한국 퍼지 및 지능시스템 학회 2002, Vol. 12, No. 6, pp. 565-570

Fuzzy H-continuous Mappings and Fuzzy Strongly Closed Graphs

K.Hur [a], Y.S.Ahn [b], and J.H. Ryou [c]

[a] Division of Mathematics and Informational Statistics
Wonkwang University Iksan, Chonbuk, Korea 579-792
E-mail:kulhur@wonkwang.ac.kr
[b] Dept. of Computer Information Science
Dongkang College Kwangju, Korea 500-714
E-mail:youngsin@dongkang.ac.kr
[c] Division of Mathematics and Informational Statistics
Wonkwang University Iksan, Chonbuk, Korea 579-792
E-mail:donggni@hanmail.net

Abstract

We introduce the concepts of fuzzy H-continuity and fuzzy strongly closed graph, respectively and investigate some of their properties.

Key words and phrases : fuzzy almost continuous, fuzzy T_{2w}, fuzzy H-closed set, fuzzy H-continuous, fuzzy strongly closed graph, fuzzy closed graph.

1. Introduction and preliminaries

Since the introduction of fuzzy sets by Zadeh in his classic paper[17] of 1965 and fuzzy topological spaces by Chang[3] in 1968, certain mappings between topological spaces, weaker than usual open or continuous ones, have been generalized to and studied in fuzzy topological spaces by different authors[1, 2, 5, 6, 11, 13, 14, 16]. In this paper, we extend the notion of H-continuity introduced by P.E.Long and T.R. Hamlett[9] to fuzzy topological spaces.

In order to make the exposition self-contained as far as practicable, we list some definitions and results that will be used in the sequel. Let X be a non-empty(ordinary) set and let I the unit interval $[0,1]$. A *fuzzy set* A in X is a mapping from X into I[17]. A *fuzzy point* x_λ in X is a fuzzy set in X defined by : for each $y \in X$,

$$x_\lambda(y) = \begin{cases} \lambda, & \text{if } y = x, \\ 0, & \text{if } y \neq x, \end{cases}$$

where $x \in X$ and $\lambda \in (0,1]$ are respectively called the *support* and the *value* of x_λ[12,14]. A fuzzy point x_λ is said to *belong to* a fuzzy set A in X iff $\lambda \le A(x)$[12]. A fuzzy set A in X is the union of all fuzzy points which belong to A[12]. A subfamily T of I^X is called a *fuzzy topology* on X[3] if (i) $\emptyset, X \in T$, (ii) for any $\{U_\alpha\}_{\alpha \in \Lambda} \subset T$, $\bigcup_{\alpha \in \Lambda} U_\alpha \in T$ and (iii) for any $A, B \in T$, $A \cap B \in T$. In this case, each member of T is called a *fuzzy open*(in short, *F-open*) set in X and its complement a *fuzzy closed*(in short, *F-closed*) set in X. The pair (X, T) is called a *fuzzy topological space*(in short, *fts*). For a fts X, $FO(X)$ and $FC(X)$ denote the collection of all F-open sets and F-closed sets in X, respectively. For a fuzzy set A in a fts X, the closure $\mathrm{cl}A$ and the interior $\mathrm{int}A$ of A are defined respectively as $\mathrm{cl}A = \bigcap\{V \in I^X : A \subset V$ and $V^c \in FO(X)\}$ and $\mathrm{int}A = \bigcup\{V \in FO(X) : V \subset A\}$[12]. We will use the notion of fuzzy compactness in the sense of S. Ganguly and S. Saha[7].

