Journal of the Korean Statistical Society (2002), **31**: *4, pp 483–490*

A Weak Convergence for a Linear Process with Positive Dependent Sequences[†]

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ABSTRACT

A weak convergence is obtained for a linear process of the form $X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}$, where $\{\epsilon_t\}$ is a strictly stationary sequence of associated random variables with $E\epsilon_t = 0$ and $E\epsilon_t^2 < \infty$ and $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$. We also apply this idea to the case of linearly positive quadrant dependent sequence.

Keywords. Weak convergence, linear process, associated random variables, linearly positive quadrant dependent random variables, Wiener process.
AMS 2000 subject classifications. Primary 60F05; Secondary 60G10.

1. Introduction

Consider a linear process of the form

$$X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j} \tag{1.1}$$

defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\{\epsilon_t\}$ is a strictly stationary sequence of random variables with $E\epsilon_t = 0$ and $E\epsilon_t^2 < \infty$ and $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$. The linear processes have found in time-series analysis and they arise in a wide variety of contexts (see *e.g.* Hannan, 1970). Many important time-series models, such as ARMA process (see *e.g.* Brockwell and Davis, 1987), have the form (1.1) with $\sum_{j=0}^{\infty} |a_j| < \infty$.

Received May 2002; accepted August 2002.
[†]This paper was supported by research fund of Chungwoon University in 2002.
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A finite collection of random variables $\{\epsilon_1, \ldots, \epsilon_m\}$ is said to be associated if

$$\mathrm{Cov}\{f(\epsilon_1, \ldots, \epsilon_m), g(\epsilon_1, \ldots, \epsilon_m)\} \geq 0$$

for any two coordinatewise nondecreasing functions f, g on $\mathbf{R}^m$ such that the covariance is defined. An infinite collection of random variables is associated if every finite subcollection is associated. This concept was introduced by Esary, Proschan and Walkup (1967). Many authors also have studied this concept providing interesting results and applications. See for example, Newman and Wright (1981), Newman (1984), Cox and Grimmett (1984), Birkel (1988), Roussas (1994) and Matula (1998).

In this paper we prove a weak convergence (see Theorem 1.1) for a linear process of the form (1.1) generated by a strictly stationary sequence of associated random variables, which is a generalization of Newman and Wrights' (1981) invariance principle to the linear process. We also apply this idea to the linearly positive quadrant dependent sequence.

Theorem 1.1. *Let $\{a_j\}$ be a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$ and $\{\epsilon_t\}$ a stationary sequence of associated random variables with $E\epsilon_t = 0$ and $E\epsilon_t^2 < \infty$ and satisfying*

$$0 < \sigma^2 = E\epsilon_1^2 + 2\sum_{t=2}^{\infty} E(\epsilon_1 \epsilon_t) < \infty. \tag{1.2}$$

Let X_t be a linear process of the form (1) and $\tau^2 = (\sum_{j=0}^{\infty} a_j)^2 \sigma^2$. Then, we have, for $s \in [0,1]$

$$\frac{1}{\tau\sqrt{n}} \sum_{t=1}^{[ns]} X_t \Rightarrow W, \tag{1.3}$$

where $\Rightarrow$ denotes weak convergence and W denotes a standard Wiener process.

2. Proof of Theorem

Newman and Wright (1981, Theorems 2 and 3) showed the following: If $\{\epsilon_t\}$ is a strictly stationary sequence of associated random variables with $E\epsilon_t = 0$ and $E\epsilon_t^2 < \infty$ and satisfying (1.2) then

$$E \max_{1 \leq k \leq m} \left(\sum_{i=1}^{k} \epsilon_i \right) \leq E(\epsilon_1 + \cdots + \epsilon_m)^2 \tag{2.1}$$

and

$$\frac{1}{\sigma\sqrt{n}}\sum_{t=1}^{[ns]}\epsilon_t \Rightarrow W. \tag{2.2}$$

Proof of Theorem 1.1. For every fixed m, put

$$X_{1,t}=\sum_{j=0}^{m}a_j\epsilon_{t-j} \quad \text{and} \quad X_{2,t}=\sum_{j=m+1}^{\infty}a_j\epsilon_{t-j}. \tag{2.3}$$

From Fuller (1996) we have, for any $n\geq 1$,

$$\begin{aligned}\sum_{t=1}^{n}X_{1,t} &= \sum_{t=1}^{n}\sum_{j=0}^{m}a_j\epsilon_{t-j}\\ &= \sum_{j=0}^{m}a_j\sum_{t=1}^{n}\epsilon_t+\sum_{k=1}^{m}\epsilon_{1-k}\sum_{j=k}^{m}a_j-\sum_{k=0}^{m-1}\epsilon_{n-k}\sum_{j=k+1}^{m}a_j\\ &= \sum_{j=0}^{m}a_j\sum_{t=1}^{n}\epsilon_t+I-II, \end{aligned} \tag{2.4}$$

where $I=\sum_{k=1}^{m}\epsilon_{1-k}\sum_{j=k}^{m}a_j$, $II=\sum_{k=0}^{m-1}\epsilon_{n-k}\sum_{j=k+1}^{m}a_j$.

