Journal of the Korean Statistical Society (2002), **31**: *4, pp 491-507*

Balking Phenomenon in the $M^{[x]}/G/1$ Vacation Queue

Kailash C. Madan[1]

ABSTRACT

We analyze a single server bulk input queue with optional server vacations under a single vacation policy and balking phenomenon. The service times of the customers as well as the vacation times of the server have been assumed to be arbitrary (general). We further assume that not all arriving batches join the system during server's vacation periods. The supplementary variable technique is employed to obtain time-dependent probability generating functions of the queue size as well as the system size in terms of their Laplace transforms. For the steady state, we obtain probability generating functions of the queue size as well as the system size, the expected number of customers and the expected waiting time of the customers in the queue as well as the system, all in explicit and closed forms. Some special cases are discussed and some known results have been derived.

Keywords. Compound Poisson arrival process, Bernoulli schedule server vacations, single vacation policy, balking, general service times, general vacation times, steady state.
AMS 2000 subject classifications. Primary 60K25; Secondary 90B22.

1. Introduction

A number of researchers including Levy and Yechiali [13], Fuhrman [9], Doshi [7, 8], Keilson and Servi [10], Cramer [6], Madan [14, 15], Choi and Park [4], Takagi [18, 19] and many others have studied vacation queues with different vacation policies with single or multiple server vacations. The different vacation policies include Bernoulli schedules, exhaustive service, generalized vacations, among others. In the present paper, we study a batch arrival vacation queue $M^{[x]}/G/1$ with a balking phenomenon. Batch arrival vacation queues have also been studied by many authors including Baba [1], Rosenberg and Yechiali [17], Lee *et al.* [11, 12],

Received April 2002; accepted August 2002.
[1]Department of Statistics, Yarmouk University, Irbid, Jordan (e-mail : Kailashm@yu.edu.jo)

Chae and Lee [3], Borthakur and Choudhury [2] and Choudhury [5]. In our system $M^{[x]}/G/1$, we assume optional server vacations based on Bernoulli schedules, which means that on completion of each service the server may take a vacation or continue staying in the system. Under the Bernoulli schedules, it is not that the server tosses a coin at the completion of each service and either takes a vacation or not. Instead, this kind of policy has wider implications. Under this option, the server is not obliged to always take a vacation ($p = 1$) after completing each service or never to take a vacation ($p = 0$).

Rather, this flexible option may enable him to take a vacation for a preventive maintenance including an overhauling of the system or being sent to take another more important task. Queueing systems that allow the server to be on vacation arise in many computer and communication systems. The server may require a vacation due to lack of work, server failure, preventive maintenance or another task being assigned to the server. Some of the applications which can be modeled using these vacation systems are computer maintenance and testing, CPU scheduling, priority queues, polling systems or cyclic queues.

Under multiple vacation policy, it is often assumed that on returning back from a vacation if the server finds the system empty then he takes another vacation. But unlike this assumption of repeated vacations, we assume that whenever the server takes a vacation, it is always a single vacation. Our other key assumption is the balking phenomenon during periods of server vacations under which we assume that not all arriving batches join the system during the server vacations. One may encounter many such queueing situations in which batches of customers may go back elsewhere on finding the server missing from the system. We further assume that both, the service times of customers as well as the vacation times of the server, have arbitrary (general) distributions. The mathematical model of our study is briefly described in the next section.

2. The Mathematical Model

Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i dt$ $(i = 1, 2, 3, \dots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$ where $0 \le c_i \le 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. Customers are provided one by one service on a 'first come-first served' basis and their service time S follows a general (arbitrary) distribution with distribution function $G(s)$ and the density function $g(s)$. Let $\mu(x)dx$ be the conditional probability of completion of a service during the interval $(x,\ x+dx]$, given that the elapsed service time is x, so that

$$\mu(x) = \frac{g(x)}{1-G(x)}, \tag{1}$$

and, therefore,

$$g(s) = \mu(s)e^{-\int_0^s \mu(x)dx}. \tag{2}$$

As soon as the service of a customer is complete, then with probability p the server may decide to take a vacation or with probability $1-p$ he may decide to continue to be available for the next service. The server's vacation time V also follows a general (arbitrary) distribution with distribution function $B(v)$ and the density function $b(v)$. Let $\beta(x)dx$ be the conditional probability of completion of a vacation during the interval $(x,\ x+dx]$, given that the elapsed vacation time is x, so that

$$\beta(x) = \frac{b(x)}{1-B(x)}, \tag{3}$$

and, therefore,

$$b(v) = \beta(v)e^{-\int_0^v \beta(x)dx}. \tag{4}$$

Next, we assume that during server's vacation period, not all arriving batches will join the queue. It has been assumed that during server's vacation period an arriving batch will join the queue with probability π and balks (leaves as soon as it arrives) with probability $1-\pi$. Further, we assume that all stochastic processes involved in the system are independent of each other.