Definition 1.1[17]. Let f be a mapping from a set X into a Y. $A \in I^X$ and $B \in I^Y$. Then :

(i) The *image* of A under f, $f(A)$ is a fuzzy set in Y defined by for each $y \in Y$,

$$[f(A)](y) = \sup_{x \in f^{-1}(y)} A(x) \quad \text{if } f^{-1}(y) \neq \emptyset,$$
$$= 0 \quad \text{otherwise},$$

where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

(ii) The *inverse image* of B under f, $f^{-1}(B)$ is a fuzzy set in X denoted by for each $x \in X$,

접수일자 : 2002년 3월 5일
완료일자 : 2002년 9월 16일
[a] Corresponding author
2000 Mathematies subject classification of AMS: 54A40, 54B20, 54C60
This paper was supported by Wonkwang University in 2002

$$f^{-1}(B)(x) = B(f(x)).$$

Result 1.A[3,16]. Let $f: X \to Y$ be a mapping. Then :

(1) $f^{-1}(B^c) = [f^{-1}(B)]^c$ for each $B \in I^Y$.
(2) $[f(A)]^c \subset f(A^c)$ for each $A \in I^X$.
(3) If $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in I^Y$.
(4) If $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$, where $A_1, A_2 \in I^X$.
(5) $f(f^{-1}(B)) \subset B$ for each $B \in I^Y$.
In particular, if f is surjective, then $f(f^{-1}(B)) = B$ for each $B \in I^Y$.
(6) $A \subset f^{-1}(f(A))$ for each $A \in I^X$.
In particular, if f is injective, then $f^{-1}(f(A)) = A$ for each $A \in I^X$.
(7) If $\{B_\alpha\}_{\alpha \in \Lambda} \subset I^Y$, then
$$f^{-1}(\cup_{\alpha \in \Lambda} B_\alpha) = \cup_{\alpha \in \Lambda} f^{-1}(B_\alpha)$$
and
$$f^{-1}(\cap_{\alpha \in \Lambda} B_\alpha) = \cap_{\alpha \in \Lambda} f^{-1}(B_\alpha).$$
(8) If $\{A_\alpha\}_{\alpha \in \Lambda} \subset I^X$, then
$$f(\cup_{\alpha \in \Lambda} A_\alpha) = \cup_{\alpha \in \Lambda} f(A_\alpha).$$
(9) Let $g: Y \to Z$ be a mapping.
Then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ for each $C \in I^Z$.

Result 1.B[4]. Let $f: X \to Y$ be a mapping. Then :

(1) $f(x_\lambda) = [f(x)]_\lambda$ for each $x_\lambda \in F_p(X)$.
(2) If $A \in I^X$ and $x_\lambda \in A$, then $f(x_\lambda) \in f(A)$.
(3) If $A \in I^X$ and $y_\lambda \in f(A)$, then there exists $x \in X$ such that $f(x) = y$ and $x_\lambda \in A$.
(4) If $B \in I^Y$, $y \in f(X)$ and $y_\lambda \in B$, then for each $x \in f^{-1}(y)$, $x_\lambda \in f^{-1}(B)$.
(5) If $B \in I^Y$ and $x_\lambda \in f^{-1}(B)$, then $[f(x)]_\lambda \in B$.

Definition 1.2[12]. Let (X, T) be a fts and let Y a crisp subset of X. Then the family $T_Y = \{A|_Y : A \in T\}$ is a fuzzy topology on Y. In this case, T_Y is called the *fuzzy relative topology* or *fuzzy subspace topology* of T to Y and the pair (Y, T_Y) is called a fuzzy subspace of (X, T).

It is clear that $A|_Y = A \cap Y$.

Definition 1.3[8]. Two fuzzy sets A and B in a set X are said to be *disjoint* if $A \odot B = \emptyset$, where $(A \odot B)(x) = \max[0, A(x) + B(x) - 1]$ for each $x \in X$.

It is clear that $A \odot B = \emptyset$ if and only if $A \bar{q} B$, i.e., $A \subset B^c$.

Definition 1.4[5]. A fts X is said to be *fuzzy* T_{2w}(in short, FT_{2w}) if for any two distinct fuzzy points x_λ and y_μ in X, there exist $U, V \in FO(X)$ such that $x_\lambda \in U$, $y_\mu \in V$ and $U \odot V = \emptyset$.

