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풀흐름라인에서 변동성전파원리에 대한 증명 : 존재와 측정

최 상 웅*

Proof of the Variability Propagation Principle in a Pull Serial Line : Existence and Measurement

Sang-Woong Choe*

■ Abstract ■

In this study, we consider infinite supply of raw materials and backlogged demands as given two boundary conditions. And we need not make any specific assumptions about the inter-arrival of external demand and service time distributions. Under these situations, the ultimate objective of this study is to prove the variability propagation principle in a pull serial line and is to measure it in terms of the first two moments of the inter-departure process subject to number of cards in each cell. Two preparations are required to achieve this objective : The one is to derive a true lower bound of variance of the inter-departure process. The other is to establish a constrained discrete minimax problem for the no backorder (backlogging) probabilities in each cell. We may get some fundamental results necessary to a completion for the proof through the necessary and sufficient conditions for existence of optimal solution of a constrained discrete minimax problem and the implicit function theorem. Finally, we propose a numeric model to measure the variability propagation principle. Numeric examples show the validity and applicability of our study.

Keyword : Lower Bound of Variance, Constrained Discrete Minimax Problem

2002년도 추계 한국경영과학회 학술대회(2002년 10월 19일) 최우수논문상(이론부문) 수상논문으로 소정의 심사 과정을 거쳐 게재추천되었음.

* School of e-Business and Tourism, Pohang College

1. Introduction

In the last two decades, there has been considerable interest in the study and analysis of the pull systems. The models used include analytic approaches as well as simulation approaches. Analytic solutions exist almost exclusively for the pull serial lines with deterministic or exponentially distributed times (see e.g. Bardinelli, 1992 ; Bitran *et al.*, 1987 ; Buzacott, 1989 ; Deleersnyder *et al.*, 1989 ; Kim, 1985 ; Mitra *et al.*, 1990, 1991 ; Spearman, 1992 ; Tayur, 1993). On the other hand, more complex systems are investigated by simulation(see e.g. Aytug *et al.*, 1998 ; Hum *et al.*, 1988 ; Blair *et al.*, 1991 ; Huang *et al.*, 1983 ; Sarker *et al.*, 1988, 1989 ; Philipoom *et al.*, 1987). Simulation by itself can not solve any optimization problem. They have investigated important steady state performance measures such as throughput, average WIP and average flow time under the ergodicity. The majority of pull researches has treated the performance analysis problem. And they have adopted the strong assumption such as the infinite supply of raw material and the infinite external demand process.

This study deals with derivation of the unique fundamental structural property in a pull serial line, which implies the variability propagation principle (abbreviated VPP). And we consider infinite supply of raw materials and backlogged demands as given two boundary conditions. Furthermore, we need not make any specific assumptions about the inter-arrival of external demand and service time distributions.

In a push type ordering system, there is one fundamental principle that seems remarkably robust in explaining performances of stages in series : the VPP. Suresh and Whitt (1990) have not defined but described the VPP in the ***open*** tandem queues (stages) with ***infinite*** buffer capacities as the following statement :

"*Increased variability in the arrival process or the service times of a queue (stage) tends to propagate to the departure process from that queue (stage) and thus to the arrival process to a subsequent queue (stage).*"

But Suresh and Whitt (1990) have not shown conditions for existence of the VPP in the ***open*** tandem queues (stages) with ***infinite*** buffer capacities and have not considered the case of ***finite*** buffer capacities necessary to a pull serial line.

On the other hand, in a pull type ordering system, Kimura *et al.* (1981) and Muramatsu *et al.* (1985) have pointed out characteristics of the amplifications (propagations) of a production quantity, an order quantity and an inventory quantity in each cell (stage) without explicit proofs and conditions for existence of the amplifications (propagations).

Although the results of many theories and applications are very promising, it is clear that there is still a need for the development of quantitative models to gain insight in the mechanics of a card controlled pull system. Useful models for serial pull systems are provided by the finite-buffer literature for tandem queues. At any rate, to date little work has been done on the analytical approach to a derivation of the unique fundamental structural property in a pull serial line, the VPP.

From these motives, two major problems in a pull serial line can be identified :

i) Given the boundary conditions, does the VPP exists in a pull serial line subject to general service and demand schemes? If so, what are the necessary and sufficient conditions for existence?

ii) What model should be followed to measure the VPP?

These issues are addressed in this paper. In particular, convergence of the inter-departure process and measurement of its the first two moments are with first priority required to answer to these problems. The theories and models proposed in this study may be ultimately applied to the following major topics :

i) The Bull-whip effect in SCM (supply chain management) may be quantified.

ii) Optimal arrangement of cells (stages) or optimal assignment of servers in a pull serial line may be obtained.

iii) Any steady state performances including distributions may be easily computed, since they functionally relate to the first two moments of the inter-departure process.

iv) An equivalent push type serial line in view of the inter-departure process may be obtained.

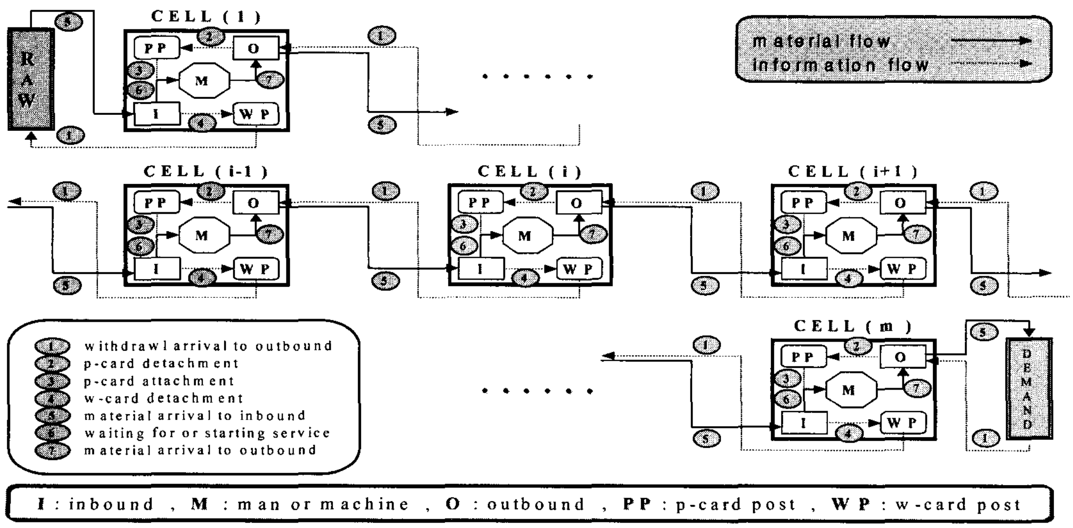
2. Model formulation of a pull serial line

Henceforth, the "pull" in this paper is meant for the pure (traditional) card (kanban) control. Although there are several ways of achieving a pull type control systems, actual physical implementation of a pull control is most often achieved by means of a card (kanban) system.

As a result, the terms "kanban (card)" and "pull" are used without distinction.

Pull systems may be either constant order quantity, nonconstant withdrawl cycle or constant withdrawl cycle, nonconstant order quantity. In particular, the latter is also called a periodic pull system. In this paper, the former is adopted. Although the order quantity is fixed, the period between "pulls" varies due to the randomness of manual or machine processing time and external demand process.

In detail a series of cell may be represented as [Figure 1], which is the same as shown in Kimura *et al.* (1981) and Schonberger (1982). In figure 1, w - card post is unnecessary since the existence of information sensing and material handling capabilities are taken for granted(Mitra



[Figure 1] Flow of cards and materials (containers) in a pull serial line *et al.* 1990).

As a result, descriptions of implementations is identical to Mitra et al. (1990) and Tayur (1993).

In this paper, a pull production line consists of a series of cell, which is composed of a manufacturing node (inbound + man or machine), a bulletinboard (p - card post) and outbound. A manufacturing node and a bulletin board can be described as the queueing model. Thus each cell in the pull serial line consists of queue & outbound.

Items flow through the cells in sequence and one operation is performed in each cell which consists of one machine or one server. The lot size and the batch size is 1 and there is only 1 type of item produced in each cell. External demand arrives in a single unit. The service time in each cell and the inter-arrival time of external demand are assumed to be i.i.d and their means and variances are known (distribution free, general distribution).

In addition to, there is no transit time for the movement of items between cells ; no scrap or defectives are produced, and there is no down time.

There is infinite supply of raw materials to the cell (1) and external demands are permitted to be backlogged. Finally, there is a finite buffer size in each cell, which is said to be a maximum inventory level or a fixed number of cards.

3. Preliminaries

3.1 Nomenclature

We adopt the following nomenclature.

(N 1) $i \in [0, m]$ ··· cell index, and note that 0 denotes raw material pool.