3. Definitions and Equations

We assume that $W_n(x,t)$ is the probability that at time t there are $(n \geq 0)$ customers in the queue excluding one customer in service and the elapsed time of this customer is x. Accordingly, $W_n(t) = \int_0^\infty W_n(x,t)dx$ denotes the probability that at time t there are $(n \geq 0)$ customers in the queue excluding one customer in service irrespective of the value of x. Next, we let $V_n(x,t)$ to be the probability that at time t there are $(n \geq 0)$ customers in the queue and the server is on vacation with elapsed vacation time x. Accordingly, $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are $(n \geq 0)$ customers in the queue and the server is on vacation irrespective of the value of x. Further, we define $P_n(t) = W_n(t)+V_n(t)$ as the probability that at time t there are $(n \geq 0)$ customers in the queue irrespective of the state of the server. And finally, we let $Q(t)$ to be probability that at time t there is no customer in the system and the server is idle. In addition, we define the following probability generating functions (pgf's):

$$W_q(x,z,t)=\sum_{n=0}^{\infty}W_n(x,t)z^n,\ \ W_q(z,t)=\sum_{n=0}^{\infty}W_n(t)z^n,$$

$$V_q(x,z,t)=\sum_{n=0}^{\infty}V_n(x,t)z^n,\ \ V_q(z,t)=\sum_{n=0}^{\infty}V_n(t)z^n,$$

$$P_q(z,t)=W_q(z,t)+V_q(z,t)=\sum_{n=0}^{\infty}\left\{W_n(t)+V_n(t)\right\}z^n,$$

$$C(z)=\sum_{i=1}^{\infty}c_iz^i,\ \ |z|\le 1. \tag{5}$$

Then connecting states of the system at time $t+dt$ with those at time t and collecting all mutually exclusive cases, we obtain the following set of difference-differential equations:

$$\frac{\partial}{\partial x}W_n(x,t)+\frac{\partial}{\partial t}W_n(x,t)+\{\lambda+\mu(x)\}W_n(x,t)=\sum_{i=1}^{n}\lambda c_iW_{n-i}(x,t),\ \ n\ge 1, \tag{6}$$

$$\frac{\partial}{\partial x}W_0(x,t)+\frac{\partial}{\partial t}W_0(x,t)+\{\lambda+\mu(x)\}W_0(x,t)=0, \tag{7}$$

$$\frac{\partial}{\partial x}V_n(x,t)+\frac{\partial}{\partial t}V_n(x,t)+\{\lambda\pi+\beta(x)\}V_n(x,t)=\sum_{i=1}^{n}\lambda\pi c_iV_{n-i}(x,t),\ \ n\ge 1, \tag{8}$$

$$\frac{\partial}{\partial x}V_0(x,t)+\frac{\partial}{\partial t}V_0(x,t)+\{\lambda\pi+\beta(x)\}V_0(x,t)=0, \tag{9}$$

$$\frac{\partial}{\partial t}Q(t)+\lambda Q(t)=(1-p)\int_0^{\infty}W_0(x,t)\mu(x)dx+\int_0^{\infty}V_0(x,t)\beta(x)dx. \tag{10}$$

Equations (6) through (10) are to be solved subject to the following boundary conditions:

$$\begin{aligned}W_n(0,t)&=(1-p)\int_0^{\infty}W_{n+1}(x,t)\mu(x)dx\\&\quad+\int_0^{\infty}V_{n+1}(x,t)\beta(x)dx+\lambda c_{n+1}Q(t),\ \ n\ge 1,\end{aligned} \tag{11}$$

$$W_0(0,t)=(1-p)\int_0^{\infty}W_1(x,t)\mu(x)dx+\int_0^{\infty}V_1(x,t)\beta(x)dx+\lambda c_1Q(t), \tag{12}$$

$$V_n(0,t) = p\int_0^\infty W_n(x,t)\mu(x)dx, \ n \geq 0. \tag{13}$$

We assume that the system starts when there is no customer in the queue and the server is idle so that the initial conditions are

$$Q(0) = 1, \ W_n(x,0) = V_n(x,0) = 0, \ n \geq 0. \tag{14}$$

Further, we define the Laplace transform (LT) of a function $f(t)$ as

$$LT(f(t)) = f^*(s) = \int_0^\infty e^{-st} f(t)dt, \ \operatorname{Re}(s) > 0, \tag{15a}$$

$$LT\left(\frac{d}{dt}f(t)\right) = sf^*(s) - f(0). \tag{15b}$$

4. Time-dependent PGF of the Queue Size

We take LT of equations (6) through (13), use (14), (15a), (15b) and simplify. Thus we obtain

$$\frac{\partial}{\partial x}W_n^*(x,s) + \{s+\lambda+\mu(x)\}W_n^*(x,s) = \sum_{i=1}^{n}\lambda c_i W_{n-i}^*(x,s), \ n \geq 1, \tag{16}$$

$$\frac{\partial}{\partial x}W_0^*(x,s) + \{s+\lambda+\mu(x)\}W_0^*(x,s) = 0, \tag{17}$$

$$\frac{\partial}{\partial x}V_n^*(x,s) + \{s+\lambda\pi+\beta(x)\}V_n^*(x,s) = \sum_{i=1}^{n}\lambda\pi c_i V_{n-i}^*(x,s), \ n \geq 1, \tag{18}$$

$$\frac{\partial}{\partial x}V_0^*(x,s) + \{s+\lambda\pi+\beta(x)\}V_0^*(x,s) = 0, \tag{19}$$

$$(s+\lambda)Q^*(s) = 1 + (1-p)\int_0^\infty W_0^*(x,s)\mu(x)dx + \int_0^\infty V_0^*(x,s)\beta(x)dx, \tag{20}$$