Thus it follows from (2.2), (2.3) and (2.4) that, for every fixed $m\geq 1$ and $s\in[0,1]$,

$$\frac{1}{\tau\sqrt{n}}\sum_{t=1}^{[ns]}X_t=\left(\sum_{j=0}^{m}a_j\right)\frac{1}{\tau\sqrt{n}}\sum_{t=1}^{[ns]}\epsilon_t+\frac{1}{\tau\sqrt{n}}(I-II)+\frac{1}{\tau\sqrt{n}}\sum_{t=1}^{[ns]}X_{2,t}, \tag{2.5}$$

where $\tau^2=(\sum_{j=0}^{\infty}a_j)^2\sigma^2$. From (2.2) and the fact that

$$\left(\sum_{j=0}^{m}a_j\right)^2\sigma^2 \to \tau^2 \tag{2.6}$$

as $m\to\infty$, the first term of the right-hand side of (2.5) converges weakly to W.

According to Theorems 4.1 and 4.2 of Billingsley (1968), to prove (1.3) it suffices to show that, for any $\epsilon>0$ and every fixed $m\geq 1$,

$$\limsup_{n\to\infty}P\Big(\sup_{0\leq s\leq 1}|I|\geq\epsilon\tau\sqrt{n}\Big)=0, \tag{2.7}$$

$$\limsup_{n\to\infty}P\Big(\sup_{0<s<1}|II|\geq\epsilon\tau\sqrt{n}\Big)=0 \tag{2.8}$$

and

$$\lim_{m\to\infty}\limsup_{n\to\infty} P\left\{\sup_{0\le s\le 1}\left|\sum_{t=1}^{[ns]} X_{2,t}\right| \ge \epsilon\tau\sqrt{n}\right\} = 0. \tag{2.9}$$

Note that

$$\frac{1}{\tau\sqrt{n}}\max_{-n\le t\le n}|\epsilon_t| \;\to\; 0 \tag{2.10}$$

in probability by the assumption. From (2.10) and $\sum_{j=0}^{\infty}|a_j| < \infty$ we have

$$\frac{1}{\tau\sqrt{n}}\sup_{0\le s\le 1}|I| \le \frac{1}{\tau\sqrt{n}}\max_{-m<t\le n}|\epsilon_t|\sum_{k=1}^{m}\left(\sum_{j=k}^{m}|a_j|\right) \to 0, \tag{2.11}$$

$$\frac{1}{\tau\sqrt{n}}\sup_{0\le s\le 1}|II| \le \frac{1}{\tau\sqrt{n}}\max_{n-m<t\le n}|\epsilon_t|\sum_{k=1}^{m}\left(\sum_{j=k+1}^{m}|a_j|\right) \to 0 \tag{2.12}$$

in probability. Hence, from (2.11) and (2.12), (2.7) and (2.8) follow respectively. It remains to show (2.9). First from (2.3) we have, for any $n \ge 1$

$$\sum_{t=1}^{n} X_{2,t} = \sum_{j=m+1}^{\infty} a_j \sum_{t=1}^{n}\epsilon_{t-j}. \tag{2.13}$$

And, from (1.2) and (2.1) we have

$$\sup_j E\max_{1\le k\le n}\left(\sum_{t=1}^{k}\epsilon_{t-j}\right)^2 \le cn\sigma^2, \tag{2.14}$$

for some constant c since the random variables are stationary.

By applying Minkowski's inequality, it follows from (2.13) and (2.14) that

$$\begin{aligned}
E\sup_{0\le s\le 1}\left(\sum_{t=1}^{[ns]} X_{2,t}\right)^2 &= E\sup_{0\le s\le 1}\left(\sum_{j=m+1}^{\infty} a_j\sum_{t=1}^{[ns]}\epsilon_{t-j}\right)^2 \\
&\le E\left(\sum_{j=m+1}^{\infty}|a_j|\sup_{0\le s\le 1}\left|\sum_{t=1}^{[ns]}\epsilon_{t-j}\right|\right)^2 \\
&\le \left(\sum_{j=m+1}^{\infty}\left\{E\sup_{0\le s\le 1}\left|\sum_{t=1}^{[ns]}|a_j|\epsilon_{t-j}\right|^2\right\}^{1/2}\right)^2
\end{aligned}$$

$$\leq c\sigma^2 n \left(\sum_{j=m+1}^{\infty} |a_j| \right)^2 . \tag{2.15}$$

From (2.15), the Markov inequality and the fact that

$$\sum_{j=m+1}^{\infty} |a_j| \to 0 \tag{2.16}$$

as m $\to \infty$, (2.9) follows immediately. Hence, the proof is complete. □

Remark. Taking $s = 1$ in the theorem, we obtain the central limit theorem: Let $\{X_t\}$ be a linear process defined as in the theorem. Then, for $t \in [0, 1]$,

$$\frac{1}{\tau\sqrt{n}} \sum_{t=1}^{n} X_t \xrightarrow{d} N(0, 1)$$

where $\xrightarrow{d}$ denotes convergence in distribution.

