[D(1)] = \{w_2 \geq w_3 \geq w_4,\ w_2 - w_3 \geq w_3 - w_4\}$$

$$W[D(2)] = \{w_5 \geq w_6 \geq w_7,\ w_5 \geq 2w_6\}$$

$$W[D(4)] = \{w_8 \geq w_9 \geq w_{10},\ w_8 - w_9 \geq 0.2\}$$

$$W[D(9)] = \{w_{11} \geq w_{12},\ 0.4 \leq w_{11} \leq 0.7\}.$$

For example, the incompletely specified weight, $w_5 \geq 2w_6$ in $W[D(2)]$ means that attribute 5, cultural similarity, is at least twice as important as attribute 6, Electronic data interchange(EDI) capability, in view of attribute 2, cultural and communication barriers.

Suppose the purchasing department's LPI on alternative values is given in functionally independent fashion for each lowest level attribute, *IT* = { 3,5,6,7,8,10,11,12 }, the set *Vi* is shown as follows:

$V3 = \{1 = v_3(x) \geq v_3(z) \geq v_3(y) = 0,\ v_3(x) - v_3(z) \geq v_3(z) - v_3(y)\}$

$V5 = \{1 = v_5(z) \geq v_5(x) \geq v_5(y) = 0,\ v_5(z) - v_5(x) \geq 0.1,\ 0.3 \leq v_5(z) \leq 0.6\}$

$V6 = \{1 = v_6(x) \geq v_6(z) \geq v_6(y) = 0,\ v_6(x) \geq 3v_6(z)\}$

$V7 = \{1 = v_7(y) \geq v_7(z) \geq v_7(x) = 0,\ v_7(z) \geq 2v_7(x)\}$

$V8 = \{1 = v_8(x) \geq v_8(y) \geq v_8(z) = 0\}$

$V10 = \{1 = v_{10}(y) \geq v_{10}(z) \geq v_{10}(x) = 0\}$

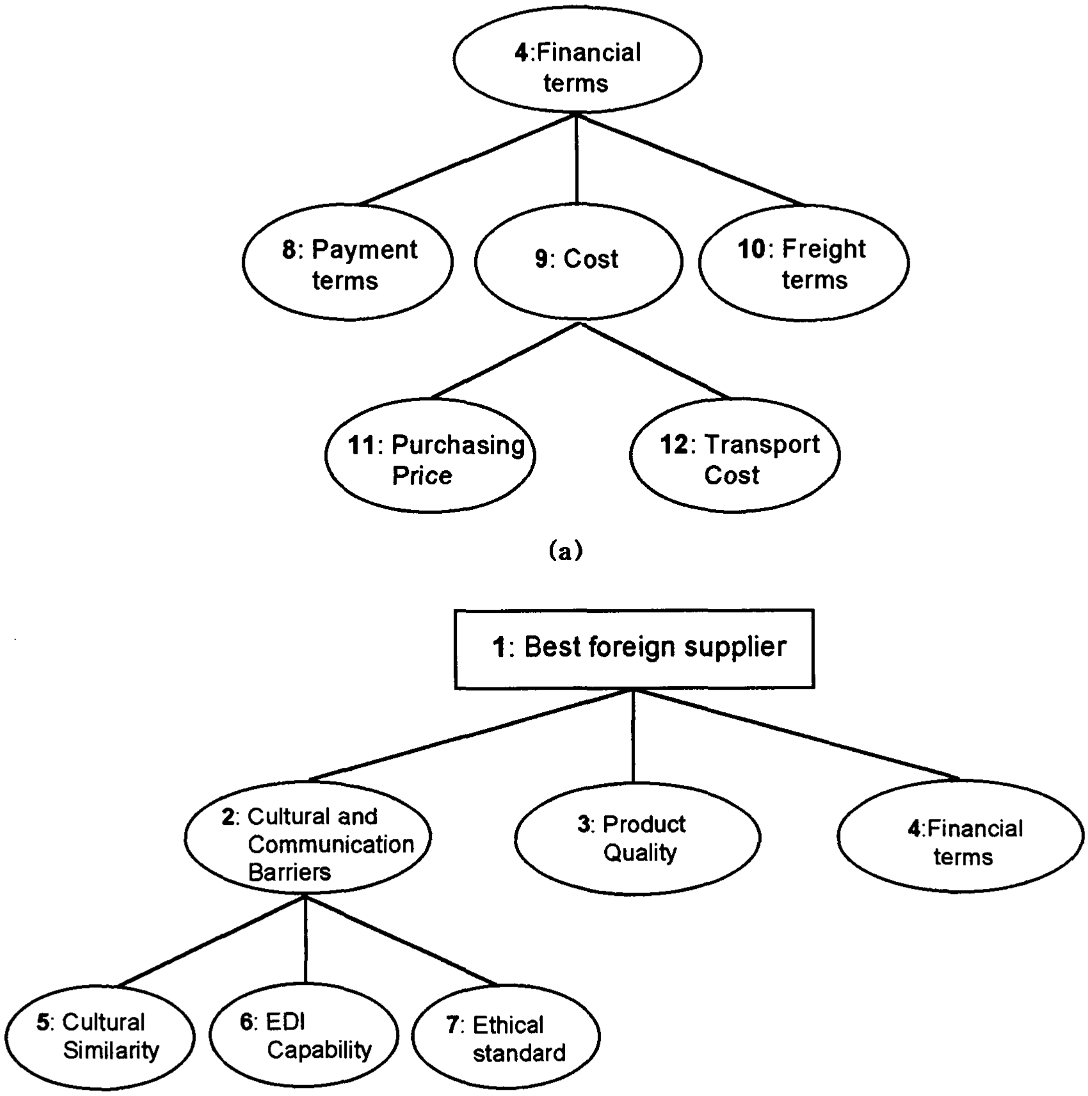
$V11 = \{1 = v_{11}(z) \geq v_{11}(x) \geq v_{11}(y) = 0\}$