(N 2) T_i, $i \neq 0$ ··· maximum of inventory levelsor number of cards in cell (i)

(N 3) $K_i = \{k \mid k \geq 1 + \sum_{a=i+1}^{m} T_a\}$

$K_{(i)} = \{k \mid k \geq 1 + \sum_{a=i}^{m} T_a\}$

(N 4) C_i^k, $k \in K_i$ ··· time at which the withdrawl order for the k_th material arrives at the outbound of cell (i)

(N 5) D_i^k, $k \in K_i$ ··· time at which the k_th material arrives at the outbound of cell (i)

(N 6) Z_i^k, $k \in K_i$ ··· time at which the k_th material departs from the outbound of cell (i), in other words, the k_th material arrives at the the queue of cell (i + 1)

(N 7) S_i^k, $i \neq 0$, $k \in K_{(i)}$ ··· processing (service) time of the k_th material at the queue of cell (i)

(N 8) $A(k)$, $k \in K_m$ ··· time at which the k_th external demand arrives at the outbund of cell (m)

(N 9) $\{\Delta C i^k = C_i^{k+1} - C_i^k \mid k \in K_i\}$ ··· increments process of C_i^k, $k \in K_i$, and note that m indicates the inter-arrivals (increments) process of external demands denoted by $\Delta A(k)$, $k \in K_m$

(N 10) $\{\Delta D_i^k = D_i^{k+1} - D_i^k \mid k \in K_i\}$ ··· increments process of D_i^k, $k \in K_i$

(N 11) $\{\Delta Z_i^k = Z_i^{k+1} - Z_i^k \mid k \in K_i\}$ ··· increments process of Z_i^k, $k \in K_i$, which denotes the inter-departutre process of cell (i)

(N 12) $U_i^k = S_i^k - \Delta Z_{i-1}^k$, $i \neq 0$, $k \in K_{(i)}$

(N 13) $W_{Q,i}^k = \text{Max}(D_i^{k-1} - Z_{i-1}^{k}, 0)$, $i \neq 0$, $k \in K_{(i)}$ ··· waiting time of the k_th material in the queue of cell (i)

(N 14) $I_i^k = \text{Max}(0, Z_{i-1}^{k+1} - D_i^k),\ i \neq 0,\ k \in K_{(i)}$ … virtual idle time of man or machine in cell (i)

(N 15) $$J_i^k = \begin{cases} 0, \text{ if } C_i^k \geq D_i^k \Leftrightarrow Z_i^k = C_i^k \\ 1, \text{ if } C_i^k < D_i^k \Leftrightarrow Z_i^k = D_i^k \\ k \in K_i \end{cases}$$

(N 16) $f_i^k = P(J_i^k = 0) = P(Z_i^k = C_i^k),\ k \in K_i$

(N 17) $N_i = M_i + B_i$

$M_i,\ i \neq 0$ … number of materials being or being served in the queue of cell (i) in the steady state

$B_i,\ i \neq 0$ … number of materials not immediately satisfied when a withdrawal order from cell (i+1) arrives at the outbound of cell (i) in the steady state

$O_i,\ i \neq 0$ … number of materials in the outbound of cell (i) in the steady state.

(N 18) $$\begin{cases} \lambda_d(\neq 0) \cdots \text{ external demand rate} \\ \mu_i(\neq 0), i \neq 0 \cdots \text{ service rate in cell (i)} \end{cases}$$

(N 19)

$$\begin{cases} C_{A,i+1}^2 = \dfrac{V(\Delta Z_i)}{[E(\Delta Z_i)]^2},\ i \in [0, m+1] \\ C_{DM}^2 = V(\Delta A)\lambda_d^2,\ C_{S,i}^2 = V(S_i)\mu_i^2,\ i \neq 0 \\ \lim\limits_{k \to \infty}(\Delta Z_i^k, \Delta A(k), S_i^k) = \Delta Z_i, \Delta A, S_i \end{cases}$$

(N 20) $T(i, i+1) = T_i + T_{i+1},\ i \neq 0$

(N 21) $$\begin{cases} \lambda_i^- = \lambda_d \cdot \sum_{n=0}^{T(i,i+1)-1} P(N_i = n) \cdots \\ \quad \text{effective arrival rate to cell (i)} \\ \mu_i^- = \mu_i \cdot \sum_{n=1}^{T(i,i+1)} P(N_i = n) \cdots \\ \quad \text{effective service rate of cell (i)} \\ \rho_i = \dfrac{\lambda_d}{\mu_i} \cdots \text{traffic intensity of cell (i)} \\ i \neq 0 \end{cases}$$

(N 22) $$\begin{cases} \mathbf{x}_i^{(v)} \cdots [x_{1i}, x_{2i}, \cdots, x_{\langle n \rangle i}]^T \\ \mathbf{x}_i^{(p)} \cdots [\rho_i, V(S_i), T_i, T_{i+1}]^T \end{cases}$$

(N 23) $$\begin{cases} E_+^n \cdots \text{ n-dimensional half space} \\ \mathbf{O} \cdots \text{ null vector} \end{cases}$$

3.2 Convergence of inter-departure process

We let R_+ denote the set of nonnegative real numbers and let B_o, the subsets of R_+ denote the class of bounded Borel sets. The stochastic processes $\{C_i^k \mid k \in K_i\}$ and $\{D_i^k \mid k \in K_i\}$ defined on the probability space (R_+, B_o, P) have independent increments, and are generated by (1) and (2) respectively.

$$C_i^k = \begin{cases} Z_{i+1}^{k-T_{i+1}},\ i \neq m \\ A(k) \quad,\ i = m \end{cases} \tag{1}$$

$$D_i^k = \text{Max}(Z_{i-1}^k, D_i^{k-1}) + S_i^k,\quad i \neq 0,\ k \in K_{(i)} \tag{2}$$

It follows from (N 4) and (N 5) that the stochastic processes $\{C_i^k \mid k \in K_i\}$, $\{D_i^k \mid k \in K_i\}$ are mutually independent. Also, by equation (1), a stochastic process $\{Z_i^k \mid k \in K_i\}$ defined on the probability space (R_+, B_o, P) has independent increments and is generated by

$$Z_i^k = \text{Max}(C_i^k, D_i^k) = \begin{cases} \text{Max}(Z_{i+1}^{k-T_{i+1}}, D_i^k),\ i \neq m \\ \text{Max}(A(k), D_m^k),\ i = m \end{cases} \tag{3}$$

In particular, the boundary condition of an infinite supply of raw materials requires $\{C_0^k \mid k \in K_0\} \equiv \{Z_0^k \mid k \in K_0\}$.

Thus, we need not consider $\{D_0^k \mid k \in K_0\}$.

$\{Z_i^k \mid i \neq 0,\ k \in K_i\}$ may be generated by

$$Z_i^k = C_{i-1}^{k+T_i},\ i \neq 0 \tag{4}$$

where $\{C_0^{k+T_1} \mid k \in K_1\}$ implies that a withdrawl order for the material arrives at the raw materials pool.

In general, the stochastic processes $\{C_i^k \mid k \in K_i\}$, $\{D_i^k \mid k \in K_i\}$ and $\{Z_i^k \mid k \in K_i\}$ have nonstationary increments since they can not be represented as a sum of i.i.d.(independent and identically distributed) random variables. In a pull serial line, if either the p-card post of cell (i) or the outbound of cell (i-1) is empty, then the server remains idle until the earliest point in time until both material and card are available. In other words, a pull serial line is subject to blocking (back order, backlogging) from time to time. This is the reason why the stochastic processes $\{C_i^k \mid k \in K_i\}$, $\{D_i^k \mid k \in K_i\}$ and $\{Z_i^k \mid k \in K_i\}$ can not be renewal processes.

Lemma 1 : If there exists a pull serial line such that there is no down time, then we get the following two results :

i) Three sequences of integrable random variables $\{W_{Q,i}^k \mid k \in K_{(i)}\}$, $\{\Delta D_i^k \mid k \in K_{(i)}\}$ $\{\Delta Z_{i-1}^k \mid k \in K_{(i-1)}\}$ defined on the probability space (R_+, B_o, P) converge in L^2 to some random variables $W_{Q,i}$, ΔD_i and ΔZ_{i-1} respectively for each i.

ii)
$$\begin{cases} E(\Delta D_i) = E(\Delta Z_{i-1}) \\ V(\Delta D_i) = 2V(S_i) + 2E(W_{Q,i})[E(S_i) \\ \qquad - E(\Delta Z_{i-1})] + V(\Delta Z_{i-1}) \end{cases} \tag{5}$$

Proof : From the assumption that there is no down time, any given pull serial line must be always stable. This implies that

$$\{(W_{Q,i}^k)^2 \mid k \in K_{(i)}\}$$

is uniformly integrable and $W_{Q,i}^k$ converges in probability to some random variable $W_{Q,i}$. That is, for every $\varepsilon > 0$,

$$\sup_k E[(W_{Q,i}^k)^2] < \infty,$$
$$\lim_{k\to\infty} P(\mid W_{Q,i}^k - W_{Q,i} \mid > \varepsilon) = 0$$

Thus, $W_{Q,i}^k$ converges in L^2 to some random variable $W_{Q,i}$. By (N 13) and (N 14), we obtain

$$\begin{cases} W_{Q,i}^{k+1} - I_i^k = D_i^k - Z_{i-1}^{k+1} = U_i^k + W_{Q,i}^k \\ W_{Q,i}^{k+1} \cdot I_i^k = 0 \end{cases} \tag{6}$$

It follows from equation (6) and (N 12) that

$$\Delta D_i^k = U_i^{k+1} + I_i^k + \Delta Z_{i-1}^{k+1} = S_i^{k+1} + I_i^k \tag{7}$$

Since S_i^{k+1} are i.i.d. random variable, it is clear that I_i^k and ΔD_i^k converges in L^2 to I_i and ΔD_i respectively. Thus U_i^k in equation (6) converges in L^2 to some random variable U_i. Similarly, ΔZ_{i-1}^k in (N 12) converges in L^2 to some random variable ΔZ_{i-1}. Since all of $W_{Q,i}^k$, I_i^k, ΔD_i^k, U_i^k and ΔZ_{i-1}^k converge in L^2, as a matter of course, they converge in L^1. Hence, based on the (6), (7) and Lebesque's Dominated Convergence Theorem, we have

$$E(I_i) = -E(U_i) \tag{8}$$
$$E(\Delta D_i) = E(S_i) - E(U_i) = E(\Delta Z_{i-1}) \tag{9}$$
$$E(W_{Q,i}) = \frac{E(I_i^2) - E(U_i^2)}{2E(U_i)} \tag{10}$$

