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A Sequential Approach for Estimating the Variance of a Normal Population Using Some Available Prior Information

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ABSTRACT

Using some available information about the unknown variance σ^2 of a normal distribution with mean μ, a sequential approach is used to estimate σ^2. Two cases have been considered regarding the mean μ being known or unknown. The mean square error (MSE) of the new estimators are compared to that of the usual estimator of σ^2, namely, the sample variance based on a sample of size equal to the expected sample size. Simulation results indicates that, the new estimator is more efficient than the usual estimator of σ^2 whenever the actual value of σ^2 is not too far from the prior information.

Keywords. Normal population, mean square error, expected sample size, efficiency, bias.
AMS 2000 subject classifications. Primary 62F10.

1. Introduction

Sequential analysis is a method of statistical inference whose characteristics feature is that the number of observations required by the procedure is not determined in advance of the experiment. This approach is applicable in practice whenever, it is difficult to obtain all the units of the sample at one time and the researcher may have to wait too long before he will get his sample ready for investigation. The decision to terminate the experiment depends, at each stage, on the results of the observations previously made. Furthermore, to obtain better results, some available information about the parameter of interest can be used with the information given by the sample.

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Suppose that the random variable X has a normal distribution with mean μ and unknown variance σ^2, $N(\mu, \sigma^2)$. We discuss here in details the case when the mean μ is known. The case when μ is unknown, needs a slight modification to our estimation procedure, and it will be discussed later in the next section. For the case when μ is known, without loss of generality, assume that $\mu = 0$. Also, assume that the maximum sample size that can be obtained is a preassigned positive integer n (maybe because the total cost of taking the sample is fixed and limited, for example). Since it is unreasonable to estimate the population variance with a random sample less than 2, the units of the random sample $X_1, X_2, \ldots, X_n$ are obtained first as an initial starting sample $X_1, X_2, \ldots, X_{n_0}$, where $2 \leq n_0 < n$ and the remaining members of the random sample, $X_{n_0+1}, \ldots, X_n$, are obtained one at a time. Furthermore, suppose that available information from different sources, about the value of σ^2 is in a form of an initial guess σ_0^2.

In order to describe the new estimator, we define some notation. For $k = n_0, n_0+1, \ldots, n$, let $S_k^2 = k^{-1}\sum_{i=1}^{k} X_i^2$, $W_k = kS_k^2/\sigma_0^2$, $C_{1k} = \chi_k^2(\alpha/2)$ and $C_{2k} = \chi_k^2(1-\alpha/2)$, where $\chi_k^2(\alpha/2)$ and $\chi_k^2(1-\alpha/2)$ are the lower and the upper $100(\alpha/2)\%$ and $100(1-\alpha/2)\%$ percentiles of the chi-square distribution function with k degrees of freedom respectively, $(0 < \alpha < 1)$. Also, let $\tau(W_k)$ be a weighted function of W_k, which will be determined later.

Now the procedure for estimating σ^2 using a sequential approach and some available information about σ^2 is described in the following steps (the case when $\mu = 0$):

(1) Observe sample units $X_1, X_2, \ldots, X_{n_0}$ from the normal distribution with mean zero and variance σ^2;

(2) Test the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ *vs.* $H_a : \sigma^2 \neq \sigma_0^2$, at a suitable significance level α, *i.e.*, reject H_0 if $W_{n_0} < C_{1n_0}$ or $W_{n_0} > C_{2n_0}$, otherwise accept H_0;

(3) If H_0 is accepted, then estimate σ^2 by $\{\tau(W_{n_0})\ \sigma_0^2 + (1-\tau(W_{n_0}))S_{n_0}^2\}$;

(4) If H_0 is rejected, then obtain another sample unit X_{n_0+1} and repeat steps (2) and (3) using W_{n_0+1} and $S_{n_0+1}^2$ to test H_0. If H_0 is accepted, then estimate σ^2 by $\{\tau(W_{n_0+1})\ \sigma_0^2 + (1-\tau(W_{n_0+1}))S_{n_0+1}^2\}$;

(5) At the k^{th} stage where we have k sample units $n_0 \leq k \leq n$, use W_k and S_k^2 to test H_0. If H_0 is accepted, then estimate σ^2 by $\{\tau(W_k)\ \sigma_0^2 + (1-\tau(W_k))S_k^2\}$;

(6) If H_0 is rejected on using the n^{th} unit, *i.e.*, the last sample unit, then estimate σ^2 by S_n^2.