$V12 = \{1 = v_{12}(y) \geq v_{12}(x) \geq v_{12}(z) = 0\}$.

Now, we decompose the attribute tree in Figure 1 into two sub-trees in Figure 2 and compute the value interval on the attribute 4 (Financial terms).

In the sub-tree of Figure 2(a), the set *DT(4)* is {8,10,11,12} and *DL(4)* is {4,8,9,10,11,12}. With the DM's LPI on attribute weights, the set of constraints *W4* in the sub-tree of Figure 2(a) is

$$W^4 = \begin{cases} w_{11} \geq w_{12} \\ 0.4 \cdot (w_{11} + w_{12}) \leq w_{11} \leq 0.7 \cdot (w_{11} + w_{12}) \\ w_8 \geq w_9 \geq w_{10} \\ w_8 - w_9 \geq 0.2 \cdot (w_8 + w_9 + w_{10}) \\ w_{11} + w_{12} = w_9 \\ w_8 + w_9 + w_{10} = w_4 \\ w_4 = 1 \\ w_i \geq 0 \qquad\qquad i = 4,8,9,10,11,12. \end{cases}$$



(b)

<Figure 2> Sub-trees for the tree of Figure 1

(a) The sub -tree for attribute 4

(b) The sub -tree for topmost attribute 1

The constraints, $w_8 - w_9 \geq 0.2$ in $W[D(4)]$ and $0.4 \leq w_{11} \leq 0.7$ in $W[D(9)]$, are normalized in the above *W4*. Since the attribute weight, w_9, can be represented by the sum of twig level attribute w_{11} and w_{12}, and w_4 can be represented by

$w_8 + w_{10} + w_{11} + w_{12}$ in the above $W4$, the constraints set W_T^4 can be represented as follows:

$$W_T^4 = \begin{cases} w_{11} \geq w_{12} \\ 0.4 \cdot (w_{11} + w_{12}) \leq w_{11} \leq 0.7 \cdot (w_{11} + w_{12}) \\ w_8 \geq w_{11} + w_{12} \geq w_{10} \\ w_8 - (w_{11} + w_{12}) \geq 0.2 \cdot (w_8 + (w_{11} + w_{12}) + w_{10}) \\ w_8 + (w_{11} + w_{12}) + w_{10} = 1 \\ w_i \geq 0 \qquad\qquad i = 8,10,11,12. \end{cases}$$

With the set, *V8, V10, V11, V12,* and *W4* or W_T^4, the value intervals for attribute 4, $[\varphi_4^-(x,y)\ \varphi_4^+(x,y)]$, is derived by using Technique 1 or 2, An example of the formulation for alternatives x and y is

$$\varphi_4^+(x,y) = \max_{W^4 or W_T^4} \{w_8\xi_8(x,y) + w_{10}\xi_{10}(x,y) + w_{11}\xi_{11}(x,y) + w_{12}\xi_{12}(x,y)\}$$

with

$$\xi_8(x,y) = \max\{v_8(x) - v_8(y) |\ 1 = v_8(x) \geq v_8(y) \geq v_8(z) = 0\}$$
$$\xi_{10}(x,y) = \max\{v_{10}(x) - v_{10}(y) |\ 1 = v_{10}(y) \geq v_{10}(z) \geq v_{10}(x) = 0\}$$
$$\xi_{11}(x,y) = \max\{v_{11}(x) - v_{11}(y) |\ 1 = v_{11}(z) \geq v_{11}(x) \geq v_{11}(y) = 0\}$$
$$\xi_{12}(x,y) = \max\{v_{12}(x) - v_{12}(y) |\ 1 = v_{12}(y) \geq v_{12}(x) \geq v_{12}(z) = 0\}.$$