Note that S_i^{k+1} and I_i^k are mutually independent. By equation (8),

$$V(\Delta D_i) = V(S_i) + V(I_i) \quad (11)$$

, which may be written as

$$\begin{aligned} V(\Delta D_i) &= V(S_i) + 2E(W_{Q,i})E(U_i) + V(U_i) \\ &= 2V(S_i) + 2E(W_{Q,i}) \cdot \\ &\quad [E(S_i) - E(\Delta Z_{i-1})] + V(\Delta Z_{i-1}) \end{aligned} \quad (12)$$

, since S_i^k and ΔZ_{i-1}^k in (N 12) are mutually independent too. This completes the proof. ■

The inter-departure process $\{\Delta Z_i^k \mid k \in K_{(i)}\}$ and the finite scheme associated with this process $\Omega_1(\Delta Z_i^k)$ are represented as

$$\Delta Z_i^k = \alpha_i^k \Delta C_i^k + (1-\alpha_i^k)\Delta D_i^k,\ 0 \le \alpha_i^k \le 1 \quad (13)$$

$$\Omega_1(\Delta Z_i^k) = \begin{cases} \Delta C_i^k, \text{ s.t. } \alpha_i^k = 1 \\ \Delta D_i^k, \text{ s.t. } \alpha_i^k = 0 \\ \alpha_i^k \Delta C_i^k + (1-\alpha_i^k)\Delta D_i^k, \\ \text{s.t. } 0 < \alpha_i^k < 1 \end{cases} \quad (14)$$

α_i^k is a mixed distributed random variable with the distribution is given by

$$\alpha_i^k = \begin{cases} 0 & \text{, w.p. } (1-f_i)^2 \\ 1 & \text{, w.p. } f_i^2 \\ (0, x_i^k] & \text{, w.p. } -4f_i(1-f_i) \times \\ & \int_0^{x_i^k} (\alpha_i^k \ln(\alpha_i^k) + (1-\alpha_i^k)\ln \\ & (1-\alpha_i^k))\, d\alpha_i^k \end{cases} \quad (15)$$

Proposition 1 : The sequences $\{f_i^k \mid k \in K_{(i)}\}$ and $\{\alpha_i^k \mid k \in K_{(i)}\}$ are i.i.d. random variables.

Proof : By the definition, it is obvious that J_i^k and J_i^{k+1} are mutually independent for each k. Let $C_{0,r}^k$, $C_{1,r}^k$ denote the number of 0's and 1's in the pair (J_i^k, J_i^{k+1}) of an event defined in (N 15). Then we obtain

$$P(J_i^k = 0) = \frac{\sum_{r=1}^4 C_{0,r}^k P_r}{\sum_{r=1}^4 2P_r} = \frac{f_i^k + f_i^{k+1}}{2}$$

$$P(J_i^k = 1) = \frac{\sum_{r=1}^4 C_{1,r}^k P_r}{\sum_{r=1}^4 2P_r} = \frac{2 - f_i^k - f_i^{k+1}}{2}$$

Since $f_i^k = \dfrac{f_i^k + f_i^{k+1}}{2}$ and $f_i^k = f_i^{k+1}$ for each k, we have

$$E(J_i^k) = E(J_i^{k+1}),\ V(J_i^k) = V(J_i^{k+1})$$

, which means J_i^k are i.i.d. random variables. Now we may set $f_i^k = f_i$ for all k. f_i is a given constant such that $0 \le f_i \le 1$.

And it is clear that α_i^k and α_i^{k+1} are mutually independent for each k. It follows from equation (15) that

$$E(\alpha_i^k) = f_i,\ V(\alpha_i^k) = \frac{11}{18} f_i(1-f_i)$$

This means α_i^k are i.i.d. random variables. ■

Under a finite scheme $\Omega_1(\Delta Z_i^k)$, we may calculate

$$E_{\Omega_1}(\Delta Z_i^k), V_{\Omega_1}(\Delta Z_i^k)$$

with the conditional expectation and variance, which are given by

$$\begin{cases} E_{\Omega_1}(\Delta Z_i^k) = f_i E(\Delta C_i^k) + (1-f_i) E(\Delta D_i^k) \\ V_{\Omega_1}(\Delta Z_i^k) = f_i^2 V(\Delta C_i^k) + (1-f_i)^2 V(\Delta D_i^k) \\ \quad + \frac{11}{18} f_i(1-f_i)(V(\Delta C_i^k) + V(\Delta D_i^k)) \\ \quad + \frac{11}{18} f_i(1-f_i)(E(\Delta C_i^k) - E(\Delta D_i^k))^2 \end{cases} \quad (15\text{-}1)$$

Lemma 2 : If there exists a pull serial line such that the external demand is permitted to be backlogged, then we get the following two main results :

i) $\{\Delta C_i^k \mid k \in K_{(i)}\}$ and $\{\Delta Z_i^k \mid k \in K_{(i)}\}$ defined on the probability space (R_+, B_o, P) converge in L^2 to some random variables ΔC_i and $\Delta Z_i = \alpha_i \cdot \Delta C_i + (1-\alpha_i) \cdot \Delta D_i$, $0 < \alpha_i \leq 1$.

$$\text{ii)} \begin{cases} E(\Delta Z_i) = f_i E(\Delta C_i) + (1-f_i) E(\Delta D_i) \\ V(\Delta Z_i) = f_i^2 V(\Delta C_i) + (1-f_i)^2 V(\Delta D_i) \\ \qquad + \frac{11}{18} f_i (1-f_i)(V(\Delta C_i) \\ \qquad + V(\Delta D_i)) + \frac{11}{18} f_i (1-f_i) \\ \qquad (E(\Delta C_i) - E(\Delta D_i))^2 \\ 0 < f_i \leq 1 \end{cases} \tag{16}$$

Proof : In Lemma 1, there is no knowing whether ΔZ_m^k converges in L^2 to some random variable ΔZ_m or not. In Lemma 2, however, by the assumption of permitting backlogged demands, it follows from equation (1) and (N 8) that $\{\Delta C_m^k \mid k\ k \in K_{(m)}\} \equiv \{\Delta A(k) \mid k \in K_{(m)}\}$. Since $\Delta A(k)$ are i.i.d. random variables, it is clear that ΔC_m^k converges in L^2 to some random variable ΔC_m. Then a finite scheme $\Omega_1(\Delta Z_i^k)$ described in equation (13) tells us that if we fix $i = m$ in equation (13), ΔZ_m^k convergence in L^2 of to some random variable ΔZ_m. In other words, note that Lemma 1 and Proposition 1. Now, applying this fact, Lemma 1, Proposition 1 and **Continuous Mapping Principle** to a finite scheme $\Omega_1(\Delta Z_i^k)$, then we obtain ΔC_i^k converges in L^2 to some random variable ΔC_i for each $i \in [0, m-1]$.
Therefore,

$$\lim_{k\to\infty} E[(\Delta C_i^k - \Delta C_i)^2] = 0 \quad \text{and}$$

$$\lim_{k\to\infty} E[(\Delta Z_i^k - \alpha_i \Delta C_i - (1-\alpha_i)\Delta D_i)^2] = 0$$

Now that ΔZ_i^k converges in L^2 to some random variable ΔZ_i, it is clear that a conditional expectation also converges in L^2 to some random variable and its some subsequence converges in w.p. 1 (i.e. almost everywhere, almost surely) to some random variable. Thus equation (16) can be directly derived from the equation (15-1). Finally,

$$E(\alpha_i^k) = f_i,\ V(\alpha_i^k) = \frac{11}{18} f_i (1-f_i)$$

implies $f_i \neq 0$, which means also $\alpha_i \neq 0$. This completes the proof. ■

4. Main results

4.1 Desirable lower bound of $V(\Delta Z_i)$

By (N 16) and (N 17), f_i must be equal to $P(B_i = 0)$. However, it is possible for us to derive the distributions of B_i, M_i and O_i only when the distribution of N_i should be given in advance.

In this paper, we need not make any specific assumptions about the inter-arrivals of external demands and service time distributions.

Consequently, only approximate distributions of the steady state random variables such as B_i, M_i, O_i and N_i may be available. This implies that there are many possibilities of approximating their distributions. Therefore, it is necessary that we should derive a true or desirable lower bound on $V(\Delta Z_i)$ applicable to any approximate to f_i or $P(B_i = 0)$ on the basis of N_i.

Also this necessity forces us to modify the equation (13). Fortunately, there is at least one mathematical technique to solve this problem, which is the Taylor's Series. Maintaining an identical expectation of ΔZ_i, we will utilize the Taylor's Series.

Proposition 2 : The lower bound on $V(\Delta Z_i)$ applicable to any given approximate to f_i or $P(B_i = 0)$ is represented as

$$\begin{cases} \Delta Z_i = f_i \Delta C_i + (1-f_i)\Delta D_i + \\ \qquad (E(\Delta C_i) - E(\Delta D_i))(\alpha_i - f_i) \\ E(\Delta Z_i) = f_i E(\Delta C_i) + (1-f_i) E(\Delta D_i) \\ V(\Delta Z_i) = f_i^2 V(\Delta C_i) + (1-f_i)^2 V(\Delta D_i) \\ \qquad + \frac{11}{18} f_i (1-f_i)(E(\Delta C_i) - E(\Delta D_i))^2 \end{cases} \tag{17}$$

Proof : From equation (13), the lower bound may be obtained by first order Taylor's at the neighborhood of

$$(\alpha_i^k, \Delta C_i^k, \Delta D_i^k) = (E(\alpha_i^k), E(\Delta C_i^k), E(\Delta D_i^k))$$

without remainder. Then we have

$$\begin{cases} \Delta Z_i^k = f_i \Delta C_i^k + (1-f_i)\Delta D_i^k + \\ \qquad (E(\Delta C_i^k) - E(\Delta D_i^k))(\alpha_i^k - f_i) \\ E(\Delta Z_i^k) = f_i E(\Delta C_i^k) + (1-f_i) E(\Delta D_i^k) \\ V(\Delta Z_i^k) = f_i^2 V(\Delta C_i^k) + (1-f_i)^2 V(\Delta D_i^k) \\ \qquad + \frac{11}{18} f_i (1-f_i)(E(\Delta C_i^k) - E(\Delta D_i^k))^2 \end{cases}$$

By Lemma 1, Lemma 2 and Proposition 1, the equation (17) can be derived from equation (18) in the steady state.

