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Fractional Surrogate-Knapsack Cuts for Integer Programs

Youngho Lee*
Department of Industrial Systems and Information Engineering, Korea University
Sung-Buk Ku Anam Dong 5-1, Seoul, 136-701, Korea

Youngjin Kim**
Department of Industrial Systems and Information Engineering, Korea University
Sung-Buk Ku Anam Dong 5-1, Seoul, 136-701, Korea

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ABSTRACT

In this paper, we explore a new class of cutting planes by extending the concept of fractional S-K(S-K) cuts. This class of cuts is derived by applying a suitable surrogate constraint analysis that incorporates a special multiplier adjustment method to the generalized Gomory's fractional cut. We present computational results to provide insights into the performance of these cuts in comparison with other well known classes of cuts.

1. INTRODUCTION

This paper deals with the class of fractional surrogate-knapsack(S-K) cuts, a strengthened form of Gomory's fractional cut [9]. Gomory proposed a fractional cutting plane algorithm in the late fifties and early sixties, to solve integer problems via the solution of a sequence of linear programs. In this research, we investigate a method for generating a more effective class of fractional S-K cuts by applying surrogate knapsack analysis to derive a strengthened form of the generalized fractional cut of Gomory as proposed by Glover [7]. The latter generalized cut is obtained by using a multiple, not necessarily integral, of an original equation

* Email: yhlee@korea.ac.kr
** Email: youngjk@korea.ac.kr

before rounding down all coefficients and the right-hand side, and then subtracting this cut from the source equation multiplied by a suitable integer. This generalized fractional cut includes the standard Gomory's fractional cut as a special case, and also subsumes other types of fractional cuts. The proposed fractional S-K cut extends this generalized fractional cut by applying the surrogate constraint analysis of Glover, Sherali, and Lee [8] to obtain a stronger cut. (Glover, Sherali and Lee [8] presented a new surrogate-constraint analysis, giving rise to a family of strong valid inequalities called surrogate-knapsack (S-K) cuts.) They presented an analytical procedure to provide a strong S-K cut subject to constraining the values of selected cut coefficients, including the right-hand side of the cut. A polynomial-time separation procedure was developed to generate an S-K cut, and the authors demonstrated that the approach can recover facets which are not available using standard lifting methods. Additionally, they proposed a strengthened form of the fractional S-K cut.

It is well known that all valid inequalities can be obtained as Chvatal-Gomory inequalities (C-G inequalities) for integer programming problems, perhaps via a repeated application of the Chvatal-Gomory process [10]. Dietrich and Escudero [6] showed that clique and cover inequalities implied from 0-1 knapsack constraints can be obtained as rank-one C-G inequalities. They also showed that 0-1 knapsack constraints obtained by the 'big M' reduction procedure can also be generated as rank-one C-G inequalities. In addition, they showed how some extended coefficient reduction based LP-tighter and 0-1- equivalent constraints can be generated as Gomory's fractional cuts.

The original Gomory's fractional cut still performs as well as, or better than, other types of fractional cuts. For example, the combined cut performs worse than the Gomory's fractional cut for several problem instances in the experiments in Ceria, Cornuejols and Dawande [4]. In this paper, we implement the fractional S-K cut to compare its performance with other fractional cuts. Our cut generation process first determines a set of optimal multipliers for deriving the generalized fractional cut of Glover [7] which yields a maximum violation with respect to a given LP relaxation solution. Following this, we implicitly determine the surrogate-multipliers for the surrogate-knapsack cut to generate a further strengthened cut in case a specified condition holds true. A result given in Glover, Sherali and Lee [8] facilitates this derivation without explicitly having to determine the surrogate multipliers, relying just on the knowledge of their existence.

The remainder of this paper is organized as follows. In Section 2, we introduce the fractional S-K cut generation scheme, based on the generalized fractional cut and the fractional S-K cut. Section 3 provides some insights into selecting appropriate multipliers for deriving deep cuts based on defined criteria. Finally, Section 4 presents computational results.

2. FRACTIONAL S-K CUTS

In this section, we introduce the class of fractional S-K cuts as a strengthened form of the generalized fractional cut of Glover [7]. Assume that the current LP tableau contains the equation

$$y + \sum_{j \in N_b} a_j x_j = a_0 , \tag{1}$$

where y is a basic variable, and x_j for $j \in N_b$, are the current nonbasic variables. Suppose that all these variables are restricted to be integers, and in particular, let $N_b^{'} \subseteq N_b$ represent the index set of binary-restricted nonbasic variables. We also suppose that a_0 is not integral, as must occur if the LP extreme point is not integer feasible.