$$\begin{aligned} W_n^*(0,s) &= (1-p)\int_0^\infty W_{n+1}^*(x,s)\mu(x)dx + \int_0^\infty V_{n+1}^*(x,s)\beta(x)dx \\ &\quad + \lambda c_{n+1}Q^*(s), \ n \geq 1, \end{aligned} \tag{21}$$

$$\begin{aligned} W_0^*(0,s) &= (1-p)\int_0^\infty W_1^*(x,s)\mu(x)dx \\ &\quad + \int_0^\infty V_1^*(x,s)\beta(x)dx + \lambda c_1 Q^*(s), \end{aligned} \tag{22}$$

$$V_n^*(0,s) = p\int_0^\infty W_n^*(x,s)\mu(x)dx, \ n \geq 0. \tag{23}$$

We multiply both sides of equation (16) by z^n, take summation over n from 1 to ∞ and add (17) to the result and use (5) and simplify. Thus we obtain

$$\frac{\partial}{\partial x}W_q^*(x,z,s)+[s+\lambda\{1-C(z)+\mu(x)\}]W_q^*(x,z,s)=0. \tag{24}$$

A similar operation on equations (18) and (19) yields

$$\frac{\partial}{\partial x}V_q^*(x,z,s)+[s+\lambda\pi\{1-C(z)+\beta(x)\}]V_q^*(x,z,s)=0. \tag{25}$$

And yet again we perform a similar operation on (21) and (22) and once again on (23) alone. Thus we obtain

$$\begin{aligned} zW_q^*(0,z,s) &= (1-p)\int_0^\infty W_q^*(x,z,s)\mu(x)dx+\int_0^\infty V_q^*(x,z,s)\beta(x)dx \\ &\quad -(1-p)\int_0^\infty W_0^*(x,s)\mu(x)dx-\int_0^\infty V_0^*(x,s)\beta(x)dx \\ &\quad +\lambda C(z)Q^*(s), \end{aligned} \tag{26}$$

$$V_q^*(0,z,s)=p\int_0^\infty W_q^*(x,z,s)\mu(x)dx. \tag{27}$$

Using (20), equation (26) can be re-written as

$$\begin{aligned} zW_q^*(0,z,s) &= (1-p)\int_0^\infty W_q^*(x,z,s)\mu(x)dx+\int_0^\infty V_q^*(x,z,s)\beta(x)dx \\ &\quad +[\lambda\{C(z)-1\}-s]\,Q^*(s)+1, \end{aligned} \tag{28}$$

Now, we integrate equations (24) and (25) with respect to x and obtain

$$W_q^*(x,z,s)=W_q^*(0,z,s)\exp\left\{-[s+\lambda\{1-C(z)\}]x-\int_0^x\mu(t)dt\right\}, \tag{29}$$

$$V_q^*(x,z,s)=V_q^*(0,z,s)\exp\left\{-[s+\lambda\pi\{1-C(z)\}]x-\int_0^x\beta(t)dt\right\}, \tag{30}$$

where $W_q^*(0,z,s)$ and $V_q^*(0,z,s)$ are given by (28) and (27) respectively.

We again integrate equations (29) and (30) by parts and use (2) and (4). Thus we obtain

$$W_q^*(z,s)=W_q^*(0,z,s)\left\{\frac{1-G^*[s+\lambda\{1-C(z)\}]}{s+\lambda\{1-C(z)\}}\right\}, \tag{31}$$

$$V_q^*(z,s)=V_q^*(0,z,s)\left\{\frac{1-B^*[s+\lambda\pi\{1-C(z)\}]}{s+\lambda\pi\{1-C(z)\}}\right\}, \tag{32}$$

where

$$G^*\left[s+\lambda\{1-C(z)\}\right] = \int_0^\infty \exp\Big\{-[s+\lambda\{1-C(z)\}]x\Big\}dG(x),$$
$$B^*\left[s+\lambda\pi\{1-C(z)\}\right] = \int_0^\infty \exp\Big\{-[s+\lambda\pi\{1-C(z)\}]x\Big\}dB(x)$$

are the Laplace-Stieltjes Transforms of the service time and the vacation time respectively.

Now, we multiply equation (29) and (30) by $\mu(x)$ and $\beta(x)$ respectively and integrate with respect to x, use (2) and (4). Thus we obtain

$$\int_0^\infty W_q^*(x,z,s)\mu(x)dx = W_q^*(0,z,s)G^*\left[s+\lambda\{1-C(z)\}\right], \tag{33}$$
$$\int_0^\infty V_q^*(x,z,s)\beta(x)dx = V_q^*(0,z,s)B^*\left[s+\lambda\pi\{1-C(z)\}\right]. \tag{34}$$

Using equations (33) and (34) into (27), (28), we obtain on simplifying

$$V_q^*(0,z,s) = pW_q^*(0,z,s)G^*\Big[s+\lambda\{1-C(z)\}\Big], \tag{35}$$

$$\begin{aligned}&\Big\{z-(1-p)G^*\left[s+\lambda\{1-C(z)\}\right]\Big\}W_q^*(0,z,s)\\ &= V_q^*(0,z,s)B^*\left[s+\lambda\pi\{1-C(z)\}\right] + \left[\lambda\{C(z)-1\}-s\right]Q^*(s)+1.\end{aligned} \tag{36}$$

