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COMPLETELY RIGHT PROJECTIVE SEMIGROUPS

EUNHO L. MOON

ABSTRACT. We here characterize semigroups (which are called completely right projective semigroups) for which every S-automaton is projective, and then examine some of the relationships with the semigroups (which are called completely right injective semigroups) in which every S-automaton is injective.

1. INTRODUCTION

Definition 1. A *semigroup* S is a set of elements together with an associative binary operation defined on S.

Definition 2. A *right congruence* ρ on a semigroup S is an equivalence relation ρ such that if a, b, c in S and $a \rho b$, then $ac \rho bc$.

Definition 3. An *automaton* (or a deterministic automaton), $A = (A, S, \delta)$, is a triple where A is a nonempty set, S is a nonempty semigroup and δ is a function mapping $A \times S$ into A such that $\delta(a, st) = \delta(\delta(a, s), t)$ (*i. e.*, $a(st) = (as)t$) for $a \in A$ and $s, t \in S$.

We will use with the standard notations for automàta and semigroups (*cf.* Clifford & Preston [2]). Let S^1 be the semigroup adjoined S with identity $\mathbf{1}$. All automata in the following will be unitary S-automata where S is the their input semigroup unless specified and a unitary automaton means the identity element of S^1 which acts as the identity operator.

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2. COMPLETELY RIGHT PROJECTIVE SEMIGROUPS

Definition 4. A *basis* for an automaton A is a subset B of A such that for every a in A there is a unique pair (x, s) such that $xs = a$ where $x \in B$ and $s \in S^1$. An automaton A over a semigroup S is called *free* if it has a basis.

For a given semigroup S and a nonempty set X, we can construct the free automaton A over X as the set $X \times S^1$ with the operation by elements of S defined as $(x, s)t = (x, st)$.

Definition 5. An automaton P is called *projective* if every diagram

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "g"] \\ L \arrow[r, "f"] & M, \end{tikzcd}$$

where f is onto can be completed (to yield a commutative diagram) by a homomorphism $h : P \to L$ such that $fh = g$.

Definition 6. An automaton A is called *cyclic* if there is an element a in A such that $A = aS^1$.

We first characterize projective automata.

Lemma 1 (Lam & Oehmke [6]). *A free automaton is projective.*

Lemma 2 (Lam & Oehmke [6]). *If an automaton P is projective, then P is the direct sum of cyclic automata.*

Theorem 3. *Let P be an automaton over a semigroup S. Then P is projective if and only if*

(1) *P is the direct sum of cyclic automata, i.e., $P = \oplus_{i \in I} A_i$ for some indexing set I;*

(2) *For all $i \in I$, there are $a_i \in A_i$ and $e_i \in S^1$ such that $A_i = a_i S^1$ and*

$$a_i e_i = a_i, a_i s = a_i t \Longleftrightarrow e_i s = e_i t$$

for all $s, t \in S^1$.

Proof. $(\Rightarrow)$ Assume that P is a projective automaton. Then (1) is done by Lam & Oehmke [6]. To show (2), let F be a free automaton generated by $P \times \{\mathbf{1}\}$ where $\mathbf{1}s = s = s\mathbf{1}$ for all $s \in S$. Consider the diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle I_P} \\ F & \xrightarrow{\alpha} & P \end{array}$$

where I_P is the identity map on P and α is the canonical map which sends (p, s) to ps. Since α is clearly onto, there is a homomorphism $\beta : P \to F$ such that $\alpha\beta = I_P$. But since $P = \oplus_{i \in I} A_i$ where A_i is cyclic, consider the generator c_i of A_i, and let $\beta(c_i) = (a_i, s_i)$. Since $\alpha\beta = I_P$, $c_i = I_P(c_i) = \alpha\beta(c_i) = \alpha(a_i, s_i) = a_i s_i$. If x is an element of A_i, then $x = c_i t$ for some $t \in S^1$, so $\beta(x) = \beta(c_i t) = \beta(c_i)t = (a_i, s_i)t = (a_i, s_i t)$. Now consider an element a_i in P. Since $P = \oplus_{i \in I} A_i$, a_i belongs to A_j for some j. But since $c_i = a_i s_i \in A_i \cap A_j$, j must be equal to i. Hence $a_i \in A_i$ and for any $x \in A_i, x = I_P(x) = \alpha\beta(x) = \alpha(a_i, s_i t) = a_i(s_i t)$.