The value $\xi_i(a_x, a_y)$, by solving 4 LPs, is (1,-1,1,0). By substitution of the value $\xi_i(a_x, a_y)$ into the top objective function and solving the LP, $\varphi_4^+(x,y)$ is 1.0. $\varphi_4^-(x,y)$ = -0.4 is obtained by using "min" in the place of "max". Similarly, $[\varphi_4^-(x,z), \varphi_4^+(x,z)]$ =[0.0133,1.0] and $[\varphi_4^-(y,z), \varphi_4^+(y,z)]$=[-0.16,1.0] can be derived.

In sub-tree of Figure 2(b), the set *DT(1)* is {3,4,5,6,7} and *DL(1)* is {1,2,3,4,5,6,7}

The set of constraints *W1* in sub-tree of Figure 2(b) is:

$$W^1 = \begin{cases} w_2 \ge w_3 \ge w_4 \\ w_2 - w_3 \ge w_3 - w_4 \\ w_5 \ge 2w_6 \\ w_5 \ge w_6 \ge w_7 \\ w_5 + w_6 + w_7 = w_2 \\ w_2 + w_3 + w_4 = w_1 \\ w_1 = 1 \\ w_i \ge 0 \qquad\qquad i = 1,2,3,4,5,6,7. \end{cases}$$

The value interval obtained from sub-tree in Figure 2(a) is used as *V4* in the sub-tree of Figure 2(b). With the set, *W1, V3, V4, V5, V6,* and *V7,* the value interval on the topmost attribute, $[\varphi_1^-(x,y), \varphi_1^+(x,y)]$, can be derived. An example of the formulation for alternatives x and y is

$$\varphi_1^+(x,y) = \max_{W^1} \{w_3\xi_3(x,y) + w_4\xi_4(x,y) + w_5\xi_5(x,y) + w_6\xi_6(x,y) + w_7\xi_7(x,y)\}$$

with

$$\xi_3(x,y) = \max\{v_3(x) - v_3(y) | \ 1 = v_3(x) \ge v_3(z) \ge v_3(y) = 0, \\ v_3(x) - v_3(z) \ge v_3(z) - v_3(y)\}$$

$$\xi_6(x,y) = \max\{v_6(x) - v_6(y) | \ 1 = v_6(x) \ge v_6(z) \ge v_6(y) = 0, \ v_6(x) \ge 3v_6(z)\}.$$

$$\xi_7(x,y) = \max\{v_7(x) - v_7(y) | \ 1 = v_7(y) \ge v_7(z) \ge v_7(x) = 0, \ v_7(z) \ge 2v_7(x)\}$$

The value $\xi_i(a_x, a_y)$ is (1,1,0.6,1,-1). By substitution of the value $\xi_i(a_x, a_y)$ into the top objective function and solving LP, $\varphi_1^+(x,y)$ is obtained as 0.9111. Similarly, φ_1^- and φ_1^+ for the other pairs of alternatives can be derived. Thus the following yields

$$[\varphi_1^-(x,y), \varphi_1^+(x,y)] = [0.15,\ 0.9111]$$

$$[\varphi_1^-(y,z), \varphi_1^+(y,z)] = [-1,\ 0.25]$$

$$[\varphi_1^-(x,z), \varphi_1^+(x,z)] = [-0.7,\ 0.6889].$$

With the value interval on the topmost attribute, the dominance relation between alternatives is to be established. Alternative x strictly dominates y, because $\varphi_1^-(x,y) \geq 0$. Due to the fact that the relation $\varphi_1^-(x,y) = -\varphi_1^+(y,x)$ is always satisfied for each pair of alternatives, $\varphi_1^-(z,y)$ is -0.25 and $\varphi_1^-(z,x)$ is -0.6889. Alternative z weakly dominates x and y because $\varphi_1^-(z,y) \geq \varphi_1^-(y,z)$ and $\varphi_1^-(z,x) \geq \varphi_1^-(x,z)$. Accordingly, we can obtain the preference relation, $z \succ x \succ y$.