Many authors considered similar approach of estimation. For example, Katti (1962) used a two stage sampling technique and introduced the idea of testing priors and providing an estimate of μ when prior information is available in the form of an initial guess value μ_0. Waiker *et al.* (1984) gave another estimate of μ based on a pretesting of the prior information. Al-Saleh and Muttlak (1995) extended Waiker *et al.* (1984) using a sequential approach and two shrinkage factors, namely $|t| = \sqrt{k}|\overline{X_k}|/Z_{\alpha/2}$ and t^2. Pandey (1979) considered a double stage sampling method to estimate the variance σ^2 of a normal population, using an initial guess value σ_0^2 of σ^2. Also, Al-Saleh (1994) used a two stage estimation method to estimate σ^2 based on a pretesting of some available information.

In this paper we use some available information about the variance and the sequential approach to estimate σ^2. Computer simulation is used to compare the MSE of this estimator with the MSE of the usual estimator via the sample variance. In Section 2, we present our estimators. The expected sample size and the efficiency will be introduced in Section 3. In Section 4, we describe the simulation plan. Simulation results and conclusions are presented in Section 5.

2. The Estimators

2.1. Case 1 : The mean μ is known

For given n_0 and n, let $X_1, X_2, \ldots, X_{n_0}, n_0 \leq k \leq n$, be a random sample from a normal distribution with known mean μ (we may assume without loss of generality $\mu = 0$) and unknown variance σ^2, and let S_k^2, W_k, C_{1k} and C_{2k} be as defined earlier. Based on S_k^2 and W_k, the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ *vs.* $H_a : \sigma^2 \neq \sigma_0^2$ is tested using a suitable α. H_0 is accepted if $C_{1k} \leq W_k \leq C_{2k}$. Based on the result of this test, the following estimator of σ^2 is considered:

$$\widehat{\sigma}^2 = \begin{cases} \tau(W_k)\sigma_0^2 + (1-\tau(W_k))S_k^2, & \\ & \text{if } H_0 \text{ is rejected for the stages } n_0,\ n_0+1, \ldots, k-1 \\ & \quad \text{but accepted at the } k^{th} \text{ stage, } k \geq n_0, \\ S_n^2, & \text{if } H_0 \text{ is rejected for the stages } n_0, n_0+1, \ldots, n. \end{cases}$$

The available information is accepted at the k^{th} stage if S_k^2 is close to σ_0^2 so that W_k is close to k. Thus, the closer W_k to k, the more weight should be given to σ_0^2. As W_k departs from k in either direction, more weight should be given to S_k^2. Intuitively, $\tau(W_k)$ should satisfy the following conditions at the k^{th} stage:

a) $\tau(W_k)$ has a maximum value (preferably 1) at $W_k = 1$;

b) $\tau(W_k)$ increases to 1 for $C_{1k} < W_k < k$ and decreases for $k < W_k < C_{2k}$;

c) $\tau(C_{1k}) = \tau(C_{2k}) = 0$;

d) $0 \leq \tau(W_k) \leq 1$.