Definition 1.5[12,14]. Let $\{(X_\alpha, T_\alpha) : \alpha \in \Lambda\}$ be a family of fts's, let $X = \prod_{\alpha \in \Lambda} X_\alpha$ the usual Cartesian product of $\{X_\alpha\}_{\alpha \in \Lambda}$ and let π_α the projection form X onto X_α for each $\alpha \in \Lambda$. Let $\mathcal{P}_f(\Lambda)$ be the family of all finite subsets of Λ, let $\mathcal{S} = \{\pi_\alpha^{-1}(B) : B \in T_\alpha, \alpha \in \Lambda\}$ and let $\mathcal{B} = \{\cap_{\alpha \in F} \pi_\alpha^{-1}(U_\alpha) : U_\alpha \in T_\alpha, F \in \mathcal{P}_f(\Lambda)\}$. Then there exists a unique fuzzy topology T on X for which $\mathcal{B}$ is a base for T and $\mathcal{S}$ is a subbase for T. In fact, T is the family of all unions of members of $\mathcal{B}$.

In this case, T is called the *fuzzy product topology* on X and the pair (X, T) is called the *fuzzy product topological space*(in short, *product fts*).

Definition 1.6[2]. Let A be a fuzzy set in a fts X. Then :

(1) A is called a *fuzzy regular open set in* X if $A = \text{int}(\text{cl} A)$.
(2) A is called a *fuzzy regular closed set in* X if $A = \text{cl}(\text{int} A)$.

We denote the collection of all fuzzy regular open[resp. closed] set in X as $FRO(X)$ [resp. $FRC(X)$].

It is clear that $FRO(X) \subset FO(X)$ and $FRC(X) \subset FC(X)$.

Result 1.C[1, Lemma 3.1]. Let A be a fuzzy set in a fts X. Then :

(1) $\text{int}(\text{cl} A) \in FRO(X)$.
(2) $\text{cl}(\text{int} A) \in FRC(X)$.

Definition 1.7[2]. Let X and Y be fts's. Then a mapping $f: X \to Y$ is said to be *fuzzy almost continuous*(in short, *fal-continuous*) if for each $V \in FRO(Y)$, $f^{-1}(V) \in FO(X)$.

It is clear that every F-continuous mapping is fal-continuous.

Result 1.D[11, Theorem 3.3]. Let X and Y be fts's. Then a mapping $f: X \to Y$ is fal-continuous if and only if for each $x_\lambda \in F_p(X)$ and each $V \in FO(Y)$ with $f(x_\lambda) \in V$, there exists $U \in FO(X)$ such that $x_\lambda \in U$ and $f(U) \subset \text{int}(\text{cl} V)$.

2. Fuzzy H-continuous mappings

From now on, we consider X, Y, Z as fts's.

Definition 2.1. Let $A \in I^X$. Then A is said to be *fuzzy H-closed relative to* X (in short, *fH-closed*) if for each F-open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of A in X, there exists a finite subfamily Λ_0 of Λ such that $A \subset \cup_{\alpha \in \Lambda_0}(\text{cl} V_\alpha)$. The fts X is said to be a *fH-closed space* if for each

F-open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X, there exists a finite subfamily Λ_0 of Λ such that $\cup_{\alpha\in\Lambda_0}(\text{cl}\, V_\alpha)=X$.

Lemma 2.2. Let X be a FT_{2w}-space. If B is fH-closed in X, then $B\in FC(X)$.

Proof. Assume that $\text{cl}B\not\subset B$. Then there exists an $x_\lambda\in F_p(X)$ such that $x_\lambda\in \text{cl}\,B$ but $x_\lambda\not\in B$. Let $y_\mu\in B$. Since X is FT_{2w}, there exist U_{y_μ} and $V_{y_\mu}\in FO(X)$ such that $x_\lambda\in U_{y_\mu}$, $y_\mu\in V_{y_\mu}$ and $U_{y_\mu}\odot V_{y_\mu}=\emptyset$. Let $\mathcal{V}=\{V_{y_\mu}\in FO(X): y_\mu\in B\}$. Then clearly $\mathcal{V}$ is a fuzzy open cover of B. Since B is fH-closed in X, there exist $y_1,\cdots,y_n\in B$ such that $B\subset(\text{cl}\,V_{y_1}\cup,\cdots,\cup\,\text{cl}\,V_{y_n})=\text{cl}\,(\cup_{i=1}^n V_{y_i})$, where $\{V_{y_1},\cdots,V_{y_n}\}\subset\mathcal{V}$. For each $i=1,\cdots,n$, let $U_{y_i}\in FO(X)$ such that $x_\lambda\in U_{y_i}$ and $U_{y_i}\odot V_{y_i}=\emptyset$ i.e., $V_{y_i}\subset U^c_{y_i}$. Then $B\subset \text{cl}\,(\cup_{i=1}^n V_{y_i})\subset \text{cl}\,(\cup_{i=1}^n U^c_{y_i}) = \text{cl}\,(\cap_{i=1}^n U_{y_i})^c$. Thus $\text{cl}B\subset \text{cl}\,(\cap_{i=1}^n U_{y_i})^c$. Since $x_\lambda\in \text{cl}B$, $x_\lambda\in \text{cl}\,(\cap_{i=1}^n U_{y_i})^c$. But $x_\lambda\not\in \text{cl}\,(\cap_{i=1}^n U_{y_i})^c$. This is a contradiction. So $\text{cl}B\subset B$. Hence $B\in FC(X)$.