Based on the constraint (1), we may derive an extended form of the fractional cut, named as the generalized cut (GC):

$$s + \sum_{j \in N_b} (pa_j - \lceil ha_j \rceil) x_j = pa_0 - \lceil ha_0 \rceil, \tag{2}$$

where s is a nonnegative integer variable, p is an integer parameter such that pa_0 is non-integer, and $\lceil \cdot \rceil$ is the rounded-up integer. Also, note that h is a parameter chosen to satisfy two conditions : $p-1 < h \le p$, and $h > (\lceil pa_0 \rceil - 1)/a_0$. The former condition assures that $p = \lceil h \rceil$, which contributes toward eliminating that the basic variable y from (2), and along with the latter condition, it assures that the right-hand side of (2) is negative so that (2) eliminates the current LP fractional vertex. Note that for $h = p$, Equation (2) yields the customary Gomory's scaled fractional cut (GFC):

$$s + \sum_{j \in N_b} (-f_j x_j) = -f_0, \tag{3}$$

where $f_j = \lceil ha_j \rceil - ha_j$ for $j \in N_b \cup \{0\}$.

Now, we introduce the derivation of the proposed fractional S-K cut. First, let us multiply the source equation (1) by the parameter h to yield a scaled source equation

$$hy + \sum_{j \in N_b} ha_j x_j = ha_0. \tag{4}$$

Noting that $x_j \leq 1$ for all $j \in N_b'$, let us surrogate the above constraint along with the inequalities $-u_j x_j \geq -u_j$ for $j \in N_b'$, where $u_j \geq 0$ for all $j \in N_b'$, to obtain the constraint

$$hy + \sum_{j \in N_b} (ha_j - u_j) x_j \geq ha_0 - u_0, \tag{5}$$

where $u_j \equiv 0$ for $j \in N_b - N_b'$ for convenience, and where $u_0 \equiv \sum_{j \in N_b'} u_j$. From this constraint, the Chvatal-Gomory procedure [10] yields the following cut.

$$\lceil h \rceil y + \sum_{j \in N_b} \lceil ha_j - u_j \rceil x_j \geq \lceil ha_0 - u_0 \rceil, \tag{6}$$

or

$$-\lceil h \rceil y - \sum_{j \in N_b} \lceil ha_j - u_j \rceil x_j \leq -\lceil ha_0 - u_0 \rceil \tag{7}$$

Now, multiplying the source equation (1) by an integer p, we get

$$py + \sum_{j \in N_b} pa_j x_j = pa_0. \tag{8}$$

Selecting h and p such that $p = \lceil h \rceil$, we get upon adding (8) to (7)

$$\sum_{j \in N_b} (pa_j - \lceil ha_j - u_j \rceil) x_j \leq pa_0 - \lceil ha_0 - u_0 \rceil = 0. \tag{9}$$

Let us refer to (9) as the fractional S-K cut. This can be rewritten as

$$s + \sum_{j \in N_b} (pa_j - \lceil ha_j - u_j \rceil) x_j = pa_0 - \lceil ha_0 - u_0 \rceil, \tag{10}$$

where we can restrict the slack s to be a nonnegative integer variable, since the slack in (7) is integer-valued. The fractional surrogate cut (9) can be made strictly stronger than GC given by (2), if we choose u_j suitably to yield $\lceil ha_j - u_j \rceil < \lceil ha_j \rceil$ for some nonempty subset N_b'' of N_b', selecting $u_j = 0$ for all $j \in N_b - N_b''$ while keeping the right-hand side of the two cuts the same, that is, $\lceil ha_0 - u_0 \rceil = \lceil ha_0 \rceil$. The following result identifies such a strengthened cut under a suitable condition which accomplishes the foregoing tightening idea. For convenience, we write the generalized cut (GC) given by (2) in the following form

$$-\sum_{j\in N_b} g_j x_j \le -g_0, \tag{11}$$

where $\lceil ha_j \rceil$ for $j \in N_b \cup \{0\}$.