Infinitely many functions of W_k satisfy the above conditions. However, one simple choice of $\tau(W_k)$ is a linear function of W_k. Fitting a straight line between $(C_{1k}, 0)$ and $(k, 1)$ and another straight line between $(k, 1)$ and $(C_{2k}, 0)$ give the following weight function:

$$\tau(W_k) = \begin{cases} \dfrac{W_k - C_{1k}}{k - C_{1k}}, & \text{if } C_{1k} \leq W_k < k, \\ \dfrac{W_k - C_{2k}}{k - C_{2k}}, & \text{if } k \leq W_k \leq C_{2k}. \end{cases}$$

2.2. Case 2 : The mean μ is unknown

The procedure of estimating σ^2 will be the same as in Case 1 except that in place of S_k^2, W_k, C_{1k} and C_{2k}, we need to use S_k^{*2}, W_k^*, C_{1k}^* and C_{2k}^*, $k = n_0, n_0+1, \ldots, n$, where $S_k^{*2} = (k-1)^{-1} \sum_{i=1}^{k} (X_i - \overline{X})^2$, $\overline{X} = k^{-1} \sum_{i=1}^{k} X_i$, $W_k^* = (k-1)S_k^{*2}/\sigma_0^2$, $C_{1k}^* = \chi_{k-1}^2(\alpha/2)$ and $C_{2k}^* = \chi_{k-1}^2(1-\alpha/2)$. Note that $\chi_{k-1}^2(\alpha/2)$ and $\chi_{k-1}^2(1-\alpha/2)$ are the lower and the upper $100(\alpha/2)\%$ and $100(1-\alpha/2)\%$ percentiles of the chi-square distribution function with $k-1$ degrees of freedom respectively, $(0 < \alpha < 1)$. Now based on $S_k^{*2}$and W_k^*, the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ *vs.* $H_a : \sigma^2 \neq \sigma_0^2$ is tested using a suitable α. H_0 is accepted if $C_{1k}^* \leq W_k^* \leq C_{2k}^*$. Based on the result of this test, the following estimator of σ^2 is considered as follows:

$$\widehat{\sigma}^{*2} = \begin{cases} \tau^*(W_k^*)\sigma_0^2 + (1 - \tau^*(W_k^*))S_k^{*2}, & \\ \qquad \text{if } H_0 \text{ is rejected for the stages } n_0,\ n_0+1, \ldots, k-1 & \\ \qquad \text{but accepted at the } k^{th} \text{ stage, } k \geq n_0, & \\ S_n^{*2}, \qquad \text{if } H_0 \text{ is rejected for the stages } n_0, n_0+1, \ldots, n, & \end{cases}$$

where $\tau^*(W_k^*)$ is similar to that $\tau(W_k^*)$ defined above, *i.e.*

$$\tau^*(W_k^*) = \begin{cases} \dfrac{W_k^* - C_{1k}^*}{k - C_{1k}^*}, & \text{if } C_{1k}^* \leq W_k^* < k-1, \\ \dfrac{W_k^* - C_{2k}^*}{k - C_{2k}^*}, & \text{if } k \leq W_k^* \leq C_{2k}^*. \end{cases}$$

In the next section, we derive the expected sample size and the efficiency of our estimator for Case 1 and Case 2. Note that Case 2 needs only a slight modification.

3. Expected Sample Size and the Efficiency

3.1. Case 1 : The population mean is known ($\mu = 0$)

The expected sample size for the estimation procedure described earlier can be determined as follows. Let $A_k = \{C_{1k} < W_k < C_{2k}\}$, $B_k = \{A_{n_0}^c \cap A_{n_0+1}^c \cap \cdots \cap A_{k-1}^c \cap A_k\}$ and $B = \bigcap_{k=n_0}^{n} A_k^c$, where $k = n_0, n_0+1, \ldots, n$. Also, let $m = E(\text{sample size}|\sigma^2)$, then

$$m = \sum_{k=n_0}^{n} kP(B_k) + nP(B).$$

To calculate $P(B_k)$ and $P(B)$ we need to find, in general for any given σ^2, the joint distribution function of $W_{n_0}, W_{n_0+1}, \ldots, W_k$, $k = n_0, n_0+1, \ldots, n$. Since X_1, $X_2, \ldots, X_k$ is a random sample from $N(0, \sigma^2)$, let $Y_i = X_i^2/\sigma^2$, $i = 1, 2, \ldots, k$. Then $Y_1, Y_2, \ldots, Y_k$ are *iid* from chi-square distribution with one degree of freedom. Therefore, the joint density function of $Y_1, Y_2, \ldots, Y_k$ is given by