Definition 2.3. A mapping $f: X\to Y$ is said to be *fuzzy H-continuous*(in short, *fH-continuous*) if for each $x_\lambda\in F_p(X)$ and each $V\in FO(Y)$ such that $f(x_\lambda)\in V$ and V^c is fH-closed in Y, there exists $U\in FO(X)$ such that $x_\lambda\in U$ and $f(U)\subset V$.

Theorem 2.4. Let $f: X\to Y$ be a mapping. Then the following are equivalent:

(1) f is fH-continuous.

(2) If $V\in FO(Y)$ and V^c is fH-closed in Y, then $f^{-1}(V)\in FO(X)$.

These statements are implied by

(3) If B is fH-closed in Y, then $f^{-1}(B)\in FC(X)$.

Furthermore, if Y is FT_{2w}, then all three statements are equivalent.

Proof. (1)⇒(2): Suppose $f: X\to Y$ is fH-continuous. Let $V\in FO(Y)$ and let V^c be fH-closed in Y. Let $x_\lambda\in f^{-1}(V)$. Then $f(x_\lambda)\in V$. By the hypothesis, there exists $U\in FO(X)$ such that $x_\lambda\in U$ and $f(U)\subset V$. Thus $x_\lambda\in U\subset f^{-1}(V)$. Hence, by Proposition 1.8 in [10], $f^{-1}(V)\in FO(X)$.

(2)⇒(1): Suppose the condition (2) holds. Let $x_\lambda\in F_p(X)$ and let $V\in FO(Y)$ such that $f(x_\lambda)\in V$ and V^c is fH-closed in Y. Then, by the hypothesis, $f^{-1}(V)\in FO(X)$ and $x_\lambda\in f^{-1}(V)$. Let $U=f^{-1}(V)$. Then clearly $x_\lambda\in U\in FO(X)$ and $f(U)\subset V$. Hence f is fH-continuous.

(3)⇒(2): Suppose the condition (3) holds. Let $V\in FO(Y)$ and let V^c be fH-closed in Y. Then, by the hypothesis, $f^{-1}(V^c)=[f^{-1}(V)]^c\in FC(X)$. Hence $f^{-1}(V)\in FO(X)$.

Now assume that Y is FT_{2w} and we show that (2) implies (3). Let B be any fH-closed set in Y. Since Y is FT_{2w}, by Lemma 2.2, $B\in FC(Y)$. Then $B^c\in FO(Y)$. Thus, by the condition (2), $f^{-1}(B^c)=[f^{-1}(B)]^c\in FO(X)$. Hence $f^{-1}(B)\in FC(X)$.

In general, the condition (2) does not imply the condition (3) as shown in Example.

Example 2.5. Let $X=\{a,b,c\}$ and let $Y=\{x,y\}$. Consider the fuzzy topologies T_X and T_Y on X and Y, respectively defined by :

$$T_X=\{\emptyset, X, \{(a,0.2),(b,0.2),(c,0.3)\}\}$$

and

$$T_Y=\{\emptyset, Y, \{(x,0.2),(y,0.3)\}\}.$$

Let $f: (X, T_X)\to(Y,T_Y)$ be the mapping defined by $f(a)=f(b)=x$ and $f(c)=y$. Let $V=\{(x,0.2),(y,0.3)\}$. Then clearly $V\in FO(Y)$ and V^c is fH-closed in Y. Moreover, $f^{-1}(V)\in FO(X)$. So the condition (2) holds.