Then, in Theorem 2 of [8], we have the strengthened cut. Define $r_j = 1-(\lceil ha_j \rceil - ha_j)$ for $j \in N_b \cup \{0\}$, and let N_b'' be any subset of N_b' such that $\sum_{j\in N_b''} r_j < r_0$. Then, provided N_b'' is nonempty, a fractional S-K cut that strictly dominates GC given by (11) can be derived to yield the following :

$$\sum_{j\in N_b''}(1-g_j)\,x_j - \sum_{j\in N_b'-N_b''} g_j x_j \le -g_0 . \tag{12}$$

Example 2.1. Suppose that the current LP tableau contains the equation

$$y+\frac{11}{8}x_1 + \frac{5}{8}x_2 + \frac{7}{8}x_3 = \frac{5}{8},$$

where y is a basic integer variable, and assume that all the x variables are 0 - 1 variables. Let us choose $p = h = 1$, so that (2) is a standard Gomory's fractional cut (3). This fractional cut is given as follows :

$$-\frac{5}{8}x_1 - \frac{3}{8}x_2 - \frac{1}{8}x_3 \le -\frac{3}{8}.$$

Then we have from Theorem 2 of ([7]) that

$$r_1 = \frac{3}{8},\ r_2 = \frac{5}{8},\ r_3 = \frac{7}{8},\ r_0 = \frac{5}{8}.$$

To ensure the condition $\sum_{j\in N_b''} r_j < r_0$, we can select $N_b'' = \{1\}$. This gives the fractional S-K cut (12) in the form

$$\frac{3}{8}x_1 - \frac{3}{8}x_2 - \frac{1}{8}x_3 \le -\frac{3}{8}.$$

This cut strictly dominates the original fractional cut derived above. □

3. SELECTION OF MULTIPLIERS FOR CUT GENERATION

In this section, we explore some strategies for selecting appropriate multipliers for generating Gomory's fractional cuts (GFC), generalized cut (GC), and fractional S-K cut, as given by Equations (3), (2), and (9), respectively. The strength of a cut can be evaluated by several measures. The Euclidean distance from a cut generated to the fractional point might be the most meaningful measure, but several other criteria based on rectilinear or l_∞ measures, or based on minimizing the violation subject to some scaling of the cut coefficients are viable options (See Sherali and Shetty [11], or Balas, Ceria and Cornuejols [1, 2]). We focus here, in concept, on this last criterion.

The strongest Gomory's fractional cut (3) that maximize the violation with respect to the current solution, given the fractional value form of the cut coefficients, can be readily obtained by solving problem (13) below which maximizes the value of f_0

$$\text{Maximize } \lceil ha_0 \rceil - ha_0 \text{ with } h \text{ integer}. \tag{13}$$

Proposition 3.1. Let $a_0 = e/D$, where $e \in Z$, $D \in Z$, and e and D are relatively prime and there are integers h and q satisfying $he = qD + 1$. Then the problem (13) has an optimal objective value $(D-1)/D$.

Proof. Since e and D are relatively prime, $gcd(e, D) = 1$, where $\gcd(x, y)$ is the greatest common divisor of x and y. By using the Euclidean algorithm to find $gcd(e, D)$, we can readily find integers h and q satisfying $he - qD = gcd(e, D) = 1$. Then, we have that $\lceil ha_0 \rceil - ha_0 = \lceil he/D \rceil - he/D = (q+1) - (q + 1/D) = (D-1)/D$. □

The generalized cut (2) requires the condition $p - 1 < h \le p$ and $ha_0 > \lceil pa_0 \rceil - 1$. This implies that $\lceil ha_0 \rceil = \lceil pa_0 \rceil$, and so, the maximum violation in (2) with respect to the current solution is obtained by solving the following problem:

$$\text{Maximize } \lceil pa_0 \rceil - pa_0 \text{ with } p \text{ integer}. \tag{14}$$

The optimal objective value of (14) is the same as that for (13) leading to a cut (2) having the same right-hand side as that for (3) based on this criterion. However, in this case, since $h \in (p-1, p]$, we can obtain cuts having different coefficients depending on the choice of h, although for the situation in which $a_j \ge 0$ for all $j \in N_b$, it would be best to select $h = p$ to obtain a deepest cut.

