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INTERSECTION OF THE DEGREE-n BIFURCATION SET WITH THE REAL LINE

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ABSTRACT. Definition and some properties of the degree-n bifurcation set are introduced. It is proved that the interval formed by the intersection of the degree-n bifurcation set with the real line is explicitly written as a function of n. The functionality of the interval is computationally and geometrically confirmed through numerical examples. Our study extends the result of Carleson & Gamelin [2].

1. INTRODUCTION

The degree-n bifurcation set denoted by $\mathbf{M}$ is introduced by Devaney [4] and is defined to be a set of all complex values of c such that the critical orbit of the complex polynomial $P_c(z) = z^n + c$ does not escape to infinity. Rays of symmetry and k-periodic components are defined. Symmetry and some properties of $\mathbf{M}$ as well as the parameterized boundary equation of the main component are described. The intersection of the degree-n bifurcation set with the real line is introduced by Carleson & Gamelin [2] for the case of $n = 2$. In this paper, we investigate the intersection for general cases that $n \geq 2$. Typical cases are explored with computational results for $2 \leq n \leq 1000$. The following notations and symbols are used throughout the paper.

$\mathbb{C}$: set of all complex numbers.
$\mathbb{R}$: set of all real numbers.
$\mathbb{N}$: set of all natural numbers.
$f^k(z) = f \circ f^{k-1}(z)$: k-fold composite map of f at z with $f^0(z) = z$.
$\bar{c}$: complex conjugate of c.

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Definition 1. Let $P_c(z) = z^n + c$ for an integer $n \geq 2$, with $c, z \in \mathbb{C}$. Then the *degree-n bifurcation set* is defined to be the set

$$\mathbf{M} = \left\{ c \in \mathbb{C} : \lim_{k\to\infty} P_c^k(0) \neq \infty \right\}.$$

If $n = 2$, it reduces to the Mandelbrot set (*cf.* Devaney [3], [4]).

Definition 2. The sets $\mathbf{P}_m = \{(r, \phi_m) : r \geq 0,\ \phi_m = m\pi/(n-1)\}$ are called the *rays of symmetry* for $m \in \{1, 2, \ldots, 2n-2\}$.

Theorem 1. *$\mathbf{M}$ is symmetric about $\mathbf{P}_m$ for all $m \in \{1, 2, \ldots, 2n-2\}$ in the c-parameter plane.*

Proof. Let $c = re^{i\phi}$ with $r \geq 0$, $0 \leq \phi \leq 2\pi$, $i = \sqrt{-1}$. One can show recursive relations $T_k(\phi)$ for all $k \in \mathbb{N}$ as follows:

$$T_{k+1}(\phi) = T_1(\phi)(1 + T_k(\phi))^n, \text{ where } T_1(\phi) = c^{n-1} = r^{n-1}e^{i(n-1)\phi}.$$

It can be shown by induction that for all $k \in \mathbb{N}$ and $m \in \{1, 2, \ldots, 2n-2\}$

$$\overline{T_k(\phi)} = T_k(-\phi) = T_k(2\phi_m - \phi).$$

By means of $T_k(\phi)$, we obtain $P_c^{k+1}(0) = c(1 + T_k)$. Since

$$\begin{aligned} |P_c^{k+1}(0)| &= r|1 + T_k(\phi)| = r|\overline{1 + T_k(\phi)}| = r|1 + \overline{T_k(\phi)}| = r|1 + T_k(-\phi)| \\ &= r|1 + T_k(2\phi_m - \phi)| = |c^*(1 + T_k(2\phi_m - \phi))| = |P_{c^*}^{k+1}(0)| \end{aligned}$$

with $c^* = re^{i(2\phi_m - \phi)}$, we have $c^* \in \mathbf{M}$ whenever $c \in \mathbf{M}$. This completes the proof. □

Lemma 1. *Let $n \geq 2$ be an integer. Then*

$$\mathbf{M} = \{c \in \mathbb{C} : |P_c^k(0)| \leq 2^{\frac{1}{n-1}} \text{ for all } k \geq 1\} \subset \{c \in \mathbb{C} : |c| \leq 2^{\frac{1}{n-1}}\}.$$