On the other hand, let $B=\{(x,0.2),(y,0.3)\}$. Then B is fH-closed in Y but $f^{-1}(B)\not\in FC(X)$. Hence the condition (3) does not hold.

Theorem 2.6. A mapping $f: X\to Y$ is fH-continuous if and only if for each F-closed fH-closed B in Y, $f^{-1}(B)\in FC(X)$.

Proof. (⇒): Suppose f is fH-continuous. Let B be any F-closed fH-closed set in Y. Then $B^c\in FO(Y)$ and $(B^c)^c=B$ is fH-closed in Y. By Theorem 2.4, $f^{-1}(B^c)=[f^{-1}(B)]^c\in FO(X)$. Hence $f^{-1}(B)\in FC(X)$.

(⇐): Suppose the necessary condition holds. Let $V\in FO(Y)$ such that V^c is fH-closed in Y. Then clearly V^c is F-closed fH-closed in Y. By the hypothesis, $f^{-1}(V^c)=[f^{-1}(V)]^c\in FC(X)$. Thus $f^{-1}(V)\in FO(X)$. Hence, by Theorem 2.4, f is fH-continuous.

Lemma 2.7. If A and B are fH-closed in X, then $A\cup B$ is fH-closed in X.

Proof. Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a F-open cover of $A\cup B$ in X. Since $A\subset A\cup B$ and $B\subset A\cup B$, $\{V_\alpha\}_{\alpha\in\Lambda}$ is a F-open cover of A and B in X, respectively. Since A and B are fH-closed in X, there exist finite collections Λ_1 and Λ_2 of Λ such that $A\subset\cup_{\alpha\in\Lambda_1}(\text{cl}\,V_\alpha)$ and $B\subset\cup_{\beta\in\Lambda_2}(\text{cl}\,V_\beta)$. Then $A\cup B\subset\cup_{(\alpha,\beta)\in\Lambda_1\times\Lambda_2}\text{cl}$

$(V_\alpha \cup V_\beta)$ and $\Lambda_1 \times \Lambda_2$ is finite. Hence $A \cup B$ is fH-closed in X.

Lemma 2.8. Let (X, T) be a fts and let $\mathcal{B}^* = \{V \in T : V^c$ is fH-closed in $X\}$. Then there exists a unique fuzzy topology T^* on X for which $\mathcal{B}^*$ is a fuzzy base for T^*.

Furthermore $T^* \subset T$ and (X, T^*) is always a fuzzy H-closed space. In this case, (X, T^*) is called the *induced H-closed fuzzy topological space by* (X, T) and will be denoted by X^*.

Proof. Clearly $X \in \mathcal{B}^*$. So $\cup \mathcal{B}^* = X$. Let $B_1, B_2 \in \mathcal{B}^*$ and let $x_\lambda \in B_1 \cap B_2$. Then $B_1, B_2 \in T$, B_1^c and B_2^c are fH-closed in X. Thus, by Lemma 2.7, $B_1^c \cup B_2^c$ is fH-closed in X. Since $B_1^c \cup B_2^c = (B_1 \cap B_2)^c$, $(B_1 \cap B_2)^c$ is fH-closed in X. Moreover $B_1 \cap B_2 \in T$. Then $B_1 \cap B_2 \in \mathcal{B}^*$. Hence this completes the proof.

Result 2.A[8,Lemma 3.7]. Let (X, T) be a fts and let $\mathcal{B}_* = \{V \in T : V^c$ is F- compact in $X\}$. Then there exists a unique fuzzy topology T_* on X for which $\mathcal{B}_*$ is a fuzzy base for T_*.