Example 3.1. Suppose that the current LP tableau contains the equation

$$y + \frac{1}{45}x_1 + \frac{5}{8}x_2 + \frac{7}{8}x_3 = \frac{5}{8},$$

where y is a basic integer variable, and assume that all the x variables are 0-1 variables. Let us find p and h by solving (13). Since $a_0 = 5/8 = e/D$, we have that $h = 5$ and $q = 3$ satisfying $he = qD + 1$. Hence the optimal solution of (13) is $(D-1)/D = 7/8$. Then the strengthened Gomory's fractional cut (3) is given as follows :

$$-\frac{8}{9}x_1 - \frac{7}{8}x_2 - \frac{5}{8}x_3 \le -\frac{7}{8}.$$

Then we have from Theorem 2 of ([7]) that

$$r_1 = \frac{1}{9},\ r_2 = \frac{1}{8},\ r_3 = \frac{3}{8},\ r_0 = \frac{1}{8}.$$

To ensure the condition $\sum_{j \in N_b^{''}} r_j < r_0$ of Theorem 2 of ([7]), we can select $N_b^{''} = \{1\}$. This gives the fractional S-K cut (12) in the form

$$\frac{1}{9}x_1 - \frac{7}{8}x_2 - \frac{5}{8}x_3 \le -\frac{7}{8}.$$

This cut strictly dominates the strengthened fractional cut derived above. □

Now let us consider the generation of deep fractional S-K cuts by optimizing the value of p and h in (9), in the spirit of Theorem 2 of ([7]). Theorem 2 of ([7]) is aimed at enhancing the coefficients in (9) while keeping the right-hand side the same as that of the generalized cut in (2).

Suppose that we select p via (13), and we let $h = p$ to make ha_0 as large as possible, subject to the restrictions on h . This provides an initial cut (2) (or (11)). For this cut, u_0 must have a value satisfying $0 \le u_0 < ha_0 - (\lceil ha_0 \rceil - 1)$, which by (13), yields $0 \le u_0 < 1/D$. Accordingly, by examining the values of r_j for all $j \in N_b \cup \{0\}$ as in Theorem 2 of ([7]), we can attempt to strengthen the foregoing cut to the revised form (12). Alternatively, we would apply the surrogate knapsack analysis before determining the multipliers of GC, or to devise a method to determine all the multipliers simultaneously.

4. COMPUTATIONAL EXPERIENCE

In this section, we report some computational results using a set of 0-1 integer programming test problems taken from MIPLIB [3]. A summary of these test problems is given in Table 4.1. This is a particularly interesting set of six problems from MIPLIB, noting for example that for the instances P0033 and P0201, the combined cut of Ceria, Cornuejols and Dawande [4] performs worse than Gomory's cut. In Table 4.1, Z_{LP} represents the optimal objective function value of the linear programming relaxation at the root node, and Z_{IP} represents the optimal objective value of the integer program. Note that this test set also possesses relatively weak LP relaxations.

Table 4.1. Test problems from MIPLIB [3]

Problem	Constraints	Variables	Z_{LP}	Z_{IP}
P0033	15	33	2,520.6	3,089.0
P0201	133	201	6,875.0	7,615.0
P0282	241	282	176,867.5	258,411.0
P0291	252	291	1,705.1	5,223.8
P0548	177	548	315.3	8,691.0
P2756	756	2,756	2,688.7	3,124.0

At each iteration in our implementation, we generated the particular class of cuts being evaluated for all the rows for which the corresponding basic variables had fractional values. No previously added cuts were deleted during the course of the procedure. The linear programs encountered in this process were solved using the CPLEX 6.2 callable library [5]. All runs were performed on a SUN OS 6.0 workstation, and reported times are CPU seconds on this machine.