$$f(y_1, y_2, \ldots, y_k) = \left(\frac{1}{\sqrt{2\pi}}\right)^k \prod_{i=1}^{k} (y_i)^{-1/2}\, e^{-\sum_{i=1}^{k} y_i/2},\ 0 < y_1, y_2, \ldots, y_k < \infty.$$

Now, with $W_0 = 0$, and $W_i = Y_1 + Y_2 + \cdots + Y_i$, $i = 1, 2, \ldots, k$, which is a simple linear transformation of $Y_1, Y_2, \ldots, Y_k$, the joint density function of W_1, $W_2, \ldots, W_k$ is given by

$$g(w_1, w_2, \ldots, w_k) = \left(\frac{1}{\sqrt{2\pi}}\right)^k \prod_{i=1}^{k} (w_i - w_{i-1})^{-1/2}\, e^{-w_k/2},\ 0 < w_1, w_2, \ldots, w_k < \infty.$$

Now, the joint density function of $W_{n_0}, W_{n_0+1}, \ldots, W_k$, $k = n_0, n_0+1, \ldots, n$, can be obtained simply as follows:

$$g(w_{n_0}, w_{n_0+1}, \ldots, w_k) = \int \cdots \int g(w_1, w_2, \ldots, w_k)\, dw_1 dw_2 \cdots dw_{n_0-1}.$$

Therefore,

$$P(B_k) = \int \cdots \int_{B_k} g(w_{n_0}, w_{n_0+1}, \ldots, w_k)\, dw_{n_0} dw_{n_0+1} \cdots dw_k,$$

$$P(B) = \int_B \cdots \int g(w_{n_0}, w_{n_0+1}, \ldots, w_n) dw_{n_0} dw_{n_0+1} \ldots dw_n.$$

To compare our new estimator $\widehat{\sigma}^2$ with the usual estimator, the sample variance S_m^2, based on a sample of an equivalent size m we define the efficiency of $\widehat{\sigma}^2$ with respect to S_m^2, as

$$\text{eff}(\widehat{\sigma}^2) = \frac{\text{MSE}(\text{S}_m^2)}{\text{MSE}(\widehat{\sigma}^2)} = \frac{2\sigma^2/m}{\text{MSE}(\widehat{\sigma}^2)}.$$

To find MSE($\widehat{\sigma}^2$), note that, $S_k^2 = \sigma_0^2\ W_k/k$, $k = n_0, n_0+1, \ldots, n$. Thus, we can write the new estimator as follows:

$$\widehat{\sigma}^2 = \begin{cases} \tau(W_k)\sigma_0^2 + (1-\tau(W_k))\dfrac{\sigma_0^2 W_k}{k}, & \\ \qquad \text{if } H_0 \text{ is rejected for the stages } n_0, n_0+1, \ldots, k-1 & \\ \qquad \text{but accepted at the } k^{th} \text{ stage, } k \geq n_0, & \\ \dfrac{\sigma_0^2 W_n}{n}, \quad \text{if } H_0 \text{ is rejected for the stages } n_0, n_0+1, \ldots, n. & \end{cases}$$

Therefore,

$$\begin{aligned} &\text{MSE}(\widehat{\sigma}^2) \\ &= E(\widehat{\sigma}^2 - \sigma^2)^2 \\ &= \sum_{k=n_0}^{n} \int_{B_k} \cdots \int \left(\tau(w_k)\sigma_0^2 + (1-\tau(w_k))\frac{\sigma_0^2\ w_k}{k} - \sigma^2\right)^2 \\ &\quad \times g(w_{n_0}, w_{n_0+1}, \ldots, w_k) dw_{n_0} \cdots dw_k \\ &\quad + \int_B \cdots \int \left(\frac{\sigma_0^2\ w_n}{n} - \sigma^2\right)^2 g(w_{n_0}, w_{n_0+1}, \ldots, w_n) dw_{n_0} dw_{n_0+1} \cdots dw_n. \end{aligned}$$

3.2. Case 2 : The population mean μ is unknown

Similar to Case 1, the expected sample size for the estimation procedure described earlier, for the case when μ is unknown, can be determined as follows.