Next, we substitute for $V_q^*(0,z,s)$ from (35) into (36), and obtain on simplifying

$$\begin{aligned}W_q^*(0,z,s) = &\Big\{\left[\lambda\{C(z)-1\}-s\right]Q^*(s)+1\Big\}\\ &\div\Big\{z-(1-p)G^*\left[s+\lambda\{1-C(z)\}\right]\\ &-pG^*\left[s+\lambda\{1-C(z)\}\right]B^*\left[s+\lambda\pi\{1-C(z)\}\right]\Big\}.\end{aligned} \tag{37}$$

Then using (37) into (35) we have

$$\begin{aligned}V_q^*(0,z,s) = &\Big\{pG^*\left[s+\lambda\{1-C(z)\}\right]\{[\lambda\{C(z)-1\}-s]Q^*(s)+1\}\Big\}\\ &\div\Big\{z-(1-p)G^*\left[s+\lambda\{1-C(z)\}\right]\\ &-pG^*\left[s+\lambda\{1-C(z)\}\right]B^*\left[s+\lambda\pi\{1-C(z)\}\right]\Big\}.\end{aligned} \tag{38}$$

Now, we use (37), (38) into (31), (32) respectively and obtain

$$\begin{aligned} W_q^*(z,s) &= \Big[K_1^*(z,s)\Big\{[\lambda\{C(z)-1\}-s]Q^*(s)+1\Big\}\Big] \\ &\div\Big\{z-(1-p)G^*\left[s+\lambda\{1-C(z)\}\right] \\ &-pG^*\left[s+\lambda\{1-C(z)\}\right]B^*\left[s+\lambda\pi\{1-C(z)\}\right]\Big\}, \end{aligned} \tag{39}$$

$$\begin{aligned} V_q^*(z,s) &= \Big[K_2^*(z,s)pG^*\left[s+\lambda\{1-C(z)\}\right]\Big\{[\lambda\{C(z)-1\}-s]Q^*(s)+1\Big\}\Big] \\ &\div\Big\{z-(1-p)G^*\left[s+\lambda\{1-C(z)\}\right] \\ &-pG^*\left[s+\lambda\{1-C(z)\}\right]B^*\left[s+\lambda\pi\{1-C(z)\}\right]\Big\}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} K_1^*(z,s) &= \frac{1-G^*\left[s+\lambda\{1-C(z)\}\right]}{s+\lambda\{1-C(z)\}}, \\ K_2^*(z,s) &= \frac{1-B^*\left[s+\lambda\pi\{1-C(z)\}\right]}{s+\lambda\pi\{1-C(z)\}}. \end{aligned}$$

Further, adding (39) and (40) we obtain

$$P_q^*(z,s) = W_q^*(z,s) + V_q^*(z,s), \tag{41}$$

Now, we have to determine the unknown probability $Q^*(s)$ which appears in the numerators of the right hand sides of (39) and (40). It is easy to prove by Rouche's theorem that the denominator of the right hand side of (39) or (40) has one zero on or inside $|z| = 1$. Let this zero be denoted as z^*. Then the numerator of the right hand side of (39) or (40) must vanish for this zero, giving us $Q^*(s) = \left[s-\lambda\left\{C(z^*)-1\right\}\right]^{-1}$. Substituting this value of $Q^*(s)$ into (39) and (40) we have now completely determined the *pgf*'s $W_q^*(z,s)$ and $V_q^*(z,s)$ and, for that matter, the *pgf* $P_q^*(z,s)$.

5. Steady State PGF of the Queue Size and the System Size

Assuming that the steady state exists, we let $\lim_{t\to\infty} W_n(t) = W_n$, $\lim_{t\to\infty} V_n(t) = V_n$ and $\lim_{t\to\infty} Q(t) = Q$. Thus W_n, V_n and Q are the steady state probabilities corresponding to $W_n(t)$, $V_n(t)$ and $Q(t)$ respectively and let $W_q(z)$, $V_q(z)$ and $P_q(z)$ be the steady state *pgf*s of the queue size corresponding to the time-dependent *pgf*'s $W_q(z,t)$, $V_q(z,t)$ and $P_q(z,t)$ defined earlier in equation (5).

To derive the steady state results, we shall now apply the well-known Tauberian property

$$\lim_{s\to 0} sf^*(s) = \lim_{t\to\infty} f(t), \tag{42}$$

provided the limits exist. Thus we obtain from (39) and (40)