Proof. If $|c| > 2^{\frac{1}{n-1}}$, one can show by induction on $k \geq 1$ that

$$|P_c^{k+1}(0)| \geq |c|(|c|^{n-1} - 1)^{n^{k-1}}. \tag{1.1}$$

According to (1.1) $|P_c^k(0)| \to \infty$ as $k \to \infty$, we have $c \notin \mathbf{M}$. Thus $|c| \leq 2^{\frac{1}{n-1}}$ for $c \in \mathbf{M}$, and $\mathbf{M} \subset \{c \in \mathbb{C} : |c| \leq 2^{\frac{1}{n-1}}\}$. Now suppose that $|P_c^m(0)| = 2^{\frac{1}{n-1}} + \delta > 2^{\frac{1}{n-1}}$ with $\delta > 0$ for some $m \geq 1$. If $|c| = |P_c(0)| > 2^{\frac{1}{n-1}}$, then $c \notin \mathbf{M}$. If $|c| \leq 2^{\frac{1}{n-1}}$, then

$$\begin{aligned} |P_c^{m+1}(0)| &\geq |P_c^m(0)|^n - |c| \geq (2^{\frac{1}{n-1}} + \delta)^n - 2^{\frac{1}{n-1}} \\ &= 2^{\frac{n}{n-1}}(1 + \delta\, 2^{\frac{-1}{n-1}})^n - 2^{\frac{1}{n-1}} \geq 2^{\frac{1}{n-1}} + 2n\delta. \end{aligned}$$

Proceeding by induction, we obtain $|P_c^{m+k}(0)| \geq 2^{\frac{1}{n-1}} + (2n)^k\delta \to \infty$ as $k \to \infty$ and $c \notin \mathbf{M}$. Thus

$$\mathbf{M} = \{c \in \mathbb{C}: \ |P_c^k(0)| \leq 2^{\frac{1}{n-1}} \text{ for all } k \geq 1\}. \qquad \square$$

Definition 3. The *attractive k-periodic component* M_k' is defined as the set

$$\begin{aligned}\mathbf{M}_k' &= \{c \in \mathbb{C}: \ P_c(z) \text{ has an attractive-cycle}\} \\ &= \left\{ c \in \mathbb{C}: \ \text{there exist } z_0 \text{ such that } P_c^k(z_0) = z_0, \ \left|\frac{d}{dz}P_c^k(z)\right|_{z=z_0} < 1 \right\}.\end{aligned}$$

For integers $n \geq 2$ the equation of boundary of $\mathbf{M}_1'$ is easily derived from the elementary complex analysis Ahlfors [1]. Let $\partial\mathbf{M}_k'$ denote the boundary of $\mathbf{M}_k'$ for $k \in \mathbf{M}$. Then Lemma 2 follows:

Lemma 2. *The equation of* $\partial\mathbf{M}_1'$ *is given by*

$$\begin{aligned}&\{c \in \mathbb{C}: z_0^n + c = z_0, |nz_0^{n-1}| = 1\} \\ &\quad = \{(r,\theta): r = |c| = (1/n)^{n/(n-1)}\sqrt{n^2 + 1 - 2n\cos(n-1)\psi}, 0 \leq \psi \leq 2\pi\}\end{aligned}$$

with

$$\tan\theta = (n\sin\psi - \sin n\psi)/(n\cos\psi - \cos n\psi).$$

Proof. Parameterize

$$z_0 = \alpha e^{i\psi}, 0 \leq \psi \leq 2\pi$$

with $|z_0| = \alpha = (1/n)^{1/(n-1)}$. Then

$$c = z_0 - z_0^n = \frac{\alpha}{n}(n\cos\psi - \cos n\psi) + i\frac{\alpha}{n}(n\sin\psi - \sin n\psi).$$

After a detailed algebraic manipulation one can express r by means of ψ. $\square$

Remark. As a result of Lemma 2, when $\psi = 0$, $r = (1 - \frac{1}{n})(\frac{1}{n})^{\frac{1}{n-1}}$. Since θ is a monotone increasing function of ψ on $[0, \pi/(n-1)]$, it can be shown that $\psi = 0$ corresponds to $\theta = 0$ which states that $\partial\mathbf{M}_1'$ crosses the positive real axis at $(r, 0)$.