This completes the proof. ■

Actually, it is impossible to overestimate the importance of the lower bound on $V(\Delta Z_i)$. A concise functional form compared to the true value of $V(\Delta Z_i)$ enables us to easily manipulate problems associated with proofs and structural properties. It is not too much to say that we cannot pay too much attention to this fact.

Now, we explicitly consider two boundary conditions that infinite supply of raw materials and backlogged demands are permitted to cell (1) and cell (m) respectively. Therefore, what is the implication of these conditions? We propose the equivalent statement to these assumptions. Then, relied upon these assumptions, we are with intention of investigating into measurable relations among

$$\Delta D_i, \ \Delta Z_i \text{ and } \Delta C_i$$

representative of cell (i-1), cell (i) and cell (i+1) respectively.

Theorem 1 : Suppose that there exists a pull serial line such that infinite supply of raw materials and backlogged demands are permitted, besides $E(U_i) < 0$. Then we have the following results :

$$\begin{cases} E(\Delta Z_{i+1}) = E(\Delta Z_0) = \frac{1}{\lambda_d}, \ i \in [0, m] \\ V(\Delta Z_i)_{TRUE} = f_i^2 V(\Delta C_i) + (1-f_i)^2 V(\Delta D_i) \\ \qquad + \frac{11}{18} f_i (1-f_i)(V(\Delta C_i) \\ \qquad + V(\Delta D_i)), \ C^2_{A,m+2} = C^2_{DM} \\ V(\Delta Z_i)_{LOWER} = f_i^2 V(\Delta C_i) + (1-f_i)^2 V(\Delta D_i) \end{cases}$$

Proof : In this proof, Lemma 1 and Lemma 2 are implicitly used. It follows from equation (1) or (4) that $\Delta Z_1 = \Delta C_0 = \Delta Z_0$. Hence,

$$E(\Delta Z_1) = E(\Delta Z_0).$$

And by (4), (5) and (16), we obtain

$$f_i E(\Delta Z_{i+1}) - E(\Delta Z_i) + (1-f_i) E(\Delta Z_{i-1}) = 0$$

, which may be rewritten as

$$E(\Delta Z_{i+1}) = E(\Delta Z_0) + [E(\Delta Z_0) - E(\Delta Z_{-1})] \cdot \sum_{j=0}^{i} \left[\prod_{k=0}^{j} \left(\frac{1}{f_k - 1} \right) \right], \ i \in [0, m] \quad (20)$$

Infinite supply of raw materials implies $f_0 = 1$. Thus, from equation (20), we have

$$\sum_{j=0}^{i} \left[\prod_{k=0}^{j} \left(\frac{1}{f_k - 1} \right) \right] = 0.$$

In addition,

$$E(\Delta Z_{m+1}) = E(\Delta C_m) = E(\Delta A) = 1/\lambda_d$$

It follows from (4) and (5) that

$$E(\Delta C_i) = E(\Delta D_i).$$

Applying this relation to (16) and (17), then we are done. ■

Corollary 1 : Suppose $E(U_i) < 0$. Then the following two statements are equivalent.

i) The infinite supply of raw materials and the backlogged demands are permitted.

ii) Either backlogged demands or infinite demands are permitted, and

$$P(N_i = T(i, i+1)) = 0, P(N_i = 0) = 1 - \rho_i,$$
$$\rho_i < 1, \ \forall i \in [1, m]$$

Proof : 1) i) implies ii).

By equation (19) and (N 21), we have

$$E(\Delta Z_i) = \frac{1}{\lambda_d}, \ i \in [0, m+1]$$

2) ii) implies i).

By (N 21) and equation (9), we have

$$\lambda_i^- = \mu_i^- = \frac{1}{E(\Delta D_i)} = \frac{1}{E(\Delta Z_{i-1})}$$

Since $P(N_i = T(i, i+1)) = 0, P(N_i = 0) = 1 - \rho_i$,

$$\frac{1}{E(\Delta Z_{m+1})} = \lambda_d = \lambda_i^- = \mu_i^- = \frac{1}{E(\Delta Z_{i-1})}, \quad i \in [1, m]$$

Thus we obtain either

$$\frac{1}{E(\Delta Z_{m+1})} = \frac{1}{E(\Delta Z_m)}$$

or

$$\frac{1}{E(\Delta Z_m)} = \frac{1}{E(\Delta D_m)} = \frac{1}{E(\Delta Z_{m-1})}$$

This completes the proof. ■

4.2 Constrained discrete minimax problem for $\mathbf{f}_i$

We consider

$$P(N_i = n) = F_n(\mathbf{x}_i^{(v)} \mid \mathbf{x}_i^{(p)}) = F_n(\mathbf{x}_i^{(v)}),$$
$$n \in [0, T(i, i+1)], \ \mathbf{x}_i^{(v)} \in E_+^{\langle n \rangle}, \ \mathbf{x}_i^{(p)} \in E_+^{n-\langle n \rangle} \quad (21)$$

Note that

$$\mathbf{x}_i^{(v)} \in E_+^{\langle n \rangle} = \overline{E_+^{\langle n \rangle}}, \ \mathbf{x}_i^{(p)} \in E_+^{n-\langle n \rangle} = \overline{E_+^{n-\langle n \rangle}}$$

since the set E_+^n is closed.

To begin with, it is desirable of us to select the elements in $\mathbf{x}_i^{(v)}$ such that $\frac{\partial E(W_{Q,i})}{\partial \mathbf{x}_i^{(v)}} \geq \mathbf{O}$ for the purpose of enhancing the system performances.

Proposition 3 : Suppose that there exists a pull serial line such that infinite supply of raw materials and backlogged demands are permitted, besides $E(U_i) < 0$. Then we have the following results :

i) The necessary and sufficient conditions for $\frac{\partial E(W_{Q,i})}{\partial x_i^{(v)}} \geq O$ is given by

$$\frac{\partial P(N_i=k)}{\partial x_i^{(v)}} < O, \forall k \in [1, T_i - 1],$$

if $T_i \geq 2$ (22)

ii) $\frac{\partial E(W_{Q,i})}{\partial x_i^{(v)}} \geq O \Leftrightarrow \frac{\partial E(M_i)}{\partial x_i^{(v)}} \geq O$

Proof : By (N 17), we have

$$\begin{cases} P(N_i=k) = P(M_i=k), \forall k \in [0, T_i - 1] \\ P(M_i=T_i) = \sum_{k=T_i}^{T(i,i+1)} P(N_i=k) \end{cases} \quad (23)$$

It follows from (23) that

$$\frac{\partial E(M_i)}{\partial x_i^{(v)}} = \sum_{k=1}^{T_i-1} \left[(k - T_i) \frac{\partial P(N_i = k)}{\partial x_i^{(v)}} \right] \quad (24)$$

By the Little's Law, we know that

$$\begin{cases} E(M_i) = \lambda_i^- \cdot [E(W_{Q,i}) + E(S_i)] \\ E(W_{Q,i}) = E(\Delta D_i) \cdot E(M_i) - E(S_i) \end{cases} \quad (25)$$

and this implies that

$$\frac{\partial E(W_{Q,i})}{\partial x_i^{(v)}} = E(\Delta D_i) \left(\frac{\partial E(M_i)}{\partial x_i^{(v)}} \right) \quad (26)$$

If $T_i = 1$, then it is readily seen from (24) and (26) that $\frac{\partial E(W_{Q,i})}{\partial x_i^{(v)}} = 0$.

And we consider the case of $T_i \geq 2$.

By the (Corollary 1), we know that

$$\sum_{k=1}^{T_i-1} P(N_i=k) > \sum_{k=T_i}^{T(i,i+1)-1} P(N_i=k)$$

and this shows that

$$\sum_{k=1}^{T_i-1} \frac{\partial P(N_i=k)}{\partial x_i^{(v)}} < \sum_{k=T_i}^{T(i,i+1)-1} \frac{\partial P(N_i=k)}{\partial x_i^{(v)}}$$

Since $\sum_{k=0}^{T(i,i+1)} P(N_i = k) = 1$,

$$\sum_{k=0}^{T(i,i+1)} \frac{\partial P(N_i = k)}{\partial x_i^{(v)}} = O.$$

From the (Corollary 1), we get easily

$$\sum_{k=1}^{T(i,i+1)-1} \frac{\partial P(N_i = k)}{\partial x_i^{(v)}} = O \quad (27)$$

At any rate, equation (27) ensures $\forall T_i \geq 2$, $\rho_i < 1$,

$$\begin{cases} \sum_{k=1}^{T_i-1} \frac{\partial P(N_i = k)}{\partial x_i^{(v)}} < O \\ \frac{\partial P(N_i = k)}{\partial x_i^{(v)}} < O, \forall k \in [1, T_i - 1] \end{cases} \quad (28)$$

Therefore, by (28), we may derive

$$\sum_{k=1}^{T_i-1} \left[(k - T_i) \frac{\partial P(N_i=k)}{\partial x_i^{(v)}} \right] > O \quad (29)$$

It is readily from (24) and (26) seen that

$$\frac{\partial E(W_{Q,i})}{\partial x_i^{(v)}} \geq O.$$

Conversely,

$$\text{if } \frac{\partial P(N_i = k)}{\partial \mathbf{x}_i^{(v)}} < \mathbf{O},\ k \in [1, T_i - 1]$$

$$\forall T_i \geq 2,\ \rho_i < 1$$

then it is clear that $\frac{\partial E(W_{Q,i})}{\partial \mathbf{x}_i^{(v)}} > \mathbf{O}$.