Hence $A_i = a_i S^1$ for some $a_i \in A_i$. Also if $a_i \in A_i$, then $\beta(a_i) = (a_i, u)$ for some $u \in S^1$, so $a_i = I_P(a_i) = \alpha\beta(a_i) = \alpha(a_i, u) = a_i u$ and $(a_i, u) = \beta(a_i) = \beta(a_i u) = \beta(a_i)u = (a_i, u)u = (a_i, u^2)$. Hence we have that $a_i = a_i u$ where $u = u^2$, call u by e_i, then we have that $a_i = a_i e_i$ and $e_i^2 = e_i$. Assume that $a_i s = a_i t$ for all $s, t \in S^1$. Since $(a_i, e_i s) = (a_i, e_i)s = \beta(a_i)s = \beta(a_i s) = \beta(a_i t) = \beta(a_i, e_i t)$, it is clear that $e_i s = e_i t$ for all $s, t \in S^1$. Conversely, if $e_i s = e_i t$ for all $s, t \in S^1$, then $a_i s = a_i e_i s = a_i e_i t = a_i t$.

($\Leftarrow$) Assume that P is an automaton satisfying (1) and (2), and consider the diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \delta} \\ L & \xrightarrow{\alpha} & M \end{array}$$

where α is onto.
Since $P = \oplus_{i \in I} A_i$ where $A_i = a_i S^1$ for some $a_i \in A_i$, select an element $\ell_i \in L$ such that $\alpha(\ell_i) = \delta(a_i)$. Define a map $\beta : P \to L$ by $\beta(a_i s) = \ell_i e_i s$. If $a_i s = a_i t$, then $e_i s = e_i t$ by (2), so $\ell_i e_i s = \ell_i e_i t$. Also for every $t \in S, \beta((a_i s)t) = \beta(a_i(st)) = \ell_i e_i(st) = \beta(a_i s)t$. Hence β is a well defined homomorphism. Also since $\alpha\beta(a_i s) = \alpha(\ell_i e_i s) = \alpha(\ell_i)e_i s = \delta(a_i)e_i s = \delta(a_i s)$, we have that $\alpha\beta = \delta$ so that the diagram commutes. Thus P is a projective automaton. $\square$

Corollary 4. *Let P be a projective automaton over a semigroup S. If S has no idempotents except a left identity, then P is free.*

Proof. Let P be a projective automaton over a semigroup S, and assume that S has no idempotents except a left identity. Since P is the direct sum of cyclic automata and each direct summand is isomorphic to S, it suffices to prove that P is free by the definition. □

Lemma 5. *Let P be an automaton over a semigroup S. Then P is projective if and only if every direct summand of P is projective.*

Proof. It is easily proved by similar arguments of Theorem 3. □

Definition 7. For an automaton A, an element x of A is called *a zero element* if $xs = x$ for all s in S^1.

Lemma 6. *Let A_i be the direct summand of a projective automaton P as described in Theorem 3. Then each A_i has a zero element if and only if S^1 has a left zero element.*

Proof. Let x be a left zero element of S^1 and for every i let a_i be the generator of direct summand A_i. Since $(a_ix)s = a_i(xs) = a_ix$ for all s in S^1, each direct summand A_i has a zero element a_ix. Conversely, for each i let x_i be a zero element of A_i. If P is a projective automaton, then there is an idempotent element e_i in S^1 such that A_i is isomorphic to e_iS^1. For every i let a_i be a generator of the cyclic automaton A_i, then there is s_i in S^1 such that $x_i = a_is_i$. Since $a_is_i = a_is_it \Leftrightarrow e_is_i = e_is_it$ for all $t \in S^1$, an element e_is_i is a left zero element of S^1. □

Lemma 7. *Let $\boldsymbol{\rho}$ be a right congruence on a semigroup S. If S has a left identity, then the set of equivalence classes of $\boldsymbol{\rho}$ on S is a cyclic S-automaton.*

Proof. Let S be a semigroup with a left identity and $\boldsymbol{\rho}$ be a right congruence on S. If $a\boldsymbol{\rho}$ is the equivalence class of $\boldsymbol{\rho}$ containing a and c is any element of S, we define $(a\boldsymbol{\rho})c = (ac)\boldsymbol{\rho}$. It is easily seen that this law of composition is well-defined. Since the multiplication of S is associative, we have that $(a\boldsymbol{\rho})cd = a(cd)\boldsymbol{\rho} = (ac\boldsymbol{\rho})d = ((a\boldsymbol{\rho})c)d$ for all $d \in S$. Also if e is a left identity of S, then for every s in S, $s\boldsymbol{\rho} = (es)\boldsymbol{\rho} = (e\boldsymbol{\rho})s$. Thus it is clear that the set $S/\boldsymbol{\rho}$ of equivalence classes of a right congruence $\boldsymbol{\rho}$ is a cyclic automaton. □

From now on we call semigroups S for which every automaton over S is projective completely right projective semigroups.