Let $A_k^* = \{C_{1k}^* < W_k^* < C_{2k}^*\}$, $B_k^* = \{A_{n_0}^{*c} \cap A_{n_0+1}^{*c} \cap \cdots \cap A_{k-1}^{*c} \cap A_k^*\}$ and $B^* = \bigcap_{k=n_0}^{n} A_k^{*c}$, where $k = n_0, n_0+1, \ldots, n$. Also, let $m^* = E(\text{sample size}|\sigma^2)$, and similarly

$$m^* = \sum_{k=n_0}^{n} kP(B_k^*) + nP(B^*).$$

Again, to calculate $P(B_k^*)$ and $P(B^*)$ we need to find, in general for any given σ^2, the joint distribution function of $W_{n_0}^*, W_{n_0+1}^*, \ldots, W_k^*$, $k = n_0, n_0+1, \ldots, n$. Since $X_1, X_2, \ldots, X_k$ is a random sample from $N(\mu, \sigma^2)$, let $W_i^* = (i-1)S_i^{*2}/\sigma^2$, $i = 2, \ldots, k$, then W_i^* has chi-square distribution with $(i-1)$ degree of freedom. Now let $Y_1, Y_2, \ldots, Y_{k-1}$ be *iid* from chi-square distribution with one degree of freedom. Therefore, the joint density function of $Y_1, Y_2, \ldots, Y_{k-1}$ is given by

$$f(y_1, y_2, \ldots, y_{k-1}) = \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \prod_{i=1}^{k-1} (y_i)^{-1/2}\, e^{-\sum_{i=1}^{k-1} y_i/2},\ 0 < y_1, y_2, \ldots, y_k < \infty.$$

Now, with $W_1^* = 0$, note that $W_i^* \stackrel{d}{=} Y_1 + Y_2 + \cdots + Y_{i-1}$, $i = 2, \ldots, k$, which is a simple linear transformation of $Y_1, Y_2, \ldots, Y_{k-1}$. Then the joint density function of $W_2^*, W_3^*, \ldots, W_k^*$ is given by

$$g(w_2^*, w_3^*, \ldots, w_k^*) = \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \prod_{i=2}^{k} (w_i^* - w_{i-1}^*)^{-1/2}\, e^{-\ w_k^*/2},\ 0 < w_2, \ldots, w_k < \infty.$$

Now, the joint density function of $W_{n_0}^*, W_{n_0+1}^*, \ldots, W_k^*$, where $k = n_0, n_0 + 1, \ldots, n$, and $n_0 > 2$ can be obtained simply as follows:

$$g(w_{n_0}^*, w_{n_0+1}^*, \ldots, w_k^*) = \int \cdots \int g(w_2^*, w_3^*, \ldots, w_k^*) dw_2^* dw_3^* \cdots dw_{n_0-1}^*.$$

Therefore,

$$P(B_k^*) = \int_{B_k^*} \cdots \int g(w_{n_0}^*, w_{n_0+1}^*, \ldots, w_k^*) dw_{n_0}^* dw_{n_0+1}^* \cdots dw_k^*$$

and

$$P(B^*) = \int_{B^*} \cdots \int g(w_{n_0}^*, w_{n_0+1}^*, \ldots, w_n^*) dw_{n_0}^* dw_{n_0+1}^* \cdots dw_n^*.$$