And from (26), we obtain

$$\frac{\partial E(W_{Q,i})}{\partial \mathbf{x}_i^{(v)}} \geq \mathbf{O} \text{ is equvalent to } \frac{\partial E(M_i)}{\partial \mathbf{x}_i^{(v)}} \geq \mathbf{O}$$

This completes the proof. ■

Now we can define a closed set $\Lambda_i \subseteq E_+^{\langle n \rangle}$ as

$$\Lambda_i = \overline{\Lambda_i} = \left\{ \mathbf{x}_i^{(v)} \,\middle|\, \frac{\partial E(W_{Q,i})}{\partial \mathbf{x}_i^{(v)}} \geq \mathbf{O},\ \mathbf{x}_i^{(v)} \in E_+^{\langle n \rangle} \right\} \quad (30)$$

such that $\Lambda_i \neq \emptyset$ and $\Lambda_i = \overline{\Lambda_i} = E_+^{\langle n \rangle} = \overline{E_+^{\langle n \rangle}}$.

Lemma 3 : Suppose that there exists a pull serial line such that infinite supply of raw materials and backlogged demands are permitted, besides $E(U_i) < 0$. Then we have

$$\begin{cases} V(\Delta D_i) - V(\Delta Z_{i-1}) \geq 0 \\ \dfrac{\partial (V(\Delta D_i) - V(\Delta Z_{i-1}))}{\partial \mathbf{x}_i^{(v)}} \leq \mathbf{O} \\ \forall\ \mathbf{x}_i^{(v)} \in \Lambda_i, \text{ and } T_i = 1 \text{ with equality} \end{cases} \quad (31)$$

Proof : By (N 12), (N 13), (N 14) and Lemma 1, we have

$$(-I_i) \geq_{st} (-\Delta Z_{i-1}) \Leftrightarrow I_i \leq_{st} \Delta Z_{i-1}$$

Since $-E(I_i) = E(U_i) < 0$ and equation (10), it follows

$$\begin{cases} \text{(a)} \ \dfrac{\mathbf{Max}[0, E([S_i]^2) - 2E(S_i)E(\Delta Z_{i-1})]}{2[E(\Delta Z_{i-1}) - E(S_i)]} \\ \quad \leq E(W_{Q,i}) \leq \dfrac{V(S_i) + V(\Delta Z_{i-1})}{2[E(\Delta Z_{i-1}) - E(S_i)]} \\ \text{(b)} \ [E(I_i)]^2 \leq E([I_i]^2) \leq \\ \quad \mathbf{Min}[E([U_i]^2),\ E([\Delta Z_{i-1}]^2)] \end{cases} \quad (32)$$

From equations (5) and (30),

$$\frac{\partial (V(\Delta D_i) - V(\Delta Z_{i-1}))}{\partial \mathbf{x}_i^{(v)}} = 2 \left| E(U_i) \cdot \frac{\partial E(W_{Q,i})}{\partial \mathbf{x}_i^{(v)}} \right| \leq \mathbf{O} \quad (33)$$

If $T_i = 1$, it follows from (5) and (33) that

$$\begin{cases} V(\Delta D_i) - V(\Delta Z_{i-1}) = 2 \cdot V(S_i) \geq 0 \\ \dfrac{\partial (V(\Delta D_i) - V(\Delta Z_{i-1}))}{\partial \mathbf{x}_i^{(v)}} = \mathbf{O} \end{cases} \quad (34)$$

And if $T_i \geq 2$, then it is evident that

$$V(\Delta D_i) - V(\Delta Z_{i-1}) \neq 0$$

Therefore, by (5) and (33), we obtain

$$\begin{aligned} &\mathbf{Min}[2V(S_i), \\ &\quad V(S_i) + E(S_i)\langle 2E(\Delta Z_{i-1}) - E(S_i) \rangle] \geq \\ &V(\Delta D_i) - V(\Delta Z_{i-1}) \geq V(S_i) - V(\Delta Z_{i-1}) \end{aligned} \quad (35)$$

This shows that

$$\begin{cases} V(S_i) - V(\Delta Z_{i-1}) > 0 \\ V(\Delta D_i) - V(\Delta Z_{i-1}) > 0 \\ \dfrac{\partial (V(\Delta D_i) - V(\Delta Z_{i-1}))}{\partial \mathbf{x}_i^{(v)}} < \mathbf{O} \end{cases} \quad (36)$$

We are done. ■

As previously stated,

$$f_i = P(B_i = 0) = \sum_{n=0}^{T_i} P(N_i = n)$$

implies $f_i = f_i(\mathbf{x}_i^{(v)})$, $\mathbf{x}_i^{(v)} \in E_+^{\langle n \rangle}$.

Thus f_i is continuous over the closed set Λ_i. Now we can define a set Λ as

$$\Lambda = \bigcap_{i=0}^{m} \Lambda_i = \left\{ \mathbf{x}^{(v)} \,\middle|\, \frac{\partial E(W_{Q,i})}{\partial \mathbf{x}^{(v)}} \geq \mathbf{O},\ \mathbf{x}^{(v)} \in \Lambda_i \right\} \tag{37}$$

such that $\Lambda = \overline{\Lambda} \subseteq \Lambda_i = \overline{\Lambda_i} \subseteq E_+^{\langle n \rangle} = \overline{E_+^{\langle n \rangle}}$.

And a function $\tau(\mathbf{x}^{(v)})$ is defined as

$$\tau(\mathbf{x}^{(v)}) = \mathrm{Max}_i \left[f_i(\mathbf{x}^{(v)}) \right],\ i \in I_X = [0, m] \tag{38}$$

Note that $\tau(\mathbf{x}^{(v)})$ is also continuous over Λ since

$$\begin{aligned} &\left| \tau(\mathbf{x}^{(v)}) - \tau(\mathbf{x}_e^{(v)}) \right| = \\ &\quad \left| \mathrm{Max}_i \left[f_i(\mathbf{x}^{(v)}) \right] - \mathrm{Max}_i \left[f_i(\mathbf{x}_e^{(v)}) \right] \right| \\ &\quad \leq \mathrm{Max}_i \left| f_i(\mathbf{x}^{(v)}) - f_i(\mathbf{x}_e^{(v)}) \right|, \\ &\quad i \in I_X,\ \forall\, \mathbf{x}_e^{(v)} \in \Lambda \end{aligned} \tag{39}$$

Finally, $J(\mathbf{x}^{(v)})$ and a directional derivative are given by

$$J(\mathbf{x}^{(v)}) = \{ i \mid f_i(\mathbf{x}^{(v)}) = \tau(\mathbf{x}^{(v)}) \} \subset I_X \tag{40}$$

$$\begin{cases} \dfrac{\partial \tau(\mathbf{x}_e^{(v)})}{\partial \mathbf{d}} = \lim\limits_{h \to +0} \dfrac{\tau(\mathbf{x}_e^{(v)} + h \cdot \mathbf{d}) - \tau(\mathbf{x}_e^{(v)})}{h} \\ \| \mathbf{d} \|^2 = \mathbf{d}^T \mathbf{d},\ \forall\, \mathbf{x}_e^{(v)},\ \mathbf{d} \in \Lambda \end{cases} \tag{41}$$

Now we consider the constrained discrete minimax problem (P1) defined as

$$\begin{aligned} &\mathrm{Inf}_{\mathbf{x}^{(v)} \in \Lambda}\ \mathrm{Max}_{i \in I_X} \left[f_i(\mathbf{x}^{(v)}) \right] \\ &= \mathrm{Inf}_{\mathbf{x}^{(v)} \in \Lambda} \left[\tau(\mathbf{x}^{(v)}) \right] \end{aligned} \tag{P1}$$

Lemma 4 : Let the function $f_i(\mathbf{x}^{(v)})$ be twice differentiable in a neighborhood $N_\delta(\mathbf{x}_e^{(v)})$ of $\mathbf{x}_e^{(v)}$. Then, $\tau(\mathbf{x}^{(v)})$ is differentiable at $\mathbf{x}_e^{(v)}$ in any direction $\mathbf{d} \in \Lambda$ and

$$\begin{cases} \dfrac{\partial \tau(\mathbf{x}_e^{(v)})}{\partial \mathbf{d}} = \\ \mathrm{Max}_{i \in J(\mathbf{x}_e^{(v)})} \left[\left\langle \dfrac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial \mathbf{x}^{(v)}},\ \mathbf{d} \right\rangle \right] \\ \langle \mathbf{A},\ \mathbf{B} \rangle = \mathbf{A}^T \mathbf{B} \end{cases} \tag{42}$$

Proof : Using the Taylor's series, we obtain

$$\begin{aligned} &f_i(\mathbf{x}_e^{(v)} + h \cdot \mathbf{d}) = f_i(\mathbf{x}_e^{(v)}) + \\ &h \cdot \left\langle \frac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial \mathbf{x}^{(v)}},\ \mathbf{d} \right\rangle + O_i(h;\, \mathbf{x}_e^{(v)},\, \mathbf{d}) \\ &\forall\, i \in I_X,\ h \in (0, \delta) \end{aligned} \tag{43}$$

If we fix $\mathbf{x}_e^{(v)}$ and $\mathbf{d}$, then

$$\begin{aligned} &g_i(h) = f_i(\mathbf{x}_e^{(v)} + h \cdot \mathbf{d}) \\ &\forall\, i \in I_X,\ h \in (0, \delta) \end{aligned} \tag{44}$$

By some calculations, we have

$$\frac{dg_i(h)}{dh} = \left\langle \frac{\partial f_i(\mathbf{x}_e^{(v)} + h \cdot \mathbf{d})}{\partial \mathbf{x}^{(v)}},\ \mathbf{d} \right\rangle \tag{45}$$

$$\begin{aligned} g_i(h) = g_i(0) + h \cdot \frac{dg_i(0)}{dh} + \\ \int_0^h \left(\frac{dg_i(h)}{dh} - \frac{dg_i(0)}{dh} \right) dh \end{aligned} \tag{46}$$