$$\begin{aligned}
W_q(z) &= \lim_{s\to 0} sW_q^*(z,s)\\
&= \Big\{ \lim_{s\to 0} s\big[K_1^*(z,s)\big\{[\lambda\{C(z)-1\}-s]Q^*(s)+1\big\}\big]\Big\}\\
&\quad \div\Big[\lim_{s\to 0}\{z-(1-p)G^*[s+\lambda\{1-C(z)\}]\\
&\quad -pG^*[s+\lambda\{1-C(z)\}]\,B^*[s+\lambda\pi\{1-C(z)\}]\,\}\Big]\\
&= \Big[\lim_{s\to 0} sK_1^*(z,s)\lim_{s\to 0} s\big\{[\lambda\{C(z)-1\}-s]Q^*(s)+1\big\}\Big] \qquad (43)\\
&\quad \div\Big[\lim_{s\to 0}\{z-(1-p)G^*[s+\lambda\{1-C(z)\}]\\
&\quad -pG^*[s+\lambda\{1-C(z)\}]\,B^*[s+\lambda\pi\{1-C(z)\}]\,\}\Big]\\
&= \Big[K_1(z)\lambda\{C(z)-1\}Q\Big]\\
&\quad \div\Big\{z-(1-p)G^*[\lambda\{1-C(z)\}]\\
&\quad -pG^*[\lambda\{1-C(z)\}]\,B^*[\lambda\pi\{1-C(z)\}]\Big\},\\
V_q(z) &= \lim_{s\to 0} sV_q^*(z,s)\\
&= \Big[\lim_{s\to 0} s\big\{K_2^*(z,s)pG^*[s+\lambda\{1-C(z)\}]\\
&\quad \times\{[\lambda\{C(z)-1\}-s]Q^*(s)+1\}\,\big\}\Big]\\
&\quad \div\Big[\lim_{s\to 0}\{z-(1-p)G^*[s+\lambda\{1-C(z)\}]\\
&\quad -pG^*[s+\lambda\{1-C(z)\}]\,B^*[s+\lambda\pi\{1-C(z)\}]\,\}\Big]\\
&= \Big\{\lim_{s\to 0} sK_2^*(z,s)\lim_{s\to 0} spG^*[s+\lambda\{1-C(z)\}] \qquad (44)\\
&\quad \times[\{\lambda(z-1)-s\}\,Q^*(s)+1]\Big\}\\
&\quad \div\Big[\lim_{s\to 0}\{z-(1-p)G^*[s+\lambda\{1-C(z)\}]\\
&\quad -pG^*[s+\lambda\{1-C(z)\}]\,B^*[s+\lambda\pi\{1-C(z)\}]\,\}\Big]
\end{aligned}$$

$$= \Big\{K_2(z)pG\left[\lambda\{1 - C(z)\}\right]\lambda(z-1)Q\Big\}$$
$$\div\Big\{z - (1-p)G^*\left[\lambda\{1 - C(z)\}\right]$$
$$-pG^*\left[\lambda\{1 - C(z)\}\right]B^*\left[\lambda\pi\{1 - C(z)\}\right]\Big\},$$

where

$$K_1(z) = \lim_{s\to 0} K_1^*(z,s) = \frac{1 - G^*\left[\lambda\{1 - C(z)\}\right]}{\lambda\{1 - C(z)\}},$$
$$K_2(z) = \lim_{s\to 0} K_2^*(z,s) = \frac{1 - B^*\left[\lambda\pi\{1 - C(z)\}\right]}{\lambda\pi\{1 - C(z)\}}.$$

Further adding (43) and (44) we have

$$\begin{aligned} &P_q(z) \\ &= W_q(z) + V_q(z) \\ &= \frac{\{K_1(z) + K_2(z)pG^*\left[\lambda\{1 - C(z)\}\right]\}\lambda(C(z)-1)Q}{z - (1-p)G^*\left[\lambda\{1 - C(z)\}\right] - pG^*\left[\lambda\{1 - C(z)\}\right]B^*\left[\lambda\pi\{1 - C(z)\}\right]}. \end{aligned} \tag{45}$$

Now we shall determine Q. For that purpose, we shall use the normalizing condition $P_q(1) + Q = 1$. However, since $W_q(z)$ in (43) and $V_q(z)$ in (44) are both indeterminate of the zero/zero form at $z = 1$ and hence we use L'Hospital's rule on (43) and obtain on simplifying

$$\begin{aligned} W_q(1) &= \lim_{z\to 1} W_q(z) \\ &= \frac{\lambda E(S)Q}{1 - \lambda E(S)E(I) - p\lambda\pi E(V)E(I)}. \end{aligned} \tag{46}$$

Similarly equation (44) yields

$$\begin{aligned} V_q(1) &= \lim_{z\to 1} V_q(z) \\ &= \frac{p\lambda E(V)Q}{1 - \lambda E(S)E(I) - p\lambda\pi E(V)E(I)}. \end{aligned} \tag{47}$$

Then we use (46) and (47) in the normalizing condition $W_q(1) + V_q(1) + Q = 1$ and obtain on simplifying

$$Q = \frac{1 - \lambda E(S)E(I) - p\lambda\pi E(V)E(I)}{1 + \lambda E(S) - \lambda E(S)E(I) + p\lambda E(V) - p\lambda\pi E(V)E(I)}, \tag{48}$$

which is the steady state probability that the server is idle.

Note that (48) also yields the condition for the existence of the steady state. This condition is given by

$$\lambda E(S)E(I) + p\lambda\pi E(V)E(I) < 1. \tag{49}$$

Next, we substitute for Q from (48) into (46) and (47) and obtain

$$W_q(1) = \frac{\lambda E(S)}{1 + \lambda E(S) - \lambda E(S)E(I) + p\lambda E(V) - p\lambda\pi E(V)E(I)}, \tag{50}$$

which is the steady state probability that the server is busy providing service.

$$V_q(1) = \frac{p\lambda E(V)}{1 + \lambda E(S) - \lambda E(S)E(I) + p\lambda E(V) - p\lambda\pi E(V)E(I)}, \tag{51}$$

which is the steady state probability that the server is away on vacation.

We further note that $W_q(1)$ found in equation (50) is the proportion of time the server remains busy in the system. Therefore, the system's utilization factor ρ is also given by (50).