Also,

$$\text{eff}(\widehat{\sigma}^{*2}) = \frac{\text{MSE}(S_m^{*2})}{\text{MSE}(\widehat{\sigma}^{*2})} = \frac{2\sigma^2/(m^*-1)}{\text{MSE}(\widehat{\sigma}^{*2})}.$$

To find $\text{MSE}(\widehat{\sigma}^{*2})$, note that, $S_k^{*2} = \sigma_0^2 W_k^*/(k-1)$, $k = n_0, n_0+1, \ldots, n$. Thus, we can write the new estimator as follows:

$$\widehat{\sigma}^{*2} = \begin{cases} \tau(W_k^*)\sigma_0^2 + (1 - \tau(W_k^*))\dfrac{\sigma_0^2\, W_k^*}{k-1}, & \\ \qquad \text{if } H_0 \text{ is rejected for the stages } n_0, n_0+1, \ldots, k-1 & \\ \qquad\quad \text{but accepted at the } k^{th} \text{ stage, } k \geq n_0, & \\ \dfrac{\sigma_0^2 W_n^*}{n-1}, \quad \text{if } H_0 \text{ is rejected for the stages } n_0, n_0+1, \ldots, n. & \end{cases}$$

Therefore,

$$\begin{aligned}
&\mathrm{MSE}(\widehat{\sigma}^{*2})\\
&= E(\widehat{\sigma}^{*2}-\sigma^2)^2\\
&= \sum_{k=n_0}^{n} \int\cdots\int_{B_k^*} \left(\tau(w_k^*)\sigma_0^2 + (1-\tau(w_k^*))\frac{\sigma_0^2\, w_k^*}{k-1} - \sigma^2\right)^2\\
&\quad \times g(w_{n_0}^*, w_{n_0+1}^*, \ldots, w_k^*)dw_{n_0}^* \cdots dw_k^*\\
&\quad + \int\cdots\int_{B} \left(\frac{\sigma_0^2\, w_n^*}{n-1} - \sigma^2\right)^2 g(w_{n_0}^*, w_{n_0+1}^*, \ldots, w_n^*)dw_{n_0}^* dw_{n_0+1}^* \cdots dw_n^*.
\end{aligned}$$

Next, we discuss the plan of the simulation study.

4. Simulation

Computer simulation was used to gain knowledge about the efficiency of the estimator $\widehat{\sigma}^2$ compared with the usual estimator S_m^2 (which is based on the expected sample size m). The normal distribution with mean μ and variance σ^2 and two significance levels $\alpha = 0.01, 0.05$ were used in our study. In case of μ assumed to be known only $\mu = 0$ was investigated. In case when μ assumed to be unknown $\mu = 2$ were investigated. We considered different values of σ^2 ranging from 0.01 to 10. Sample sizes ranging from 10 to 30 were considered in the simulation. Without loss of generality, we assumed that $\sigma_0^2 = 1$. For each combination of sample size, value of μ and value of σ^2, 10000 data sets were generated from the suitable normal distribution. For each generated data set, steps (1) to (6) of Section 1 were computed using the units of the selected sample. In case of the i^{th} iteration if H_0 was accepted at the k_i^{th} stage, where $k_i < n$, then $\widehat{\sigma}_i^2$ will be calculated using

$$\widehat{\sigma}_i^2 = \begin{cases} \tau(W_{k_i})\sigma_0^2 + (1-\tau(W_{k_i}))S_{k_i}^2, & \text{if } H_0 \text{ is rejected for the stages } n_0, n_0+1, \ldots, k-1 \\ & \quad \text{but accepted at the } k^{th} \text{ stage, } k \geq n_0, \\ S_n^2, & \text{if } H_0 \text{ is rejected for the stages } n_0, n_0+1, \ldots, n. \end{cases}$$