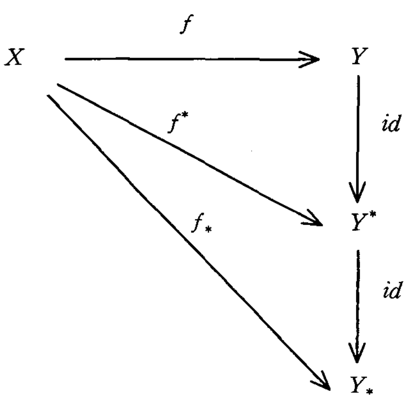
Furthermore $T_* \subset T$ and (X, T_*) is always a compact fuzzy topological space. In this case, (X, T_*) is called the *induced compact fuzzy topological space by* (X, T) and will be denoted by X_*.

It is clear that $T_* \subset T^* \subset T$.

Proposition 2.9. Let $f: X \to Y$ a mapping. Define the mappings $f^*: X \to Y^*$ and $f_*: X \to Y_*$ as follows, respectively : for each $x \in X$,

$$f^*(x) = f(x) \text{ and } f_*(x) = f(x).$$

Consider the following diagram :



Then the following hold.

(1) f is fH-continuous if and only if f^* is F-continuous.

(2) [8,Proposition 3.8] f is fc-continuous if and only if f_* is F-continuous.

(3) $id: Y \to Y^*$ and $id: Y^* \to Y_*$ are F-continuous.

(4) $id^{-1}: Y^* \to Y$ is fH-continuous.

(5) $id^{-1}: Y_* \to Y^*$ is fc-continuous.

Theorem 2.10. Let $f: X \to Y$ be fH-continuous. If $f^*: X \to Y^*$ is F-closed(resp. F-open), then f is F-closed(resp. F-open).

Proof. Suppose $f^*: X \to Y^*$ is F-closed(resp. F-open). Let $F \in FC(X)$ (resp. $F \in FO(X)$). Then, by the hypothesis, $f^*(F)$ is F-closed(resp. F-open) in Y^*. By Proposition 2.9(3), $id: Y \to Y^*$ is F-continuous. Then $id^{-1}(f^*(F))$ is F-closed(resp. F-open) in Y. But $id^{-1}(f^*(F)) = f(F)$. So $f(F)$ is F-closed(resp. F-open) in Y. Hence f is F-closed(resp. F-open).

Theorem 2.11. If $f: X \to Y$ is fH-continuous and $A \in P(X)$, then $f|_A: A \to Y$ is fH-continuous.

Proof. Let $V \in FO(Y)$ such that V^c is fH-closed in Y. Then, by Theorem 2.4(2), $f^{-1}(V) \in FO(X)$. Thus $(f|_A)^{-1}(V) = f^{-1}(V) \cap A \in FO(A)$. Hence $f|_A$ is fH-continuous.

Theorem 2.12. If $f: X \to Y$ is F-continuous and $g: Y \to Z$ is fH-continuous, then $g \circ f: X \to Z$ is fH-continuous.

Proof. Let $W \in FO(Z)$ such that W^c is fH-closed in Z. Then, by Theorem 2.4(2), $g^{-1}(W) \in FO(Y)$. Since f is F-continuous, $f^{-1}(g^{-1}(W)) \in FO(X)$. But $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Thus $(g \circ f)^{-1}(W) \in FO(X)$. Hence, by Theorem 2.4(2), $g \circ f$ is fH-continuous.

3. Fuzzy strongly closed graphs

Let $f: X \to Y$ be a mapping. Then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the Cartesian product $X \times Y$ is called the *graph of* f.

Definition 3.1. Let X and Y be fts's. Then a mapping $f: X \to Y$ is said to *have a fuzzy strongly closed graph*(in short, *F-strongly closed graph*) or the graph $G(f)$ is said to be *fuzzy strongly closed* (in short, *F-strongly closed*) in $X \times Y$ if for each $(x_\lambda, y_\mu) \not\in F_p(G(f))$, there exist $U \in FO(X)$ and $V \in FO(Y)$ such that $x_\lambda \in U$, $y_\mu \in V$ and $(U \times \text{cl}\, V) \odot G(f) = \emptyset$.

Definition 3.2. Let X and Y be fts's and let $f: X \to Y$ be a mapping. Then the graph $G(f)$ of f is said to *have an upper fuzzy point* in $X \times Y$ provided that for each $(x_\lambda, y_\mu) \not\in F_p(G(f))$, there exist $U \in FO(X)$ and $V \in FO(Y)$ such that $x_\lambda \in U$, $y_\mu \in V$ and if $(U \times \text{cl}\, V) \odot G(f) \neq \emptyset$, then there exists

$(a, b) \in G(f)$ such that $(U \times \mathrm{cl}\, V)(a, b) > \frac{1}{2}$.