After substituting the value of Q from (48) into (43) and (44) we have now completely and explicitly determined the *pgf*'s $W_q(z)$ and $V_q(z)$ and, for that matter, also the *pgf* $P_q(z)$ in (45).

Now, let $P_s(z)$ denote the steady state probability generating function of the system size. Then we have

$$P_s(z) = Q + zP_q(z), \tag{52}$$

where $P_q(z)$ and Q have been obtained in (45) and (48) respectively.

6. Steady State Expected Queue Size and Expected System Size

Let L_q denote the mean number of customers in the queue. Then, we have from (45), $L_q = (d/dz)P_q(z)$, at $z = 1$. We note that at $z = 1$, $P_q(z)$ in equation (45) is indeterminate of the 0/0 form. Therefore, to find L_q, we proceed as follows:

Let $P_q(z) = N(z)/D(z)$, where $N(z)$ and $D(z)$ denote the numerator and denominator of the right hand side of (45). Then we use double differentiation and obtain

$$\begin{aligned} L_q &= P_q'(1) = \lim_{z\to 1} \frac{d}{dz} P_q(z) \\ &= \lim_{z\to 1} \frac{D'(z)N''(z) - N'(z)D''(z)}{2\{D'(z)\}^2} \\ &= \frac{D'(1)N''(1) - N'(1)D''(1)}{2\{D'(1)\}^2}. \end{aligned} \tag{53}$$

We carry out the desired derivatives at $z = 1$ and after some algebra and simplification, we have

$$N'(1) = \lambda\{E(S) + E(V)\}Q, \tag{54}$$

$$N''(1) = \lambda^2 \left[E(I)E(S^2) + p\pi E(I)E(V^2) + 2pE(I)E(S)E(V)\right] Q, \tag{55}$$

$$D'(1) = 1 - \lambda E(I)\left\{E(S) + p\pi E(V)\right\}, \tag{56}$$

$$D''(1) = -\lambda \begin{bmatrix} \lambda(E(I))^2E(S^2) + p\lambda\pi^2\{E(I)\}^2E(V^2) + E(I(I-1))E(S) \\ +p\pi E(I(I-1)E(V) + 2p\lambda\pi\{E(I)\}^2E(S)E(V) \end{bmatrix}, \tag{57}$$

where Q is given by (48) and $E(S)$, $E(S^2)$ are respectively the first and second moments of service time, $E(V)$, $E(V^2)$ are respectively the first and second moments of vacation time and $E(I)$, $E(I(I-1))$ are respectively the average batch size and the second factorial moment of the batch size of arriving customers.

Note that in the above calculations, we have used the following facts:

$$K_1(1) = \lim_{z\to 1} K_1(z) = E(S), \; K_1'(1) = \lim_{z\to 1} K_1'(z) = \frac{\lambda E(I)E(S^2)}{2},$$

$$K_2(1) = \lim_{z\to 1} K_2(z) = E(V), \; K_2'(1) = \lim_{z\to 1} K_2'(z) = \frac{\lambda\pi E(I)E(V^2)}{2},$$

$$G^*[0] = 1, \; G^{*'}(0) = \lambda E(I)E(S),$$

$$G^{*''}(0) = \lambda^2((E(I))^2E(S^2) + \lambda E(I(I-1)E(S),$$

$$B^*[0] = 1, \; B^{*'}(0) = \lambda\pi E(I)E(V),$$

$$B^{*''}(0) = \lambda^2\pi^2((E(I))^2E(V^2) + \lambda\pi E(I(I-1)E(V).$$

On using (54), (55), (56) and (57) into (53) we have now determined L_q in explicit and closed form. Using the values of L_q from (53), we can easily find L, the expected number of customers in the system and also W_q and W, the mean waiting time in the queue and the system respectively by Little's formulas $L = L_q + \rho$, $W_q = L_q/\lambda_a$, where λ_a is the actual arrival rate of customers which is given by $\lambda_a = \lambda(W_q(1) + Q) + \lambda\pi V_q(1)$, where $W_q(1)$, $V_q(1)$ and ρ have already been found.

We can easily verify that when there are no server vacations then with $p = 0$, we have $\lambda_a = \lambda$, as it should be.

7. Special Cases

Case 1 : Single arrivals in an $M/G/1$ vacation queue with balking during vacations. In this case we let $c_1 = 1$, and $c_i = 0$ for $i \neq 1$. Consequently $C(z) = z$, $E(I) = 1$ and $E(I(I-1)) = 0$. With these values the main results yield

$$P_q(z) = \frac{\{[G^*\{\lambda(1-z)\} - 1] + p\,[B^*\{\lambda(1-z)\} - 1]\,G^*\{\lambda(1-z)\}\}\,Q}{z - (1-p)G^*\{\lambda(1-z)\} - pG^*\{\lambda(1-z)\}\,B^*\{\lambda\pi(1-z)\}}, \tag{58}$$

$$Q = \frac{1 - \lambda E(S) - p\lambda\pi E(V)}{1 + p\lambda E(V) - p\lambda\pi E(V)}, \tag{59}$$

$$\rho = \frac{\lambda E(S)}{1 + p\lambda E(V) - p\lambda\pi E(V)}, \tag{60}$$

provided

$$\lambda E(S) + p\lambda\pi E(V) < 1. \tag{61}$$

Further, L_q is given by (53), where

$$N'(1) = \lambda\{E(S) + E(V)\}Q, \tag{62}$$

$$N''(1) = \lambda^2\left\{E(S^2) + p\pi E(V^2) + 2pE(S)E(V)\right\}Q, \tag{63}$$

$$D'(1) = 1 - \lambda\left\{E(S + p\pi E(V)\right\}, \tag{64}$$

$$D''(1) = -\lambda^2\left\{E(S^2) + p\pi^2 E(V^2) + 2p\pi E(S)E(V)\right\}. \tag{65}$$