where $k_i = n_0, n_0+1, \ldots, n$ is the number of units that were used until we accept H_0 in a given iteration $i = 1, 2, \ldots, 10000$. Also, $S_{m_i}^2$ is estimated on each iteration by $S_{m_i}^2 = S_{k_i}^2$ if H_0 was accepted at the k_i^{th} stage, where $k_i < n$, or by $S_{m_i}^2 = S_n^2$ if $k_i = n$. Based on these 10000 sampling replications, and since k_i is a random variable in our simulation, we estimated the MSE of $\widehat{\sigma}^2$ and its absolute bias as well as the MSE of S_m^2 and the expected sample size as follows:

$$\widehat{\text{MSE}}(\widehat{\sigma}^2) = \frac{1}{N}\sum_{i=1}^{10000} k_i(\widehat{\sigma}_i^2 - \sigma^2)^2, \ \widehat{\text{MSE}}(S_m^2) = \frac{1}{N}\sum_{i=1}^{10000} k_i(S_{k_i}^2 - \sigma^2)^2,$$

$$|\text{bias}| = \left|\overline{\sigma}^2 - \sigma^2\right| \text{ and } m = \frac{1}{10000}\sum_{i=1}^{10000} k_i,$$

where $N = \sum_{i=1}^{10000} k_i$ and $\overline{\sigma}^2 = N^{-1}\sum_{i=1}^{10000} k_i\widehat{\sigma}_i^2$.

Note that it is easy to see by using the concept of double expectation, that the above simulated estimates are unbiased and consistent estimates for MSE, bias, and m. The efficiency of the new estimator $\widehat{\sigma}^2$ was estimated by $\widehat{\text{eff}}(\widehat{\sigma}^2) = \widehat{\text{MSE}}(S_m^2)/\widehat{\text{MSE}}(\widehat{\sigma}^2)$ for each combination of sample size, value of μ and value of σ^2. The efficiency of the of our estimator in Case 2 can be obtained in a similar manner.

5. Simulation Results and Conclusions

The main results of Case 1 when the mean is known are in Table 1:

(1) The efficiency of $\widehat{\sigma}^2$ with respect to S_m^2 is more than 1.40 for $\alpha = 0.05$ and more than 2.0 for $\alpha = 0.01$ when $\sigma^2 = 1$, *i.e.*, when our prior information is accurate;

(2) For any sample size n, $\widehat{\sigma}^2$ continues to be more efficient than S_m^2 for $1 \leq \sigma^2 < 3$;

(3) The efficiency and |bias| of $\widehat{\sigma}^2$ increases for smaller α;

(4) The |bias| is relatively negligible for $\alpha = 0.05$;

(5) For σ^2 >3 the efficiency will be less than or equal to one. However, for very large σ^2, the efficiency will approach one; this is intuitively clear since large value of σ^2 produce samples that lead to the rejection of H_0, which means that there will be a slight difference between our estimator and the usual sample variance;

(6) The expected sample size m increases as σ^2 increases away from 1.

Table 2 contains our simulation for Case 2, when the mean is assumed unknown. However, similar exactly similar conclusions to those of Case 1 can be drawn from Case 2. Therefore, based on these observations, when the prior information is accurate, we can conclude that the estimator $\widehat{\sigma}^2$ is better than the usual estimator of σ^2. Hence, we recommend the use of this estimator in situation when we feel that the available information is accurate.

This sequential approach is applicable in situations where it is not possible to get all sample units at one time and the researcher has to wait a long time before he finishes sampling. In this case, at each stage whatever sample points are available, one can decide (after a preliminary test) whether to quit or to continue sampling. Areas where this is applicable can be found in medicine and agriculture, where sampling units are too expensive and time consuming to be obtained.