3. Applications

We will prove a weak convergence for a linear process with linearly positive quadrant dependent sequence by using the idea based on the proof of the theorem in Section 2.

Definition 3.1. *A sequence $\{\epsilon_t, t \in Z_+\}$ of random variables is said to be linearly positive quadrant dependent(LPQD) if for any disjoint subsets $A, B \subset \mathbf{Z}_+$ and positive r_j's $\sum_{i\in A} r_i\epsilon_i$ and $\sum_{j\in B} r_j\epsilon_j$ are PQD.*

Newman (1984) introduced this concept and also established the central limit theorem. Birkel (1993) obtained a weak convergence for LPQD sequences as follows.

Theorem 3.1. **(Birkel, 1993)** *Let $\{\epsilon_t\}$ be a stationary sequence of LPQD random variables with $E\epsilon_t = 0$, $E\epsilon_t^2 < \infty$ and satisfying (1.2). Assume for some $\rho > 0$ and $s > 2$*

$$\sum_{t=n+1}^{\infty} E(\epsilon_1\epsilon_t) = O(n^{-\rho}), \tag{3.1}$$

$$E|\epsilon_t|^s < \infty. \tag{3.2}$$

Then, for $s \in [0,1]$

$$(\sigma\sqrt{n})^{-1}\sum_{t=1}^{[ns]}\epsilon_t \Rightarrow W,$$

where $\Rightarrow$ *denotes weak convergence and* W *denotes a standard Wiener process.*

Theorem 3.2. *Let* X_t *be a linear process of the form (1.1), where* $\{a_j\}$ *is a sequence of real numbers with* $\sum_{j=0}^{\infty}|a_j| < \infty$ *and* $\{\epsilon_t\}$ *is a stationary sequence of LPQD random variables with* $E\epsilon_t = 0$, $E\epsilon_t^2 < \infty$ *and satisfying (1.2), (3.1) and (3.2). Then, for* $s \in [0,1]$

$$\frac{1}{\tau\sqrt{n}}\sum_{t=1}^{[ns]} X_t \Rightarrow W,$$

where $\tau^2 = (\sum_{j=0}^{\infty} a_j)^2\sigma^2$.

Proof. As in the proof of Theorem 1.1 we have (2.5). By Theorem 3.1 and (2.6) the first term of right-hand side of (2.5) converges weakly to W. By Theorems 4.1 and 4.2 obtained in Billingsley (1968) it suffices to show that (2.7), (2.8) and (2.9) hold. But (2.7) and (2.8) still hold here as in the proof of Theorem 1.1. To prove (2.9) we consider (2.13), *i.e.*

$$\sum_{t=1}^{n} X_{2,t} = \sum_{j=m+1}^{\infty} a_j \sum_{t=1}^{n}\epsilon_{t-j} \tag{3.3}$$

for any $n \geq 1$. By Lemma 3 of Birkel (1993) and Theorem 3.7.5 of Stout (1974), it follows from (3.1) and (3.2) that for $r > 2$

$$\sup_j E \max_{1\leq k\leq n}\left(\left|\sum_{t=1}^{k}\epsilon_{t-j}\right|\right)^r = O(n^{r/2}). \tag{3.4}$$

By applying Minkowski's inequality from (3.3) and (3.4) we have

$$E\sup_{0\leq s\leq 1}\left(\left|\sum_{t=1}^{[ns]} X_{2,t}\right|\right)^r = E\sup_{0\leq s\leq 1}\left(\left|\sum_{j=m+1}^{\infty} a_j \sum_{t=1}^{[ns]}\epsilon_{t-j}\right|\right)^r$$

$$\leq E\left(\sum_{j=m+1}^{\infty}|a_j|\sup_{0\leq s\leq 1}\left|\sum_{t=1}^{[ns]}\epsilon_{t-j}\right|\right)^r$$

$$\leq\left(\sum_{j=m+1}^{\infty}\left\{E\sup_{0\leq s\leq 1}\left|\sum_{t=1}^{[ns]}|a_j|\epsilon_{t-j}\right|^r\right\}^{1/r}\right)^r$$

$$\leq c\left(\sum_{j=m+1}^{\infty}|a_j|\right)^r n^{r/2},\quad r>2. \tag{3.5}$$

Now from (3.5), the Markov inequality and (2.16), (2.9) follows immediately. Hence the proof of Theorem 3.2 is complete. □

Acknowledgements

The authors wish to thank the referees for a very thorough review and comment of this paper.

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