Lemma 8. *If a semigroup S is completely right projective, then S has both an identity and a left zero element.*

Proof. Assume that a semigroup S is completely right projective and let P be the singleton automaton. If P is projective by hypothesis, then there is an idempotent e in S^1 such that P is isomorphic to eS^1. Thus e is a left zero element of S^1. Next we suppose that S has no identity element $\mathbf{1}$ and consider the S-automaton S^1 which is the semigroup obtained from S adjoined with $\mathbf{1}$. Then, by Lemma 2, there is some element a in S such that $S^1 = aS^1 \subseteq S$. Thus S has an identity element $\mathbf{1}$. □

In characterizing completely right projective semigroups we use the more standard result for general semigroups.

Lemma 9 (Bulman-Fleming [1]). *Let $\boldsymbol{\rho}$ be the smallest right congruence containing (a,b), $\boldsymbol{\rho} = \langle (a,b) \rangle$, and let $(x,y) \in \boldsymbol{\rho}$. Then there exist $s_1, \dots, s_n \in S$ such that $x \in \{as_1, bs_1\}$, $y \in \{as_n, bs_n\}$, and $\{as_i, bs_i\} \cap \{as_{i+1}, bs_{i+1}\} \neq \varnothing$ for $i = 1, 2, \dots, n-1$.*

Theorem 10. *If S is a completely right projective semigroup, then S is either a trivial group or a trivial group adjoined with $\mathbf{0}$.*

Proof. If S is a completely right projective semigroup, then it has shown that S has a left zero element and each direct summand of an automaton has a zero element. Let I be the set of all non-generators of S. If $I = \varnothing$, then S is right simple. But since S has a left zero element, it implies that S is the singleton. If $I \neq \varnothing$, let $x \in I$ and $s \in S$. If xs were not in I, then $S = (xs)S \subseteq xS \subset S$ which is a contradiction. Thus xs is in I so that I is a proper right ideal of S. Define Rees right congruence λ_I on S by $s\lambda_I t$ if and only if either $s = t$ or $s, t \in I$. The set S/λ_I of equivalence classes of λ on S is the cyclic automaton containing only one nontrivial equivalence class I. If S/λ_I were generated by I, then all generators of S should be in I that contradicts to the choice of x. Thus S/λ_I is generated by some element not in I and hence S/λ_I is isomorphic to S. Since it means that λ_I is the identity congruence, the right ideal I has only one element x(actually a left zero element of S). If s is any element of S, then sx is too a left zero elementand x is actually a zero element of S, say $x = \mathbf{0}$. Let $T = S \backslash \{\mathbf{0}\}$ and we claim that if T is nonempty then it is a subsemigroup of S. If ab were equal to $\mathbf{0}$ for any a, b in T, then $\{\mathbf{0}\} = a(bS) = aS = S$. Hence ab is nonzero and then T is a right simple subsemigroup. But the fact that completely right projective semigroups contain an identity element shows that T is a group. Thus S is a group adjoined with $\mathbf{0}$. Let $\boldsymbol{\rho}$ be any right congruence on S. If the cyclic automaton $S/\boldsymbol{\rho}$ is projective by hypothesis, then it is isomorphic to either S or $\{\mathbf{0}\}$. Thus a right congruence $\boldsymbol{\rho}$ on S is either the identity or the universal congruence.

For any element x in T, we consider the smallest right congruence $\boldsymbol{\rho}$ generated by $(\mathbf{1}, x)$. If $\boldsymbol{\rho}$ were the universal congruence, then $\mathbf{0}\ \boldsymbol{\rho}\ \mathbf{1}$ and there are elements $s_1, \dots, s_n$ in S such that $\mathbf{0} \in \{\mathbf{1}s_1, xs_1\}$, $\mathbf{1} \in \{\mathbf{1}s_n, xs_n\}$ and

$$\{\mathbf{1}s_i, xs_i\} \cap \{\mathbf{1}s_{i+1}, xs_{i+1}\} \neq \varnothing \quad \text{for all} \quad i = 1, \dots, n-1.$$