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TABLE 1 *The efficiency of* $\widehat{\sigma}^2$ *with respect to* S_m^2*, when* μ *is known*

		$\alpha = 0.05$				$\alpha = 0.01$		
σ^2		n=10	n=20	n=30		n=10	n=20	n=30
0.01	eff	1.00	1.00	1.00		1.00	1.00	1.00
	|bias|	0.00	0.00	0.00		0.00	0.00	0.00
	m	10.00	20.00	30.00		10.00	20.00	30.00
0.51	eff	1.00	0.98	1.01		1.10	1.06	1.09
	|bias|	0.10	0.10	0.09		0.13	0.13	0.12
	m	5.36	5.73	5.97		5.10	5.10	5.14
0.76	eff	1.74	1.77	1.79		2.23	2.25	2.24
	|bias|	0.05	0.05	0.05		0.05	0.05	0.05
	m	5.14	5.22	5.26		5.02	5.03	5.04
1.00	eff	1.71	1.74	1.78		2.26	2.27	2.30
	|bias|	0.02	0.01	0.01		0.03	0.04	0.05
	m	5.13	5.21	5.24		5.03	5.04	5.03
1.20	eff	1.46	1.51	1.52		1.83	1.86	1.86
	|bias|	0.05	0.05	0.05		0.11	0.09	0.11
	m	5.23	5.47	5.55		5.06	5.10	5.10
1.60	eff	1.10	1.13	1.15		1.23	1.27	1.30
	|bias|	0.09	0.07	0.08		0.17	0.17	0.16
	m	5.64	6.50	7.10		5.25	5.55	5.75
1.90	eff	1.00	1.00	1.00		1.04	1.04	1.05
	|bias|	0.09	0.08	0.06		0.19	0.15	0.16
	m	6.03	7.54	9.07		5.48	6.26	6.77
4.00	eff	0.96	0.95	0.96		0.90	0.90	0.90
	|bias|	0.06	0.04	0.02		0.10	0.07	0.05
	m	8.22	14.36	20.66		7.50	12.29	16.79
7.00	eff	0.99	0.99	0.99		0.97	0.97	0.97
	|bias|	0.01	0.01	0.04		0.03	0.04	0.02
	m	9.31	17.81	26.55		8.92	16.63	24.55
10.00	eff	0.99	0.99	1.00		0.99	0.99	0.99
	|bias|	0.08	0.01	0.03		0.02	0.02	0.04
	m	9.66	18.97	28.28		9.44	18.23	27.12

TABLE 2 *The efficiency of* $\widehat{\sigma}^2$ *with respect to* S_m^2, *when* μ *is unknown*

		$\alpha = 0.05$				$\alpha = 0.01$		
σ^2		n=10	n=20	n=30		n=10	n=20	n=30
0.01	eff	1.00	1.00	1.00		1.00	1.00	1.00
	|bias|	0.00	0.00	0.00		0.00	0.00	0.00
	m	10.00	20.00	30.00		10.00	20.00	30.00
0.61	eff	1.37	1.38	1.38		1.55	1.55	1.57
	|bias|	0.09	0.09	0.08		0.11	0.11	0.11
	m	6.19	6.44	6.56		6.03	6.08	6.08
0.81	eff	1.85	1.86	1.80		2.49	2.39	2.41
	|bias|	0.03	0.03	0.04		0.03	0.04	0.03
	m	6.10	6.18	6.21		6.01	6.02	6.03
1.01	eff	1.73	1.75	1.78		2.17	2.30	2.26
	|bias|	0.03	0.02	0.03		0.04	0.04	0.05
	m	6.11	6.20	6.24		6.02	6.04	6.03
1.41	eff	1.23	1.27	1.32		1.45	1.52	1.40
	|bias|	0.10	0.08	0.10		0.16	0.15	0.17
	m	6.37	6.84	7.19		6.11	6.27	6.49
2.01	eff	0.98	0.97	0.97		1.04	1.00	1.00
	|bias|	0.10	0.13	0.11		0.22	0.19	0.22
	m	7.02	8.77	10.62		6.39	7.39	8.05
10	eff	1.00	1.00	1.00		0.99	0.98	0.99
	|bias|	0.10	0.14	0.11		0.23	0.04	0.05
	m	9.72	19.01	28.30		9.54	18.36	27.13

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