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THEOREMS OF LIOUVILLE TYPE FOR QUASI-STRONGLY p-HARMONIC MAPS

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ABSTRACT. In this article, we prove various properties and some Liouville type theorems for quasi-strongly p-harmonic maps. We also describe conditions that quasi-strongly p-harmonic maps become p-harmonic maps. We prove that if $\phi : M \to N$ is a quasi-strongly p-harmonic map ($p \geq 2$) from a complete noncompact Riemannian manifold M of nonnegative Ricci curvature into a Riemannian manifold N of non-positive sectional curvature such that the $(2p-2)$-energy, $E_{2p-2}(\phi)$ is finite, then ϕ is constant.

1. INTRODUCTION

Let (M^n, g) and (N^m, h) be complete Riemannian manifolds of dimension n and m, respectively, and let $p \geq 2$. A C^1 map $\phi : M \to N$ is called a *p-harmonic map* if it is a critical point of the p-energy functional

$$E_p(\phi) = \int_\Omega |\,\mathrm{d}\,\phi|^p \, dv_g \tag{1}$$

for any bounded domain $\Omega \subset M$. It is well-known Baird & Gudmundsson [1] that a C^2 map $\phi : M \to N$ is a p-harmonic map if and only if it satisfies the p-harmonic map equation

$$\text{Trace}\left(\nabla(|\,\mathrm{d}\,\phi|^{p-2}\,\mathrm{d}\,\phi)\right) = 0. \tag{2}$$

We call a 2-harmonic map just a harmonic map.

The classical Liouville theorem says that any bounded harmonic function defined on the whole plane must be a constant. Yau [7] generalized the Liouville theorem to harmonic functions on Riemannian manifolds of nonnegative Ricci curvature. And

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Cheng [2] and Schoen & Yau [5] proved theorems of Liouville type for harmonic maps from a Riemannian manifold into a Riemannian manifold (see also Hildebrandt [3]).

Note that the notion of p-harmonic map is a parallel generalization of harmonic map and some Liouville type theorems for p-harmonic maps are known. For example, Takeuchi [6] proved that if $\phi : M \to N$ is a p-harmonic map from a complete noncompact Riemannian manifold M of nonnegative Ricci curvature into a Riemannian manifold N of nonpositive sectional curvature such that $E_{2p-2}(\phi) < \infty$, then ϕ is a constant. Nakauchi [4] showed if $E_p(\phi) < \infty$ with the same curvature hypothesis, then ϕ is a constant.

On the other hand, denoting the adjoint operator of the exterior differential d by δ, and the Hodge-DeRham Laplacian by $\Delta = \mathrm{d}\,\delta + \delta\,\mathrm{d}$, one can easily see that the equation (2) is equivalent to

$$\delta(|\,\mathrm{d}\,\phi|^{p-2}\,\mathrm{d}\,\phi) = 0. \tag{3}$$

But a p-harmonic map does not satisfy $\mathrm{d}(|\,\mathrm{d}\,\phi|^{p-2}\,\mathrm{d}\,\phi) = 0$ in general. In case $p = 2$, harmonic maps satisfy the equation (3) obviously.

In this article, we describe various properties and some Liouville type theorems for maps $\phi : M \to N$ between Riemannian manifolds satisfying

$$\Delta(|\,\mathrm{d}\,\phi|^{p-2}\,\mathrm{d}\,\phi) = 0,$$

which is called the quasi-strongly p-harmonic maps.

2. QUASI-STRONGLY p-HARMONIC MAPS

Definition 2.1. Let $p \geq 2$. A map $\phi : (M^n, g) \to (N^m, h)$ between Riemannian manifolds is called a *quasi-strongly* p-harmonic map if it satisfies

$$\Delta(|\,\mathrm{d}\,\phi|^{p-2}\,\mathrm{d}\,\phi) = 0. \tag{4}$$

A quasi-strongly p-harmonic map is not in general a p-harmonic map. This is the reason that we used the terminology quasi-strongly rather than strongly. It follows from Definition 2.1 that a map $\phi : (M^n, g) \to (N^m, h)$ is a harmonic map if and only if it is a quasi-strongly 2-harmonic map (or just quasi-strongly harmonic map). Note that it follows from the ellipticity of the Laplacian Δ, a quasi-strongly p-harmonic map becomes smooth. As an immediate consequence we have the following lemma from equation (2).

Lemma 2.1. *If $\phi : (M^n, g) \to (N^m, h)$ is a p-harmonic map satisfying*

$$\mathrm{d}(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi) = 0,$$

then ϕ is quasi-strongly p-harmonic.