5. Conclusion

We presented techniques to establish dominance relation with the DM's LPI in a hierarchically structured attribute tree. The techniques are based on a linear programming model. It allows the DM to express any type of LPI on attribute weights and alternative values. Also, to handle the tree structure, we break down an attribute tree into sub-trees. The optimization results from sub-trees can be utilized in computing the value interval on the upper level attribute, without changing the optimal value interval on the upper level attribute. The efficiency of the proposed technique is dependent upon how one breaks an attribute tree into sub-trees.

Once pairwise dominance relations are established, a set of more than two alternatives may be ranked based on the transitivity of preferences. The related studies have been found in the literature [18, 25].

Although commercial software associated with LP such as *What's Best* and *LINDO* can check the inconsistency of the DM's LPI, we assume a consistent set of LPI. So this still leaves a systematic approach to be developed which can check the consistency of the DM's LPI.

Appendix:Proof for the Property of Technique3.

If we let ρ_i^k be the normalized weight of an attribute i in a sub-tree k, then the following equation is satisfied:

$$\sum_{i \in DT(k)} w_i [v_i(x) - v_i(y)] = \sum_{i \in DT(k)} \rho_i^k [v_i(x) - v_i(y)] .$$

In addition, if we break down the tree which has the topmost attribute, w_k, into sub-trees at level L_p of the attribute hierarchy, then newly defined upper level tree has the normalized attribute weight, $\rho_p^{k^N}, p \in P(k) = L_p$, and the other lower level sub-trees have the normalized attribute weight, ρ_i^P. Therefore the following is satisfied:

$$\begin{aligned}
&\sum_{i \in DT(k)} \rho_i^k [v_i(x) - v_i(y)] \\
&= \sum_{p \in P(k)} \sum_{i \in DT(p)} \rho_p^{k^N} \rho_i^p [v_i(x) - v_i(y)] \\
&= \sum_{p \in P(k)} \rho_p^{k^N} \sum_{i \in DT(p)} \rho_i^p [v_i(x) - v_i(y)] \\
&\le \sum_{p \in p(k)} \rho_p^{k^N} \varphi_p^+(x,y) \left(\text{also, } \ge \sum_{p \in p(k)} \rho_p^{k^N} \varphi_p^-(x,y) \right).
\end{aligned}$$

Accordingly $\sum_{p \in p(k)} \rho_p^{k^N} \varphi_p^-(x,y) \le \sum_{i \in DT(k)} w_i [v_i(x) - v_i(y)] \le \sum_{p \in p(k)} \rho_p^{k^N} \varphi_p^+(x,y)$

.

Applying these (in)equalities gives

$$\sum_{i\in DT(k)} w_i\left[v_i(x)-v_i(y)\right]$$

$$= \sum_{i\in DT(k)} \rho_i^k\left[v_i(x)-v_i(y)\right]$$

$$= \sum_{p\in P(k)}\sum_{i\in DT(p)} \rho_p^{k^N}\cdot\rho_i^p\cdot\left[v_i(x)-v_i(y)\right] + \sum_{\substack{i\in DT^N(k)\\ i\notin P(k)}} \rho_i^{k^N}\left[v_i(x)-v_i(y)\right]$$

$$= \sum_{p\in P(k)} \rho_p^{k^N}\sum_{i\in DT(p)} \rho_i^p\cdot\left[v_i(x)-v_i(y)\right] + \sum_{\substack{i\in DT^N(k)\\ i\notin P(k)}} \rho_i^{k^N}\left[v_i(x)-v_i(y)\right]$$

$$\leq \sum_{p\in p(k)} \rho_p^{k^N}\varphi_p^+(x,y) + \sum_{\substack{i\in DT^N(k)\\ i\notin P(k)}} \rho_i^{k^N}\xi_i^+(x,y) \quad \left(= \sum_{i\in DT^N(k)} \rho_i^{k^N}\theta_i^+(x,y)\right)$$

$$\geq \sum_{i\in P(k)} \rho_p^{k^N}\varphi_p^-(x,y) + \sum_{\substack{i\in DT^N(k)\\ i\notin P(k)}} \rho_i^{k^N}\xi_i^-(x,y) \quad \left(= \sum_{i\in DT^N(k)} \rho_i^{k^N}\theta_i^-(x,y)\right)$$

also, $\theta_i^+(x,y)$ means $\varphi_i^+(x,y)$ on attribute $i\in P(k)$ and $\xi_i^+(x,y)$ on attribute $i\notin P(k)$.)

Accordingly,

$$\sum_{i\in DT^N(k)} \rho_i^{k^N}\theta_i^-(x,y) \leq \sum_{i\in DT(k)} w_i\left[v_i(x)-v_i(y)\right] \leq \sum_{i\in DT^N(k)} \rho_i^{k^N}\theta_i^+(x,y)$$

.

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