But since $\mathbf{0} = s_1 = \cdots = s_n = \mathbf{1}$ and it contradicts to the fact that T is nonempty, the right congruence $\boldsymbol{\rho}$ is the identity, and it suffices to prove that T is the singleton. Thus S is either $\{\mathbf{1}\}$ or $\{\mathbf{1}, \mathbf{0}\}$. □

Lemma 11. *For a semigroup S, every S-automaton is free if and only if S is trivial.*

Proof. Assume that every automaton over a semigroup S is free. If a singleton automaton A is free, then it is isomorphic to the copies of S so that S is trivial. Conversely if A is an automaton over S where $S = \{x\}$, then for every $a \in A$, there is an element b in A such that $a = bx$ and then $ax = (bx)x = b(xx) = bx = a$. It suffices to prove that A is free. □

Corollary 12. *If a semigroup S has no idempotents except a left identity, then the followings are equivalent.*

(1) *Every automaton is free.*
(2) *Every automaton is projective.*
(3) *S is trivial.*

Proof. It suffices to prove by previous results. □

Remark. We've already seen that every S-automaton is projective if S is trivial. Thus when $S = \{\mathbf{1}, \mathbf{0}\}$, we consider the automaton A containing three elements a, b and c with $a\mathbf{0} = b\mathbf{0} = c$. Now A is the union of two cyclic subautomata $\{a, c\}$ and $\{b, c\}$ containing $\{c\}$ in common. It suffices to show that the automaton $A = \{a, b, c\}$ is not projective. Thus we next have the weak converse of Theorem 10 as follow.

Theorem 13. *Let S be a semigroup having a left identity. Then every cyclic automaton is projective if and only if S is either a trivial group or a trivial group adjoined with $\mathbf{0}$.*

Proof. For a semigroup S with a left identity we assume that every cyclic automaton is projective and let I be the set of nongenerators of S. If $I = \varnothing$, it is easily seen that

S is a right group. For any element x in S, let $\boldsymbol{\rho}$ be the right congruence generated by (e, x) where e is a left identity of S. If the cyclic automaton $S/\boldsymbol{\rho}$ is projective by assumption, it shows that all elements of S are left identities, that is, S is the set of left identities. For any two elements x, y in S let σ be the right congruence on S generated by (x, y). Since the cyclic automaton S/σ is too projective by assumption, it suffices to say that S is a singleton. If $I \neq \varnothing$, it has shown that I is a proper right ideal of S. Consider the cyclic automaton S/λ_I where λ_I is Rees right congruence on S with respect to I. If S/λ_I is projective, there is an idempotent g in S such that S/λ_I is isomorphic to gS^1. But since S/λ_I is generated by some element not in I, it says that S/λ_I is isomorphic to S so that the right ideal I is the singleton. Thus all elements of S except one, say x, generate S and the element x should be a (left) zero element of S. Set $x = \mathbf{0}$, and let $T = S \backslash \{\mathbf{0}\}$. Then by the same arguments in proof of Theorem 10, T is trivial so that S is a trivial group adjoined with $\mathbf{0}$.

Conversely, we assume that S is either a trivial group or a trivial group adjoined with $\mathbf{0}$. If S is trivial, then every automaton is projective and hence every cyclic automaton is trivially projective. Hence we only consider the projectivity for cyclic automata over the trivial group S adjoined with $\mathbf{0}$. If A is a cyclic automaton generated by an element a, then for any x in A there is an element s in S such that $x = as$, hence x is either a or $a\mathbf{0}$. If a is equal to $a\mathbf{0}$, then A is the singleton, hence clearly a projective automaton. If a is not equal to $a\mathbf{0}$, then there exists an isomorphism ϕ from S onto A given by $\phi(s) = as$ and this isomorphism suffices to say that A is free, so projective. □

3. Completely right injective semigroups

Completely right injective semigroups are semigroups in which every automaton is injective. Thus we now examine some of the relationships between completely right projective semigroups and completely right injective semigroups. We first give the definition of an injective automaton and its property.

Definition 8. An automaton J is called *injective* if for every monomorphism $\alpha : L \to M$ and a homomorphism $\beta : L \to J$, there is a homomorphism $f : M \to J$ such that $f\alpha = \beta$.

Lemma 14 (Moon [7]). *Let J be an automaton over a semigroup S. If S has a zero element, then J is an injective automaton if and only if for every right ideal I and S-homomorphism $\phi : I \to J$ there is an element y in J such that $\phi(s) = ys$ for all $s \in I$.*

Before we examine the relationship between completely right projective semigroups and completely right injective semigroups, we need to characterize completely right injective semigroups S. It has shown that semigroups of this type have a zero and every right ideal of S is generated by an idempotent. In addition, semigroups of this type are the disjoint union of right groups(*cf.* Feller & Gantos [3] and Moon [7]).