Case 2 : $M^{[x]}/G/1$ vacation queue when all arriving batches balk during vacations (No batch joins the system). In this case we have $\pi = 0$ and hence

$$K_2(z) = \lim_{\pi \to 0} \frac{1 - B^*\left[\lambda\pi\{1 - C(z)\}\right]}{\lambda\pi\{1 - C(z)\}} = E(V),$$

and therefore, with these substitutions in the main results we obtain

$$P_q(z) = \frac{\{K_1(z) + pE(V)G^*[\lambda\{1 - C(z)\}]\}\,\lambda\{C(z) - 1\}Q}{z - pG^*[\lambda\{1 - C(z)\}]}, \tag{66}$$

$$Q = \frac{1 - \lambda E(S)E(I)}{1 + \lambda E(S) - \lambda E(S)E(I) + p\lambda E(V)}, \tag{67}$$

$$\rho = \frac{\lambda E(S)}{1 + \lambda E(S) - \lambda E(S)E(I) + p\lambda E(V)}, \tag{68}$$

provided

$$\lambda E(S)E(I) < 1. \tag{69}$$

Further, L_q is given by (53), where

$$N'(1) = \lambda\{E(S) + E(V)\}Q, \quad (70)$$
$$N''(1) = \lambda^2 \left\{E(I)E(S^2) + 2pE(I)E(S)E(V)\right\} Q, \quad (71)$$
$$D'(1) = 1 - \lambda E(I)E(S), \quad (72)$$
$$D''(1) = -\lambda \left[\lambda\{E(I)\}^2 E(S^2) + E(I(I-1))E(S)\right]. \quad (73)$$

Case 3 : $M^{[x]}/G/1$ vacation queue when no arriving batch balks during vacation (All arriving batches join at all times). In this case we have $\pi = 1$ and hence

$$K_2(z) = \frac{1 - B^*\left[\lambda\{1 - C(z)\}\right]}{\lambda\{1 - C(z)\}},$$

and therefore, with these substitutions in the main results we obtain

$$P_q(z) = \frac{\{K_1(z) + K_2(z)pG^*[\lambda(1 - C(z))]\}\,\lambda\{C(z) - 1\}Q}{z - (1-p)G^*[\lambda\{1 - C(z)\}] - pG^*[\lambda\{1 - C(z)\}]B^*[\lambda\{1 - C(z)\}]}, \quad (74)$$
$$Q = \frac{1 - \lambda E(S)E(I) - p\lambda E(V)E(I)}{1 + \lambda E(S) - \lambda E(S)E(I) + p\lambda E(V) - p\lambda E(V)E(I)}, \quad (75)$$
$$\rho = \frac{\lambda E(S)}{1 + \lambda E(S) - \lambda E(S)E(I) + p\lambda E(V) - p\lambda E(V)E(I)}, \quad (76)$$

provided

$$\lambda E(S)E(I) + p\lambda E(V)E(I) < 1. \quad (77)$$

Further, L_q is given by (53), where

$$N'(1) = \lambda\{E(S) + E(V)\}Q, \quad (78)$$
$$N''(1) = \lambda^2 \left\{E(I)E(S^2) + pE(I)E(V^2) + 2pE(I)E(S)E(V)\right\} Q, \quad (79)$$
$$D'(1) = 1 - \lambda E(I)\left\{E(S + pE(V))\right\}, \quad (80)$$
$$D''(1) = -\lambda \left[\begin{array}{c} \lambda\{E(I)\}^2 E(S^2) + p\lambda\{E(I)\}^2 E(V^2) + E(I(I-1))E(S) \\ +pE(I(I-1))E(V) + 2p\lambda\{E(I)\}^2 E(S)E(V) \end{array}\right]. \quad (81)$$

Case 4 : $M^{[x]}/G/1$ queue with no server vacations. In this case we let $p = 0$ in the main results and obtain on simplifying

$$P_q(z) = \frac{\{G^*[\lambda\{1 - C(z)\}] - 1\}\,Q}{z - G^*[\lambda\{1 - C(z)\}]}, \quad (82)$$

$$Q = \frac{1-\lambda E(S)E(I)}{1+\lambda E(S)-\lambda E(S)E(I)}, \tag{83}$$

$$\rho = \frac{\lambda E(S)}{1+\lambda E(S)-\lambda E(S)E(I)}, \tag{84}$$

$$\lambda E(S)E(I) < 1. \tag{85}$$

Also using (82) and (83) into (52) and simplifying we can find the *pgf* of the system size

$$\begin{aligned} P_s(z) &= Q + zP_q(z) \\ &= \frac{(z-1)}{z-G^*[\lambda\{1-C(z)\}]}\left\{\frac{1-\lambda E(S)E(I)}{1+\lambda E(S)-\lambda E(S)E(I)}\right\}. \end{aligned} \tag{86}$$