Lemma 3.3. Let X and Y be fts's, let $f: X \to Y$ a mapping and let $G(f)$ have an upper fuzzy point in $X \times Y$. Then f has a F-strongly closed graph if and only if for each $x_\lambda \in F_p(X)$ and each $y_\mu \in F_p(Y)$ such that $y \neq f(x)$, there exist $U \in FO(X)$ and $V \in FO(Y)$ such that $x_\lambda \in U$, $y_\mu \in V$ and $f(U) \odot \mathrm{cl}\, V = \emptyset$.

Proof. ($\Rightarrow$): Suppose f has a F-strongly closed graph. Let $x_\lambda \in F_p(X)$ and let $y_\mu \in F_p(Y)$ such that $y \neq f(x)$. Then clearly $(x_\lambda, y_\mu) \notin F_p(G(f))$. By the hypothesis, there exist $U \in FO(X)$ and $V \in FO(Y)$ such that $x_\lambda \in U$, $y_\mu \in V$ and $(U \times \mathrm{cl}\, V) \odot G(f) = \emptyset$. Assume that $f(U) \odot \mathrm{cl}\, V \neq \emptyset$. Then $f(U) q (\mathrm{cl}\, V)$. Thus there exists $b \in Y$ such that $f(U)(b) + \mathrm{cl}\, V(b) > 1$, i.e., $\sup_{z \in f^{-1}(b)} U(z) + \mathrm{cl}\, V(b) > 1$. So there exists an $a \in X$ such that $b = f(a)$ and $U(a) + \mathrm{cl}\, V(b) > 1$, i.e., $(U \times \mathrm{cl}\, V)(a, b) > 0$. Since $(U \times \mathrm{cl}\, V) \odot G(f) = \emptyset$ and $(a, b) \in G(f)$, $(U \times \mathrm{cl}\, V)(a, b) = 0$. This is a contradiction. Hence $f(U) \odot \mathrm{cl}\, V = \emptyset$.

($\Leftarrow$): Suppose the necessary condition holds. Let $(x_\lambda, y_\mu) \notin F_p(G(f))$. Then clearly $x_\lambda \in F_p(X)$, $y_\mu \in F_p(Y)$ and $y \neq f(x)$. By the hypothesis, there exist $U \in FO(X)$ and $V \in FO(Y)$ such that $x_\lambda \in U$, $y_\mu \in V$ and $f(U) \odot \mathrm{cl}\, V = \emptyset$. Assume that $(U \times \mathrm{cl}\, V) \odot G(f) \neq \emptyset$. Since $G(f)$ has an upper fuzzy point in $X \times Y$, there exists $(a, b) \in G(f)$ such that $(U \times \mathrm{cl}\, V)(a, b) > \frac{1}{2}$, i.e., $\min[U(a), \mathrm{cl}\, V(b)] > \frac{1}{2}$. Then :

$$\begin{aligned} f(U)(b) + \mathrm{cl}\, V(b) - 1 = & \sup_{z \in f^{-1}(b)} U(z) \\ & + \mathrm{cl}\, V(b) - 1 \\ & \geq U(a) + \mathrm{cl}\, V(b) - 1 \\ & \geq 2 \min[U(a), \\ & \mathrm{cl}\, V(b)] - 1 \\ & > 0. \end{aligned}$$

Thus $f(U) \odot \mathrm{cl}\, V \neq \emptyset$. This is a contradiction. So $(U \times \mathrm{cl}\, V) \odot G(f) = \emptyset$. Hence f has a F-strongly closed graph.

Remark 3.4. If $f: X \to Y$ has a F-strongly closed graph, then for each $x_\lambda \in F_p(X)$ and each $y_\mu \in F_p(Y)$ such that $y \neq f(x)$, there exist $U \in FO(X)$ and $V \in FO(Y)$ such that $x_\lambda \in U$, $y_\mu \in V$ and $f(U) \odot \mathrm{cl}\, V = \emptyset$.