It follows from (44), (45) and (46) that

$$f_i(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})=f_i(\mathbf{x}_e^{(v)})+h\cdot\left\langle\frac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial\mathbf{x}^{(v)}},\mathbf{d}\right\rangle+\int_0^h\left\langle\frac{\partial f_i(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})}{\partial\mathbf{x}^{(v)}}-\frac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial\mathbf{x}^{(v)}},\mathbf{d}\right\rangle dh \quad (47)$$

From (43) and (47), we obtain

$$O_i(h;\mathbf{x}_e^{(v)},\mathbf{d})=\int_0^h\left\langle\frac{\partial f_i(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})}{\partial\mathbf{x}^{(v)}}-\frac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial\mathbf{x}^{(v)}},\mathbf{d}\right\rangle dh \quad (48)$$

which implies that

$$\lim_{h\to+0}O_i(h;\mathbf{x}_e^{(v)},\mathbf{d})=\lim_{h\to+0}\frac{O_i(h;\mathbf{x}_e^{(v)},\mathbf{d})}{h}=0 \quad (49)$$

Since $f_i(\mathbf{x}_e^{(v)})$ is continuous over Λ, $\forall i\in I_X$, there exists a positive number h_e for all $\mathbf{d}\in\Lambda$ such that

$$\tau(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})=\mathrm{Max}_{i\in I_X}[f_i(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})]=\mathrm{Max}_{i\in J(\mathbf{x}_e^{(v)})}[f_i(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})],\ \forall h\in(0,h_e) \quad (50)$$

Thus, it follows from (49) and (50) that there exists $0<h_e<\delta$ such that

$$\tau(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})=\mathrm{Max}_{i\in J(\mathbf{x}_e^{(v)})}[f_i(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})]\le\tau(\mathbf{x}_e^{(v)})+h\cdot\mathrm{Max}_{i\in J(\mathbf{x}_e^{(v)})}\left[\left\langle\frac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial\mathbf{x}^{(v)}},\mathbf{d}\right\rangle\right]+\mathrm{Max}_{i\in I_X}[O_i(h;\mathbf{x}_e^{(v)},\mathbf{d})],\ \forall h\in(0,h_e) \quad (51)$$

Similarly, we obtain

$$\tau(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})=\mathrm{Max}_{i\in I_X}[f_i(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})]\ge\tau(\mathbf{x}_e^{(v)})+h\cdot\mathrm{Max}_{i\in J(\mathbf{x}_e^{(v)})}\left[\left\langle\frac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial\mathbf{x}^{(v)}},\mathbf{d}\right\rangle\right]+\mathrm{Min}_{i\in I_X}[O_i(h;\mathbf{x}_e^{(v)},\mathbf{d})],\ \forall h\in(0,h_e) \quad (52)$$

Therefore, by (51) and (52), we have

$$\mathrm{Min}_{i\in I_X}[O_i(h;\mathbf{x}_e^{(v)},\mathbf{d})]\le\tau(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})-\tau(\mathbf{x}_e^{(v)})-h\cdot\mathrm{Max}_{i\in J(\mathbf{x}_e^{(v)})}\left[\left\langle\frac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial\mathbf{x}^{(v)}},\mathbf{d}\right\rangle\right]\le\mathrm{Max}_{i\in I_X}[O_i(h;\mathbf{x}_e^{(v)},\mathbf{d})] \quad (53)$$

$$\forall h\in(0,h_e),\ \mathbf{x}_e^{(v)}\in\Lambda,\ \mathbf{d}\in\Lambda$$

which implies that

$$\begin{cases}\lim_{h\to+0}\dfrac{\tau(\mathbf{x}_e^{(v)}+h\cdot\mathbf{d})-\tau(\mathbf{x}_e^{(v)})}{h}-\mathrm{Max}_{i\in J(\mathbf{x}_e^{(v)})}\left[\left\langle\dfrac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial\mathbf{x}^{(v)}},\mathbf{d}\right\rangle\right]=0\\ \dfrac{\partial\tau(\mathbf{x}_e^{(v)})}{\partial\mathbf{d}}-\mathrm{Max}_{i\in J(\mathbf{x}_e^{(v)})}\left[\left\langle\dfrac{\partial f_i(\mathbf{x}_e^{(v)})}{\partial\mathbf{x}^{(v)}},\mathbf{d}\right\rangle\right]=0\\ \forall h\in(0,h_e),\ \mathbf{x}_e^{(v)}\in\Lambda,\ \mathbf{d}\in\Lambda\end{cases} \quad (54)$$

This completes the proof. ■

If we set

$$\begin{cases}\mathbf{x}_*^{(v)}\cdots\text{stationary point of }\tau(\mathbf{x}^{(v)})\\ \Omega(\mathbf{x}^{(v)})=\{\psi(\mathbf{x}_e^{(v)}-\mathbf{x}^{(v)})\mid\psi>0,\ \mathbf{x}_e^{(v)}\in\Lambda\}\end{cases} \quad (55)$$

then

$$\begin{cases} \tau(\mathbf{x}_*^{(v)}) = \text{Max}_{i\in I_X}[f_i(\mathbf{x}_*^{(v)})] = \\ \quad \text{Inf}_{\mathbf{x}^{(v)}\in\Lambda}\text{Max}_{i\in I_X}[f_i(\mathbf{x}^{(v)})] = \\ \quad \quad \text{Inf}_{\mathbf{x}^{(v)}\in\Lambda}[\tau(\mathbf{x}^{(v)})] \end{cases} \tag{56}$$

Theorem 2 : A necessary condition for a point $\mathbf{x}_*^{(v)}$ to be a minimum point of $\tau(\mathbf{x}^{(v)})$ is that

$$\begin{cases} \text{Inf}_{\mathbf{x}_e^{(v)}\in\Lambda}\text{Max}_{i\in J(\mathbf{x}_*^{(v)})}\left[\left\langle \frac{\partial f_i(\mathbf{x}_*^{(v)})}{\partial \mathbf{x}^{(v)}}, \mathbf{x}_e^{(v)} - \mathbf{x}_*^{(v)}\right\rangle\right] = 0 \\ \text{or} \\ \text{Inf}_{\mathbf{d}_e\in\overline{\Omega(\mathbf{x}_*^{(v)})}}\text{Max}_{i\in J(\mathbf{x}_*^{(v)})}\left[\left\langle \frac{\partial f_i(\mathbf{x}_*^{(v)})}{\partial \mathbf{x}^{(v)}}, \mathbf{d}_e\right\rangle\right] \geq 0 \end{cases} \tag{57}$$

Proof : i) Necessity

Suppose that the first equation in (57) fails to hold. Then there is a point $\mathbf{x}_e^{(x)}\in\Lambda$ such that

$$\text{Max}_{i\in J(\mathbf{x}_*^{(v)})}\left[\left\langle \frac{\partial f_i(\mathbf{x}_*^{(v)})}{\partial \mathbf{x}^{(v)}}, \mathbf{x}_e^{(v)} - \mathbf{x}_*^{(v)}\right\rangle\right] = \phi < 0 \tag{58}$$

The expression on the left of the first equation in (57) cannot be positive. Clearly $\mathbf{x}_e^{(x)} \neq \mathbf{x}_*^{(x)}$. We set $\mathbf{d}_e = \mathbf{x}_e^{(v)} - \mathbf{x}_*^{(v)}$. By equation (42) in Lemma 4, (58) can be written as

$$\frac{\partial\tau(\mathbf{x}_*^{(v)})}{\partial \mathbf{d}_e} = \phi < 0 \tag{59}$$

and we obtain

$$\begin{cases} \tau(\mathbf{x}_e^{(v)} + h\cdot\mathbf{d}) = \tau(\mathbf{x}_e^{(v)}) + \\ \quad h\cdot\frac{\partial\tau(\mathbf{x}_e^{(v)})}{\partial \mathbf{d}} + O(h;\mathbf{x}_e^{(v)},\mathbf{d}) \\ \lim_{h\to+0}\frac{O(h;\mathbf{x}_e^{(v)},\mathbf{d})}{h} = 0 \end{cases} \tag{60}$$

, which enables us to reach to

$$\begin{cases} \tau(\mathbf{x}_*^{(v)} + h_e\cdot\mathbf{d}_e) = \tau(\mathbf{x}_*^{(v)}) + \\ h_e\frac{\partial\tau(\mathbf{x}_*^{(v)})}{\partial \mathbf{d}_e} + O(h_e;\mathbf{x}_*^{(v)},\mathbf{d}_e),\ h_e > 0 \end{cases} \tag{61}$$

And there exists a sufficiently small positive number h_e such that

$$|\,O(h_e;\mathbf{x}_*^{(v)},\mathbf{d}_e)\,| \leq -\frac{\phi\cdot h_e}{2} \tag{62}$$

It follows from (59), (61) and (62) that

$$\begin{cases} |\,\tau(\mathbf{x}_*^{(v)} + h_e\,\mathbf{d}_e) - \tau(\mathbf{x}_*^{(v)}) - h_e\phi\,| \leq \\ \quad -\frac{\phi\cdot h_e}{2} \end{cases} \tag{63}$$

Finally, from (63), we can derive

$$\begin{cases} \tau(\mathbf{x}_*^{(v)} + h_e\cdot\mathbf{d}_e) \leq \tau(\mathbf{x}_*^{(v)}) + \\ \quad \frac{\phi\cdot h_e}{2} < \tau(\mathbf{x}_*^{(v)}) \end{cases} \tag{64}$$

(64) contradicts the assumption that $\mathbf{x}_*^{(x)}\in\Lambda$ is a minimum point(stationary point), since

$$\forall\, h_e\in[0,1],\ (\mathbf{x}_*^{(x)} + h_e\cdot\mathbf{d}_e)\in\Lambda\ .$$

ii) first equation in (57) ⇒ second equation in in (57)