Further, L_q is given by (53), where

$$N'(1) = \lambda\{E(S)+E(V)\}Q, \tag{87}$$

$$N''(1) = \lambda^2\left\{E(I)E(S^2)\right\}Q, \tag{88}$$

$$D'(1) = 1-\lambda E(I)E(S), \tag{89}$$

$$D''(1) = -\lambda\left[\lambda\{E(I)\}^2E(S^2)+E(I(I-1))E(S)\right]. \tag{90}$$

Case 5 : $M/G/1$ queue with single arrivals and no server vacations. In this we have $C(z) = z$ and $E(I) = 1$ and $E(I(I-1)) = 0$, as in case 1. With these substitutions in the results of case 4, we obtain

$$P_q(z) = \frac{[G^*\{\lambda(1-z)\}-1]\,Q}{z-G^*\{\lambda(1-z)\}}, \tag{91}$$

$$Q = 1-\lambda E(S), \tag{92}$$

$$\rho = \lambda E(S), \tag{93}$$

$$\lambda E(S) < 1, \tag{94}$$

$$\begin{aligned} P_s(z) &= Q + zP_q(z) \\ &= \frac{(z-1)\{1-\lambda E(S)\}}{z-G^*\{\lambda(1-z)\}}. \end{aligned} \tag{95}$$

Further, L_q is given by (53), where

$$N'(1) = \lambda\{E(S)+E(V)\}Q, \tag{96}$$

$$N''(1) = \lambda^2E(S^2)Q, \tag{97}$$

$$D'(1) = 1-\lambda E(S), \tag{98}$$

$$D''(1) = -\lambda^2E(S^2). \tag{99}$$

Using (96) to (99) into (53) and simplifying we obtain

$$L_q = \frac{\lambda^2 E(S^2)}{2\{1 - \lambda E(S)\}}. \tag{100}$$

Note that the results in (91) to (95) and (100) are all known results of the $M/G/1$ queue (see Medhi [16]).

Acknowledgments

The author wishes to convey his sincere thanks and gratitude to the referees and the editors for their valuable comments and suggestions to revise the paper in the present form.

References

[1] Baba, Y. (1986). "On the $M^X/G/1$ queue with vacation time", *Operations Research Letters*, **5**, 93–98.

[2] Borthakur, A. and Choudhury, G. (1997). "On a batch arrival Poisson queue with generalized vacation", *Sankhyā*, **B59**, 369–383.

[3] Chae, K. C. and Lee, H. W. (1995). "$M^X/G/1$ vacation models with N-policy : Heuristic interpretation of the mean waiting time", *Journal of the Operational Research Society*, **46**, 258–264.

[4] Choi, B. D. and Park, K. K. (1990). "The $M/G/1$ retrial queue with Bernoulli schedule", *Queueing Systems*, **7**, 219–228.

[5] Choudhury, G. (2000). "An $M^X/G/1$ queueing system with a set up period and a vacation period", *Queueing Systems*, **36**, 23–38.

[6] Cramer, M. (1989). "Stationary distributions in a queueing system with vacation times and limited service", *Queueing Systems*, **4**, 57–68.

[7] Doshi, B. T. (1986). "Queueing systems with vacations a survey", *Queueing Systems*, **1**, 29–66.

[8] Doshi, B. T. (1990). "Conditional and unconditional distributions for $M/G/1$ type queues with server vacation", *Queueing Systems*, **7**, 229–252.

[9] Fuhrman, S. (1981). "A note on the $M/G/1$ queue with server vacations", *Operations Research*, **31**, 1368.

[10] Keilson, J. and Servi, L. D. (1986). "Oscillating random walk models for $GI/G/1$ vacation systems with Bernoulli schedules", *Journal of Applied Probability*, **23**, 790–802.

[11] Lee, H. W., Lee, S. S. and Chae, K. C. (1994). "Operating characteristics of $M^X/G/1$ queue with N-policy", *Queueing Systems*, **15**, 387–399.

[12] Lee, H. W., Lee, S. S., Park, J. O. and Chae, K. C. (1994). "Analysis of $M^X/G/1$ queue with N-policy and multiple vacations", *Journal of the Operational Research Society*, **31**, 467–496.

[13] Levi, Y. and Yechiali, U. (1976). "An $M/M/s$ queue with server's vacations", *Information Systems and Operational Research*, **14**, 153–163.

[14] Madan, Kailash C. (1999). "An $M/G/1$ queue with optional deterministic server vacations", *Metron*, **57**, 83–95.

[15] Madan, Kailash C. (2001). "On a single server queue with two-stage heterogenous service and deterministic server vacations", *International Journal of Systems Science*, **32**, 837–844.

[16] Medhi, J. (1982). *Stochastic Processes*, Wiley, Eastern.

[17] Rosenberg, E. and Yechiali, U. (1993). "The $M^X/G/1$ queue with single and multiple vacations under LIFO service regime", *Operations Research Letters*, **14**, 171–179.

[18] Takagi, H. (1991). *Queueing Analysis : A Foundation of Performance Evaluation, Vol. 1*, North Holand, Amsterdam.

[19] Takagi, H. (1992). "Time dependent process of $M/G/1$ vacation models with exhaustive service", *Journal of Applied Probability*, **29**, 418–429.