Theorem 3.5. Let $f: X \to Y$ be fal-continuous and let $G(f)$ *have an upper fuzzy point in* $X \times Y$. If Y is FT_{2w}, then f has a F-strongly closed graph.

Proof.} Let $x_\lambda \in F_p(X)$ and let $y_\mu \in F_p(Y)$ such that $y \neq f(x)$. Then $y_\mu \neq f(x_\lambda)$. Since Y is FT_{2w}, there exists $V \in FO(X)$ such that $y_\mu \in V$ and $f(x_\lambda) \in (\mathrm{cl}\, V)^c$. Since $\mathrm{cl}\, V \in FRC(Y)$, $(\mathrm{cl}\, V)^c \in FRO(Y)$. Since f is fal-continuous, there exists $U \in FO(X)$ such that $x_\lambda \in U$ and $f(U) \subset (\mathrm{cl}\, V)^c$. Then $f(U) \odot \mathrm{cl}\, V = \emptyset$. Hence, by Lemma 3.3, f has a F-strongly closed graph.

Corollary 3.5. Let $f: X \to Y$ be F-continuous and let $G(f)$ have an upper fuzzy point in $X \times Y$. If Y is FT_{2w}, then f has a F-strongly closed graph.

Theorem 3.6. Let $f: X \to Y$ be a surjection with F-strongly closed graph. Then Y is FT_{2w}.

Proof. Let y_λ and z_μ be distinct fuzzy point in Y. Since f is surjective, there exists $x \in X$ such that $y_\lambda = f(x_\lambda)$. Then $(x_\lambda, z_\mu) \notin F_p(G(f))$. Since f has a F-strongly closed graph, by Remark 3.4, there exist $U \in FO(X)$ and $V \in FO(Y)$ such that $x_\lambda \in U$, $z_\mu \in V$ and $f(U) \odot \mathrm{cl}\, V = \emptyset$. Thus $f(x_\lambda) = y_\lambda \in (\mathrm{cl}\, V)^c$. Hence Y is FT_{2w}.

The following is the immediate result of Theorem 3.6 and Corollary 3.5.

Theorem 3.7. A fts X is FT_{2w} if and only if the identity mapping $id: X \to X$ has a F-strongly closed graph.

Theorem 3.8. If a mapping $f: X \to Y$ has a F-strongly closed graph, then it is fH-continuous.

Proof. Let K be any fH-closed set in Y and let $x_\lambda \in [f^{-1}(K)]^c$. Let $y_\mu \in K$. Then clearly $(x_\lambda, y_\mu) \notin F_p(G(f))$. Since f has a F-strongly closed graph, by Lemma 3.3, there exist $U_{y_\mu} \in FO(X)$ and $V_{y_\mu} \in FO(Y)$ such that $x_\lambda \in U_{y_\mu}$, $y_\mu \in V_{y_\mu}$ and $f(U_{y_\mu}) \odot \mathrm{cl}\, V_{y_\mu} = \emptyset$. Consider the family $\{V_{y_\mu} \in FO(Y): y_\mu \in K\}$. Then clearly $\{V_{y_\mu} \in FO(Y): y_\mu \in K\}$ is an F-open cover of K in Y. Since K is fH-closed in Y, there exists a finite subset K_0 of K such that $K \subset \bigcup_{y_\mu \in K_0} (\mathrm{cl}\, V_{y_\mu})$. Let $U = \bigcap \{U_{y_\mu} \in FO(X): x_\lambda \in U_{y_\mu}$ and $y_\mu \in K_0\}$. Then $x_\lambda \in U \in FO(X)$ and $U \odot f^{-1}(K) = \emptyset$, i.e., $U \subset [f^{-1}(K)]^c$. Thus $[f^{-1}(K)]^c \in FO(X)$. So $f^{-1}(K) \in FC(X)$. Hence, by Theorem 2.4, f is fH-continuous.

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저 자 소 개

K.Hur [a]

제 8권 6호 참조

Y.S.Ahn [b]

제 11권 2호 참조

J.H. Ryou [c]

제 8권 6호 참조