If this is not true, we may suppose that the first equation in (57) holds but there is a vector $\mathbf{d}_e\in\overline{\Omega(\mathbf{x}_*^{(v)})}$ such that (59). By the definition of $\overline{\Omega(\mathbf{x}_*^{(v)})}$, there is a vector $\mathbf{v}\in\Omega(\mathbf{x}_*^{(v)})$ such that

$$\text{Max}_{i\in J(\mathbf{x}_*^{(v)})}\left[\left\langle \frac{\partial f_i(\mathbf{x}_*^{(v)})}{\partial \mathbf{x}^{(v)}}, \mathbf{v}\right\rangle\right] \leq \frac{\phi}{2} \tag{65}$$

Since $\mathbf{v}\in\Omega(\mathbf{x}_*^{(v)})$, we have

$$\mathbf{v} = \psi(\mathbf{x}_e^{(v)} - \mathbf{x}_*^{(v)}),\ \psi>0,\ \mathbf{x}_e^{(v)}\in\Lambda.$$

Hence, by (65), we have

$$\begin{cases} \text{Max}_{i\in J(x_*^{(v)})}\Big[\Big\langle \dfrac{\partial f_i(x_*^{(v)})}{\partial x^{(v)}}, \\ x_e^{(v)} - x_*^{(v)}\Big\rangle\Big] \le \dfrac{\phi}{2\cdot\psi} < 0 \end{cases} \tag{66}$$

which contradicts the first equation in (57).

iii) first equation in (57) ⇒ second equation in (57)

If this is not true, we may suppose that the second equation in (57) holds but there is a point $x_e^{(v)} \in \Lambda$ such that (58). Clearly

$$x_e^{(x)} \neq x_*^{(x)}.$$

By (58),

$$\begin{cases} \text{Max}_{i\in J(x_*^{(v)})}\Big[\Big\langle \dfrac{\partial f_i(x_*^{(v)})}{\partial x^{(v)}}, \\ x_e^{(v)} - x_*^{(v)}\Big\rangle\Big] = \phi < 0 \end{cases} \tag{67}$$

(67) contradicts the second equation in (57), since $d_e \in \Omega(x_*^{(v)}),\ \overline{\Omega(x_*^{(v)})}$.

This completes the proof. ■

Suppose for all i, $f_i(x^{(v)})$ is strictly quasi-convex. Then, $\tau(x^{(v)})$ is strictly quasiconvex. If $\tau(x^{(v)})$ is strictly quasiconvex, the necessary condition, i.e. equation (57) in Theorem 2 is also sufficient.

If a set Λ_C is defined as

$$\begin{cases} \Delta(\Lambda_C) < \infty,\ \Lambda_C = \overline{\Lambda_C} \subset \Lambda \\ \Lambda_C = \{ x_e^{(v)} \mid x_e^{(v)} \in \Lambda,\ \| x_e^{(v)} - x^{(v)} \| \le 1 \} \end{cases} \tag{68}$$

then by the Heine-Borel-Lebesque theorem and the Mini-Max theorem, **(P1)** can be converted into **(P2)** given by

$$\begin{aligned} &\text{Min}_{x^{(v)}\in\Lambda_C}\text{Max}_{i\in I_x}[f_i(x^{(v)})] \\ &= \text{Min}_{x^{(v)}\in\Lambda_C}[\tau(x^{(v)})] \end{aligned} \tag{P2}$$

Lemma 5 : Let $D_i(.)$ be the implicit function of f_i defined by equation (19) and $(D_{D,i}, D_{C,i}, D_{Z,i})^T$ be the gradient vector of a function $D_i(.)$ respectively.

Suppose the $V(\Delta Z_i)_{LOWER}$. Then,

$$f_i(x^{(v)}),\ \forall i \text{ is strictly quasiconvex}$$

if and only if $D_{Z,i} > 0$, $\forall$ i and $x_i^{(v)}$.

Proof : By equation (19), the implicit function of f_i can be defined as

$$f_i = D_i(V(\Delta D_i), V(\Delta C_i), V(\Delta Z_i)) \tag{69}$$

since $f_i(V(\Delta D_i)+V(\Delta C_i)) - V(\Delta D_i) \neq 0$.

And by some calculations, the gradient vector of a function $D_i(.)$ is given by

$$\begin{cases} (-D_{Z,i}(1-f_i)^2,\ -D_{Z,i}f_i^2,\ D_{Z,i})^T, \\ D_{Z,i} = \dfrac{1}{2(V(\Delta D_i)+V(\Delta C_i))f_i - 2V(\Delta D_i)} \end{cases} \tag{70}$$

and the hessian matrix of a function $D_i(.)$ denoted by $H_{D_i}(.)$ satisfies

$$\begin{cases} v_i^T H_{D_i}(v_i)\, v_i = 0, \\ \forall\, v_i = (V(\Delta D_i), V(\Delta C_i), V(\Delta Z_i)) \neq \mathbf{0} \end{cases} \tag{71}$$

The equation (71) means that

$$\begin{cases} D_i(v_{i,2}) = D_i(v_{i,1}) + \beta_i^T(v_{i,2} - v_{i,1}) \\ \beta_i = (-D_{Z,i}(1-f_i)^2,\ -D_{Z,i}f_i^2,\ D_{Z,i}) \text{at } v_{i,1} \\ \forall\, v_{i,2} \neq v_{i,1} \end{cases} \tag{72}$$

Also some calculations shows that

$$\begin{cases} [D_i(\mathbf{v}_{i,3}) - D_i(\mathbf{v}_{i,1})][D_i(\mathbf{v}_{i,2}) - D_i(\mathbf{v}_{i,3})] > 0 \\ \mathbf{v}_{i,3} = \theta_i \mathbf{v}_{i,1} + (1-\theta_i)\mathbf{v}_{i,2}, \quad 0 < \theta_i < 1 \\ \forall\ \mathbf{v}_{i,2} \neq \mathbf{v}_{i,1} \end{cases} \quad (73)$$

Thus, equations (72) and (73) implies that for all i, $f_i(\mathbf{x}^{(v)})$ is strictly quasilinear. That is,

$$\begin{cases} \mathrm{Min}[D_i(\mathbf{v}_{i,1}), D_i(\mathbf{v}_{i,2})] < \\ \qquad D_i(\mathbf{v}_{i,3}) < \mathrm{Max}[D_i(\mathbf{v}_{i,1}), D_i(\mathbf{v}_{i,2})] \end{cases}$$

Hence it follows from equations (57), (72) and (73) that $f_i(\mathbf{x}^{(v)})$ is strictly quasiconvex if and only if $D_{Z,i} > 0$, $\forall$ i and $\mathbf{x}_i^{(v)}$.
This completes the proof. ■

Now, we propose the variability propagation principle in a pull serial line as the Theorem 3.

Theorem 3 : There exists the variability propagation in a pull serial line with the infinite supply of raw materials and the backlogged demands if and only if there is at least one element in $\mathbf{x}_i^{(v)}$ such that satisfies (30), (37) and

$$D_{Z,i} > 0,\ \forall \mathrm{i}\ \text{and}\ \mathbf{x}_i^{(v)}.$$

Proof : The equation (19) in the Theorem 1 implies that if $f_i(\mathbf{x}^{(v)})$ is strictly quasiconvex under $V(\Delta Z_i)_{LOWER}$, then $f_i(\mathbf{x}^{(v)})$ is also strictly quasiconvex under $V(\Delta Z_i)_{TRUE}$: If there exists no variability propagation under $V(\Delta Z_i)_{LOWER}$, then there also exists no variability propagation under $V(\Delta Z_i)_{TRUE}$.
By the Lemma 5,

$$f_i(\mathbf{x}^{(v)})\ \text{satisfies}\ D_{Z,i} > 0,\ \forall \mathrm{i}\ \text{and}\ \mathbf{x}_i^{(v)}.$$

Thus,

$$V(\Delta Z_{i+1}) = V(\Delta C_i) \geq V(\Delta Z_i)\ \text{since}\ f_i \leq 1.$$

By the given boundary conditions,

$$C_{A,1}^2 = C_{A,2}^2,\ C_{A,m+2}^2 = C_{DM}^2$$

Consequently, by (N 19) and (19) in Theorem 1,

$$C_{A,1}^2 = C_{A,2}^2,\ C_{A,m+2}^2 = C_{DM}^2,\ C_{A,i+2}^2 \geq C_{A,i+1}^2$$

which indicates the VPP.
And it is obvious that $C_{A,i}^2 \in \mathbf{x}_i^{(v)}$.
This completes the proof. ■

In a pull serial line with the infinite supply of raw materials and the backlogged demands, if there is at least one cell (i) such that $D_{Z,i} < 0$, $\forall\ \mathbf{x}_i^{(v)}$, then there exist subsequences for i such that the variability propagates.

5. Numeric model and examples

5.1 Numeric model

In brief, we introduce the numeric model for computing the $C_{A,i}^2$ embedded in a pull serial line with the infinite supply of raw materials and the backlogged demands.