First, we compared the performance of the original Gomory's fractional cut (GFC with $h=1$), the generalized cut (GC), and the fractional S-K cut, given respectively by (3), (2) and (9). We obtained the parameter p for GC via (14) using CPLEX 6.2 [5], and we set $h=p$. We restricted the range of p to $0 \le p \le 10$ in solving this problem to ensure a reasonable effort. The cutting plane schemes FSKC-S1 and FSKC-S2 respectively represent the strategy of strengthening the derived GFC and the GC cuts by applying Theorem 2 of ([7]). Table 4.2 summarizes the results obtained. Here, Z_{root} denotes the final objective function value obtained at the root (initial) node in each case. From this table, two observations are discernable. First, by comparing FSKC-S1 with GFC, it is evident that Theorem 2 of ([7]) significantly improves the performance of GFC. This encourages a further investigation of the effect of the S-K analysis on the quality of the cut. The second observation comes from comparing GFC and GC with FSKC-S2. For all the problems, except P0201, GFC performs better than the latter two cuts. Moreover, for two test problems, FSKC-S2 produces a mix of cuts that improves over that generated by GC itself. From these results, it is evident that the maximum violation criteria embodied by (14) is not particularly effective, and that by its nature of squeezing the admissible range of u_0, it does not provide a sufficient opportunity for Preposition 2.1 to improve the quality of the resulting cut exploration of alternative mechanisms for generating cuts based on other defined measures of cut quality.

Table 4.2. Comparison of Gomory, generalized Gomory, and surrogate–knapsack enhanced cutting plane schemes

Problem	GFC with $h = 1$		GC		FSKC-S1		FSKC-S2	
	Z_{root}	Elapsed Time	Z_{root}	Elapsed Time	Z_{root}	Elapsed Time	Z_{root}	Elapsed Time
P0033	2,909.5	0.3	2,882.9	0.6	2,910.8	0.4	2,880.6	0.8
P0201	6,925.0	11.0	6,925.0	14.9	6,948.4	10.4	6,948.4	10.5
P0282	177,399.6	14.3	178,174.5	18.6	178,542.1	16.6	178,176.5	26.8
P0291	2,376.1	12.86	2,362.2	16.0	2,465.7	20.2	2,276.3	21.8
P0548	498.0	11.8	316.7	14.6	1,576.9	24.8	316.7	25.2
P2756	2,691.0	105.3	2,691.0	111.6	2,700.8	282.1	2,691.0	269.1

Next, we compared the fractional S-K cuts using the strategy FSKC-S1 with some well-known surrogate-knapsack (S-K) cuts (see Glover, Sherali, and Lee [8]) and lifted cover (LC) cuts (see Nemhauser and Wolsey [10]). Table 4.3, reports the final objective values obtained at the root node in each case, along with the number of cuts generated. Note that FSKS-S1 sometimes performs as good as the strategres of using S-K cuts and the LC cuts, although it requires more computational time and generates more cuts. However, these results indicate that Gomory's fractional cuts when strengthened using surrogate multiplier techniques as in Theorem 2 of [8] can yield quite a competitive performance. Furthermore, noting the relative performance of the S-K cuts in Table 4.3, along with the fact that this class of cuts is also derived through an appropriate scheme of selecting surrogate multipliers and then applying the Chvatal-Gomory rounding step, we are encouraged to recommend further research on deriving more effective procedures for deriving surrogate multipliers within such a framework.

Table 4.3. Comparison of surrogate-knapsack enhanced Gomory cuts with surrogate-knapsack and lifted cover cuts

Problem	FSKC - S1			S – K Cuts			LG – Cuts		
	Z_{root}	Num. Cuts	Elapsed Time	Z_{root}	Num. Cuts	Elapsed Time	Z_{root}	Num. Cuts	Elapsed Time
P0033	2,910.8	48	0.4	2,902.6	15	0.1	2,916.2	13	0.2
P0201	6,948.4	161	10.4	7,075.0	3	0.8	7,075.0	2	0.9
P0282	178,542.1	204	16.6	252,356.0	89	2.5	180,999.7	58	0.2
P0291	2,465.7	248	20.2	5,009.2	28	1.0	1,873.8	25	0.3
P0548	1,576.9	184	24.8	3,883.7	158	8.1	4,052.9	138	2.5
P2756	2,700.8	126	282.1	2,701.8	75	16.4	2,701.7	68	10.5

5. CONCLUSIONS

In this paper, we have explored a new class of fractional S-K cuts and have compared its performance with that of other cuts including Gomory's fractional cut. Our results reveal an advantage of applying the surrogate-knapsack analysis to Gomory's fractional cut. However, we need to further investigate other multiplier adjustment methods to obtain more dominant results. This effort includes finding other measures for evaluating the quality of a cut. Ultimately, it is of interest to study the added benefit of implementing fractional S-K cuts within a branch-and-cut framework, in lieu of generating such cuts at the root node also.

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