Corollary 2.1. *If $\phi : (M^n, g) \to (N^m, h)$ is a p-harmonic map such that $|\mathrm{d}\phi|$ is constant, then ϕ is quasi-strongly p-harmonic.*

Proof. It follows from Lemma 2.1 and the fact $\mathrm{d} \circ \mathrm{d} = 0$. □

Theorem 2.1. *Let $\phi : (M^n, g) \to (N^m, h)$ be a quasi-strongly p-harmonic map from a complete Riemannian manifold M into a Riemannian manifold N. If $E_{2p-2}(\phi) < \infty$, then ϕ is a p-harmonic map and $\mathrm{d}(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi) = 0$.*

Proof. If M is compact, then it follows from integration by parts. Thus we may assume M is complete and noncompact. For $\varepsilon > 0$, from completeness of M, one can choose a cut-off function $\eta = \eta_\varepsilon$ such that

(i) $0 \le \eta \le 1$, $\operatorname{supp}(\eta)$ is compact.
(ii) $\eta^{-1}(1)$ exhausts M as $\varepsilon \to 0$.
(iii) $|\mathrm{d}\eta|^2 \le \varepsilon^2\eta$.

One obtains from integration by parts

$$\begin{aligned}
0 &= \int_M \langle \Delta(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi), \eta|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi\rangle \\
&= \int_M \langle \delta(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi), \delta(\eta|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)\rangle + \int_M \langle \mathrm{d}(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi), \mathrm{d}(\eta|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)\rangle \\
&= \int_M \eta\left[|\delta(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)|^2 + |\mathrm{d}(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)|^2\right] \\
&\quad + \int_M \langle \delta(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi), |\mathrm{d}\phi|^{p-2}\mathrm{d}\phi(\mathrm{d}\eta)\rangle + \langle \mathrm{d}(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi), \mathrm{d}\eta \wedge |\mathrm{d}\phi|^{p-2}\mathrm{d}\phi\rangle \\
&:= I_1(\varepsilon) + I_2(\varepsilon).
\end{aligned}$$

Now using the Hölder inequality, one obtains

$$\begin{aligned}
|I_2(\varepsilon)| &\le \int_M |\mathrm{d}\eta||\mathrm{d}\phi|^{p-1}\left[|\delta(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)| + |\mathrm{d}(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)|\right] \\
&\le \left(\int_M |\mathrm{d}\phi|^{2p-2}\right)^{1/2}\left(\int_M |\mathrm{d}\eta|^2\left(|\delta(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)| + |\mathrm{d}(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)|\right)^2\right)^{1/2} \\
&\le 2\varepsilon C\left(\int_M \eta\left[|\delta(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)|^2 + |\mathrm{d}(|\mathrm{d}\phi|^{p-2}\mathrm{d}\phi)|^2\right]\right)^{1/2} \\
&= 2\varepsilon C I_1(\varepsilon)^{1/2},
\end{aligned}$$

where $C = E_{2p-2}(\phi)^{1/2}$. In the last inequality, we used (iii) and the inequality

$$(a+b)^2 \leq 2(a^2+b^2).$$

Thus

$$0 = I_1 + I_2 \geq I_1 - 2\varepsilon C I_1^{1/2}.$$

Consequently, one has

$$I_1(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and so

$$\delta(|\,\mathrm{d}\,\phi|^{p-2}\,\mathrm{d}\,\phi) = 0 = \mathrm{d}(|\,\mathrm{d}\,\phi|^{p-2}\,\mathrm{d}\,\phi). \qquad \square$$

Corollary 2.2. *Let (M,g) be a complete noncompact Riemannian manifold of* $\mathrm{Ric}(M) \geq 0$ *and (N,h) a Riemannian manifold of nonpositive sectional curvature. Let $\phi : (M^n, g) \to (N^m, h)$ be a quasi-strongly p-harmonic map. If $E_{2p-2}(\phi) < \infty$, then ϕ is a constant map.*

Proof. It follows from Theorem 2.1 and Takeuchi [6, Main Theorem]. □

Corollary 2.3 (Schoen & Yau [5]). *Let (M,g) be a complete noncompact Riemannian manifold of* $\mathrm{Ric}(M) \geq 0$ *and (N,h) a Riemannian manifold of nonpositive sectional curvature. Let $\phi : (M^n, g) \to (N^m, h)$ be a harmonic map. If $E(\phi) < \infty$, then ϕ is a constant map.*

Proof. Recall that a quasi-strongly 2-harmonic map is a just harmonic map. □

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