2. Intersection of the Degree-n Bifurcation Set with the Real Line

In the following theorem, the interval formed by the intersection of the degree-n bifurcation set with the real line is explicitly written as a function of n using the results of Theorem 1 and Lemma 1.

Table 1. End points of $\mathbf{M} \cap \mathbb{R}$ as a function of n

n	$-2^{1/(n-1)}$	$-\rho$	ρ
2	-2.0000000000000000000		0.2500000000000000000
3		-0.38490017945975050967	0.38490017945975050967
4	-1.2599210498948731648		0.47247039371057743679
5		-0.53499224398113761920	0.53499224398113761920
6	-1.1486983549970350068		0.58235593230964937103
7		-0.61973145119955752250	0.61973145119955752250
8	-1.1040895136738123376		0.65012250149741493585
9		-0.67540949835697115318	0.67540949835697115318
10	-1.0800597388923061699		0.69683731441301435375
⋮	⋮	⋮	⋮
1000	-1.00069408178494375409		0.99211607219360468002

Theorem 2. *Let $n \geq 2$ be an integer. Then*

$$\mathbf{M} \cap \mathbb{R} = \begin{cases} [-2^{\frac{1}{n-1}},\ \rho], & \text{if } n \text{ is even}, \\ [-\rho,\ \rho], & \text{if } n \text{ is odd}, \end{cases}$$

where $\rho = \left(1 - \frac{1}{n}\right)\left(\frac{1}{n}\right)^{\frac{1}{n-1}}$.

Proof. Consider $c \in [-2^{\frac{1}{n-1}},\ 2^{\frac{1}{n-1}}]$ in view of the result of Lemma 1. Let x be a real fixed point of P_c such that

$$P_c(x) = x^n + c = x.$$

Case 1: When n is even. It is shown that $c = x - x^n$ assumes its maximum ρ at $x = (1/n)^{1/(n-1)}$. Hence it suffices to consider $c \in [-2^{\frac{1}{n-1}}, \rho]$. Let $a > 0$ be the largest real fixed point of P_c such that $a^n + c = a$. Indeed, one can show that $a = 2^{\frac{1}{n-1}}$. For $0 < c = P_c(0) = a - a^n \leq \rho < a$, it follows that