The nonlinear simultaneous equations in <Table 1> may be converted into a constrained optimization problem. Also it can be proved that this constrained optimization problem is unimodal, which means that there exists a unique optimal solution. In addition, we can model this problem with the augmented Lagrange multipliers and prove that the algorithm used to solve this problem has the convergence order of 1. All proofs associated with these issues are omitted for want of space (see Choe, 2002). And codes

〈Table 1〉 Nonlinear simultaneous equations for the squared coefficient of variation of inter-departure process

Find :

$$C^2_{A,h} \geq 0\ ,\ h \in [1\ ,\ m+2]\ ,\ i \in [1\ ,\ m]$$

$$\begin{cases} C^2_{A,m+2} = C^2_{DM} \\ C^2_{A,i+2} = M(i) \cdot C^2_{A,i+1} - N(i) \cdot C^2_{A,i} \\ \qquad - 2 \cdot \lambda_d^2 \cdot N(i) \cdot SS(i) \\ SS(i) = \dfrac{C^2_{S,i}}{\mu_i^2} + E(W_{Q,i}) \cdot \left(\dfrac{1}{\mu_i} - \dfrac{1}{\lambda_d}\right) \\ C^2_{A,2} = C^2_{A,1} \\ E(W_{Q,i}) = \dfrac{E(M_i)}{\lambda_d} - \dfrac{1}{\mu_i} \\ \rho_i = \dfrac{\lambda_d}{\mu_i} \end{cases}$$

$$\begin{cases} M(i) = \begin{cases} \left(\dfrac{1}{f_i}\right)^2, & \text{LOWER} \\ \dfrac{18}{7 \cdot f_i^2 + 11 \cdot f_i}, & \text{TRUE} \end{cases} \\ N(i) = \begin{cases} \left(\dfrac{1 - f_i}{f_i}\right)^2, & \text{LOWER} \\ \dfrac{7 \cdot f_i^2 - 25 \cdot f_i + 18}{7 \cdot f_i^2 + 11 \cdot f_i}, & \text{TRUE} \end{cases} \end{cases}$$

$$\begin{cases} E(M_i) = \sum_{x=0}^{T_i} x P(M_i = x) \\ f_i = 1 - \rho_i + C_{B,i} \cdot \sum_{n=1}^{T_i} P(N_i = n) \\ C_{B,i} = \dfrac{\rho_i}{\sum_{n=1}^{T_i + T_{i+1} - 1} P(N_i = n)} \\ T_{m+1} = \text{big } M \\ P(M_i = x) = \begin{cases} 1 - \rho_i\ , & x = 0 \\ C_{B,i} P(N_i = x), & x \in [1\ ,\ T_i - 1] \\ C_{B,i} \sum_{n=T_i}^{T(i,i+1)-1} P(N_i = n)\ , & x = T_i \end{cases} \end{cases}$$

$$\begin{cases} P(N_i = n) = \begin{cases} (1 - \rho_i)\delta_i\ , & n = 0 \\ \rho_i (1 - r_i) r_i^{n-1} \delta_i\ , & n \in [1, T(i, i+1) - 1] \\ \rho_i (1 - \rho_i) r_i^{T(i,i+1)-1} \delta_i\ , & n = T(i, i+1) \end{cases} \\ \delta_i = \dfrac{1}{1 - \rho_i^2 \cdot r_i^{T(i,i+1)-1}} \\ r_i = 1 - \dfrac{\rho_i}{E(L_i)} \\ E(L_i) = \dfrac{0.5\rho_i^2 (C^2_{S,i} + 1)(\rho_i^2 C^2_{S,i} + C^2_{A,i})}{(1 - \rho_i)(\rho_i^2 C^2_{S,i} + 1)} + \rho_i \end{cases}$$

implementing this algorithm are compiled with the **Borland C++(Version 3.0 or 3.1)**.

5.2 Numeric examples

To begin with, we set

$$\lambda_d = 1,\ V(\Delta A) = V(S_i) = 0.0625,$$
$$T_i = 2,\ \text{big } M = 20,\ m = 8.$$

and consider

$$[\text{IBWL}]\ \mu = (2.0, 1.5, 1.5, 1.1, 1.1, 1.5, 1.5, 2.0)$$
$$[\text{BWL}]\ \mu = (1.1, 1.5, 1.5, 2.0, 2.0, 1.5, 1.5, 1.1)$$
$$[\text{LMH}]\ \mu = (2.0, 2.0, 1.5, 1.5, 1.5, 1.5, 1.1, 1.1)$$
$$[\text{HML}]\ \mu = (1.1, 1.1, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0)$$

, where IBWL, BWL, LMH and HML denote inverse bowl, bowl, low-medium-high and high-medium-low respectively.

The results to be reported on the experiments performed with our study can be divided into the following two groups :

(1) computing the squared coefficients of variation of inter-departure process under the lower bound.

(2) if (1) is successfully performed, quantifying some issues associated with distribution including under the true value.

<Table 2> shows that there exists an optimal solution for <Table 1> in all problems. Thus, further experiments are required.

Results of experiments are given by <Table 3>~<Table 7>.

If we consider

$$\lambda_d = 1,\ V(\Delta A) = 0.0625,\ V(S_i) = 0.0,$$
$$T_i = [i \bmod 2] + 1,\ \text{big } M = 20,\ m = 20.$$

and $\mu_k = 0.2 \cdot k + 0.9, k \in [1, 20]$, then results of experiments are to be given by [Figure 2].

⟨Table 2⟩ $(C^2_{A,i})^*$ under lower bound

$(C^2_{A,i})^*$	[IBWL]	[BWL]	[LMH]	[HML]
1	0.0438	0.0412	0.0417	0.0436
2	0.0438	0.0412	0.0417	0.0436
3	0.0439	0.0473	0.0418	0.0502
4	0.0445	0.0478	0.0419	0.0580
5	0.0451	0.0485	0.0424	0.0588
6	0.0520	0.0486	0.0430	0.0597
7	0.0602	0.0488	0.0435	0.0606
8	0.0610	0.0495	0.0440	0.0616
9	0.0620	0.0501	0.0507	0.0619
10	0.0625	0.0625	0.0625	0.0625

⟨Table 3⟩ $(C^2_{A,i})^*$ under true value

$(C^2_{A,i})^*$	[IBWL]	[BWL]	[LMH]	[HML]
1	0.0623	0.0621	0.0621	0.0623
2	0.0623	0.0621	0.0621	0.0623
3	0.0623	0.0622	0.0621	0.0623
4	0.0623	0.0622	0.0621	0.0624
5	0.0623	0.0622	0.0621	0.0624
6	0.0624	0.0622	0.0622	0.0625
7	0.0625	0.0622	0.0622	0.0625
8	0.0625	0.0623	0.0622	0.0625
9	0.0625	0.0623	0.0623	0.0625
10	0.0625	0.0625	0.0625	0.0625

⟨Table 4⟩ $(f_i)^*$ under true value

$(f_i)^*$	[IBWL]	[BWL]	[LMH]	[HML]
1	0.9978	0.9115	0.9978	0.9114
2	0.9918	0.9918	0.9978	0.9114
3	0.9918	0.9918	0.9918	0.9918
4	0.9114	0.9978	0.9918	0.9918
5	0.9114	0.9978	0.9918	0.9918
6	0.9918	0.9918	0.9918	0.9918
7	0.9918	0.9918	0.9914	0.9978
8	0.9977	0.8638	0.8639	0.9977

⟨Table 5⟩ Time in system under true value

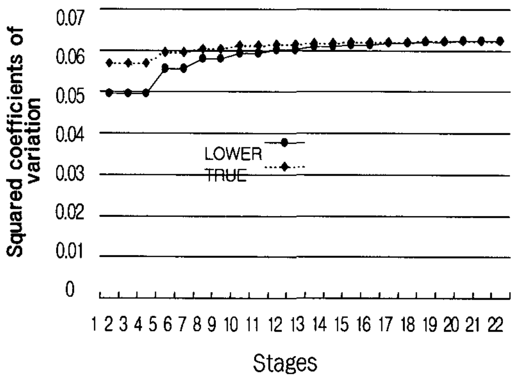
	[IBWL]	[BWL]	[LMH]	[HML]
Mean	16.0000	16.0000	16.0000	16.0000
Variance	4.7460	4.7722	4.7709	4.7474

⟨Table 6⟩ $(C_{B,i})^*$ under true value

$(C_{B,i})^*$	[IBWL]	[BWL]	[LMH]	[HML]
1	1.0002	1.0107	1.0002	1.0107
2	1.0009	1.0009	1.0002	1.0107
3	1.0009	1.0009	1.0009	1.0009
4	1.0107	1.0002	1.0009	1.0009
5	1.0107	1.0002	1.0009	1.0009
6	1.0009	1.0009	1.0009	1.0009
7	1.0009	1.0009	1.0107	1.0002
8	1.0000	1.0000	1.0000	1.0000

⟨Table 7⟩ $E(W_{Q,i})^*$ under true value

$E(W_{Q,i})^*$	[IBWL]	[BWL]	[LMH]	[HML]
1	0.0340	0.3174	0.0340	0.3176
2	0.0778	0.0777	0.0340	0.3176
3	0.0778	0.0777	0.0777	0.0778
4	0.3176	0.0340	0.0777	0.0779
5	0.3176	0.0340	0.0777	0.0779
6	0.0778	0.0778	0.0777	0.0779
7	0.0779	0.0778	0.3175	0.0341
8	0.0342	0.3519	0.3518	0.0343



[Figure 2] Variability propagation principle

6. Concluding remarks

We have proposed a numeric model and algorithm for the purpose of computing the first two moments of the inter-departure process subject to given service rate, demand rate and number of cards in each cell, and proved the existence of VPP in a pull serial line.

The necessary and sufficient conditions for existence of VPP in a pull serial line with given boundary conditions are represented as

$$f_i \geq \frac{V(\Delta D_i)}{V(\Delta D_i) + V(\Delta C_i)}, \quad \forall i \in [1, m]$$

Via some experiments, we have confirmed the validity and applicability of the proposed theories.

Through these works, some subordinate structural properties have been proved under the assumptions of infinite supply of raw materials and backlogged demands :

(1) The assumption of infinite supply of raw material results in the same throughput in each cell, which indicates that material flow in a pull serial line must be conserved.

(2) Besides, if backlogged demands are permitted, then these assumptions are equivalent to the statement that a pull serial line is stable, that is, traffic intensities of each cell must be smaller than 1. Also, the throughput in each cell is identical to the external demand rate.

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