In Moon [7], the author also shows that if the semigroup S with a left identity is completely right injective, then the necessary and sufficient condition for $L(S)$, the set of all right congruences on S, to be semiatomic is that S has no proper essential right congruences. As a lattice, $L(S)$ is semiatomic if the universal congruence is the join of its minimal right congruences on S.

Note. It is clear that every automaton over completely right projective semigroups is injective. We now characterize projective automata over completely right injective semigroups.

Theorem 15. *Assume that S is a completely right injective semigroup with a left identity. Then*

(1) *Every projective automaton over S is cyclic.*
(2) *The set of isomorphic images in S of projective automata over S is linearly ordered by inclusion.*
(3) *Every cyclic automaton which is projective is a right ideal of S.*
(4) *If every automaton over S is projective, then S is the singleton.*

Proof. Assume that a semigroup S with a left identity is completely right injective.

(1) Let P be an automaton over S. If P is projective, then it is the direct sum of cyclic automata and each direct summand of P is isomorphic to some right ideal of S. Since the set of right ideals of completely right injective semigroup S is linearly ordered by inclusion, it suffices to say that P consists of only one direct summand. Thus P is cyclic.

(2) Let A and B be projective automata over a semigroup S. If S is completely right injective, then they are cyclic by (1) and there are idempotent elements f and

g in S such that $A(B)$ is isomorphic to fS^1(*resp.* gS^1). But since the set of right ideals of a completely right injective semigroup S is linearly ordered by inclusion, we can say that the set of isomorphic images in S of projective automata is linearly ordered by inclusion.

(3) Let A be any cyclic automaton. If A is also projective, then there is some right ideal of S which is isomorphic to A. Since every right ideal of a completely right injective semigroup is projective as an S-automaton, the previous result (2) induces that a cyclic automaton A is a right ideal of S.

(4) Let I be the set of nongenerators of S and assume that I is nonempty. Since I is the largest proper right ideal of S, we define *Rees right congruence* on S with respect to I. If the cyclic automaton S/σ is projective, then there is an idempotent e in S such that S/σ is isomorphic to eS^1. If e is in I, then S/σ is isomorphic to I which is a contradiction. Thus e is not in I and S/σ is isomorphic to S so that σ is equal to the identity congruence ι. Since all elements of S except one element (actually $\mathbf{0}$) generate S, the set $T = S \setminus \{\mathbf{0}\}$ is a group. If S is again projective as an automaton, then T must be isomorphic to the copies of S. It contradicts to the construction of T. Thus I must be an empty set and so S is a right group. But since S has both a left identity and a zero, it shows that S is the singleton. $\square$

Lemma 16. *Assume that semigroups S are completely right injective and let A be an S-automaton. Then the sufficient condition for A to be projective is a cyclic automaton which is isomorphic to some right ideal of S.*

Proof. Let A be an automaton over a semigroup S and assume that every automaton over S is injective. If A is injective, then for an S-homomorphism $f : S \to A$ there is some element a in A such that $f(s) = as$ for all s in S. Thus every S-automaton contains a cyclic subautomaton. Set $B = A \setminus aS^1$. If A were projective, then B should be the disjoint union of cyclic subautomata which are isomorphic to right ideals of S. Suppose that bS^1, cS^1 are two direct summands of B. Then there are right ideals I, J of S such that $bS^1(cS^1)$ is isomorphic to $I(J)$. Since all right ideals of a completely right injective semigroup S are linearly ordered by inclusion, it suffices to say that bS^1 is equal to cS^1, that is, B is the cyclic automaton which is isomorphic to some right ideal of S. Since it says that A is the disjoint union of two cyclic automata which are isomorphic to some right ideals, it suffices to say that B is empty by the same argument above. Thus in order to be for A projective, an automaton A should be cyclic which is isomorphic to some right ideal of S. $\square$

Corollary 17. *Let S be a semigroup with a left identity.*

(1) *If every automaton over S is projective, then it is too injective.*

(2) *If every automaton over S is injective, then the cyclic automaton which is isomorphic to some right ideal of S is projective.*

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DEPARTMENT OF GENERAL EDUCATION, MYONGJI UNIVERSITY, SAN 38-2 NAM-DONG, YONGIN, GYEONGGI-DO 449-728, KOREA
Email address: ehlmoon@mju.ac.kr