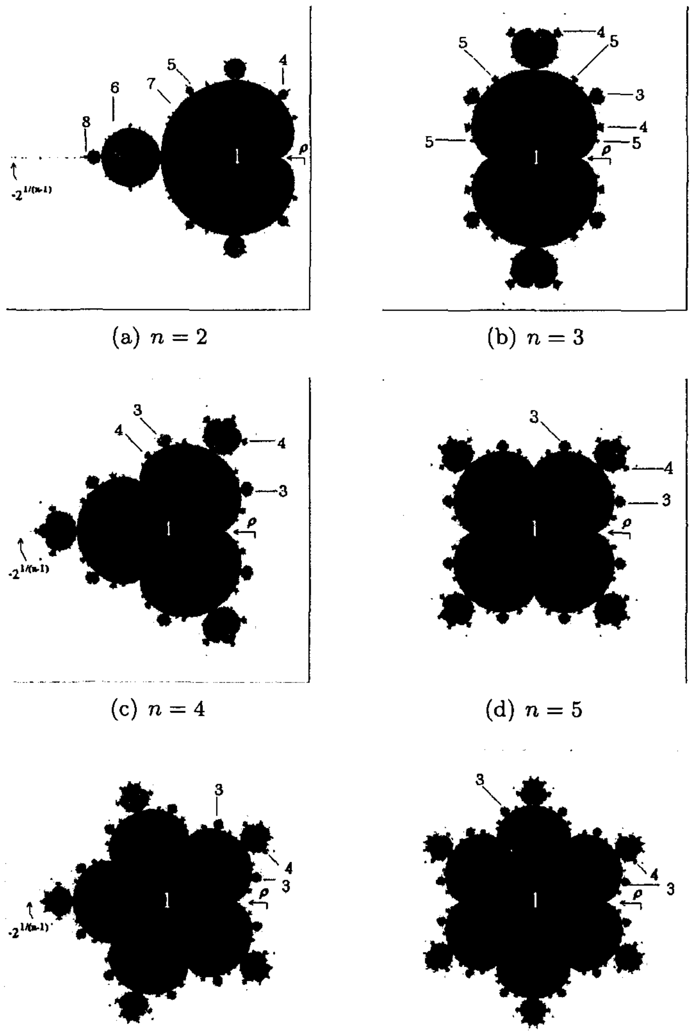
$$0 < P_c^k(0) < a^n + c = a$$

by induction on $k \geq 1$. Hence such $c \in \mathbf{M}$. For $-2^{\frac{1}{n-1}} = -a \leq c = P_c(0) \leq 0$, it is clear that for all $k \geq 1$ with even n

$$P_c^{k+1}(0) = P_c^k(0)^n + c \geq c \geq -a,$$
$$-a \leq c \leq P_c^2(0) = P_c(0)^n + c = |P_c(0)|^n + c \leq a^n + c = a.$$

Proceeding by induction, $0 \leq |P_c^k(0)| \leq a$ for all $k \in \mathbb{N}$ with $-2^{\frac{1}{n-1}} \leq c \leq 0$. Hence such $c \in \mathbf{M}$. As a result, $\mathbf{M} \cap \mathbb{R} = [-2^{\frac{1}{n-1}}, \rho]$ if n is even.

Case 2: When n is odd. Due to symmetry given by Theorem 1, it suffices to consider for $c > 0$. For $0 < \rho < c \leq 2^{\frac{1}{n-1}}$, we have $c = P_c(0) > \rho > 0$. Proceeding by



(a) $n = 2$ (b) $n = 3$

(c) $n = 4$ (d) $n = 5$

(e) $n = 6$ (f) $n = 7$

Figure 1. Typical degree-n bifurcation sets

induction on $k \geq 2$, we have $P_c^{k+1}(0) > P_c^k(0) > c > \rho$. Thus $\{P_c^k(0)\}$ is monotone increasing and not bounded above, from which $\lim_{k\to\infty} P_c^k(0) = \infty$. Hence such $c \notin \mathbf{M}$. For $0 \leq c \leq \rho$, let $a > 0$ be the largest real fixed point of P_c such that $a^n + c = a$. Proceeding by induction on $k \geq 1$, we have $0 \leq P_c^k(0) < a$ for all $k \in \mathbb{N}$. Hence such $c \in \mathbf{M}$. Consequently, the symmetry shows that $\mathbf{M} \cap \mathbb{R} = [-\rho, \rho]$ if n is odd. □

Remark. The value of ρ is alternatively derived from the remark of Lemma 2. If $n = 2$, then Theorem 2 reduces to that of Carleson & Gamelin [2]. Typical degree-n bifurcation sets are shown in Figure 1 in the c-parameter plane for $2 \leq n \leq 7$. The component $\mathbf{M}'_k$ is identified by a number k and shaded in different patterns or colors.

Some computational results are displayed in Table 1 listing end points of $\mathbf{M} \cap \mathbb{R}$ as a function of n for $\leq n \leq 1000$. It can be easily shown that the interval $[-2^{1/(n-1)}, \rho]$ as well as the interval $[-\rho, \rho]$ approaches $[-1, 1]$ as n tends to infinity. Although details of our elaborate numerical experiments are not shown here, careful measurements from Figure 1 show a good agreement with the result of Theorem 2.

REFERENCES

1. Lars V. Ahlfors: *Complex Analysis*, An introduction to the theory of analytic functions of one complex variable, 3rd ed., International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978. MR **80c**:30001
2. Lennart Carleson & Theodore W. Gamelin: *Complex Dynamics*, Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993. MR **94h**:30033
3. Robert L. Devaney: *An Introduction to Chaotic Dynamical Systems.* The Benjamin/Cummings publishing Co., Inc., Menlo Park, CA, 1986. MR **87e**:58142
4. ______: *Chaos, Fractals and Dynamics*, Computer experiments in mathematics. Science Television, New York; distributed by the American Mathematical Society, Providence, RI, 1989. MR **92a**:58093

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