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CONFORMAL VECTOR FIELDS AND TOTALLY UMBILIC HYPERSURFACES

DONG-SOO KIM, SEON-BU KIM,
YOUNG HO KIM AND SEONG-HEE PARK

ABSTRACT. In this article, we show that if a semi-Riemannian space form carries a conformal vector field V of which the tangential part V^T on a connected hypersurface M^n becomes a conformal vector field and the normal part V^N on M^n does not vanish identically, then M^n is totally umbilic. Furthermore, we give a complete description of conformal vector fields on semi-Riemannian space forms.

1. Introduction

Conformal mappings, conformal symmetries and conformal vector fields are of great importance in general relativity, as is well known since the early 1920's ([7, 15]). A vector field V satisfying $\mathfrak{L}_V g = 2\sigma g$ on a semi-Riemannian manifold (M, g) is called an infinitesimal conformal transformation or a conformal vector field on M, where $\mathfrak{L}$ denotes the Lie derivative on M and σ is a smooth function.

A totally umbilic submanifold of a semi-Riemannian manifold is the one whose first fundamental form and second fundamental form are proportional. An ordinary hypersphere $S^n(r)$ of an affine $(n+1)$-space of the Euclidean space R^m is one of the best known example of totally umbilic submanifolds of R^m. Totally umbilic submanifolds of a Riemannian space form with constant sectional curvature are well known ([4, 5]). On the other hand, there are four kinds of totally umbilic submanifolds in semi-Euclidean space (See, for example, [1]).

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In [17], Remark 1, R. Sharma and K. L. Duggal observed that if (M^n, g) is a semi-Riemannian totally umbilic submanifold of a semi-Riemannian manifold $(\bar{M}^m, \bar{g})$(that is, the induced metric $g = \bar{g}|_{M^n}$ is nondegenerate), then for any conformal vector field V on $\bar{M}^m$, the tangential part V^T of V on M^n reduces to a conformal vector field on M^n (See also Proposition 2.). Thus it would be worth considering the converse.

In this article we prove the converse of the above remark for a semi-Riemannian hypersurface of a semi-Riemannian space form $\bar{M}_k^{n+1}(\bar{c})$ with constant sectional curvature $\bar{c}$ in the following way.

THEOREM 1. *Let $(M^n, g), n \geq 2$, be a connected semi-Riemannian hypersurface of a semi-Riemannian space form $(\bar{M}_k^{n+1}(\bar{c}), \bar{g})$. Suppose that $\bar{M}_k^{n+1}(\bar{c})$ carries a conformal vector field V of which the tangential part V^T on M^n becomes a conformal vector field. Then one of the following holds:*

(i) *(M^n, g) is totally umbilic,*

(ii) *the restriction of V to M^n reduces to a tangent vector field on M^n.*

In general, in case the hypersurface (M^n, g) is lightlike (that is, the metric g is degenerate), we do not have a shape operator S satisfying $g(SX, Y) = g(X, SY)$. Furthermore, Remark 1 in [17] does not hold in general for lightlike hypersurfaces (See [6]; p.118). Henceforth all of hypersurfaces are assumed to be semi-Riemannian unless stated otherwise.

To prove Theorem 1, we derive a useful formula (2.13) for the normal part V^N of V on totally umbilic hypersurfaces of $\bar{M}_k^{n+1}(\bar{c})$, which is also used to prove Theorem 3. Theorem 3 characterizes conformal vector fields on totally umbilic hypersurface M^n of $\bar{M}_k^{n+1}(\bar{c})$ in terms of conformal vector fields on the ambient space form $\bar{M}_k^{n+1}(\bar{c})$.

In Section 3, using Theorem 3, we give a complete description of conformal vector fields on semi-Riemannian space forms (cf. [18]; pp. 336–337).

2. Main theorems

On a semi-Riemannian manifold (M^n, g) a vector field V is called conformal if it preserves the conformal class of the metric:

$$\mathfrak{L}_V g = 2\sigma g$$

for some function σ. Recall that by definition $\mathfrak{L}_V g(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$ for arbitrary tangent vectors X, Y where ∇ denotes the Levi-civita connection on M^n. The function σ is necessarily $\frac{1}{n} div(V)$, where $div(V)$ denotes the divergence of the vector field V.

PROPOSITION 2. ([17]; Remark 1) *Let (M^n, g) be a totally umbilic submanifold of a semi-Riemannian manifold $(\bar{M}^m, \bar{g})$. If V is a conformal vector field on $\bar{M}^m$, then the tangential part V^T of V on M^n is a conformal vector field on M^n.*

Proof. Suppose that V satisfies $\mathfrak{L}_V \bar{g} = 2\sigma\bar{g}$ on $\bar{M}^m$. Let V^T and V^N denote the tangential and normal part of V on M^n, respectively. Then since for all $X, Y \in TM^n$

$$\begin{aligned}&\mathfrak{L}_V \bar{g}(X,Y) = \bar{g}(\bar{\nabla}_X V, Y) + \bar{g}(X, \bar{\nabla}_Y V)\\ &= \bar{g}(\bar{\nabla}_X V^T, Y) + \bar{g}(X, \bar{\nabla}_Y V^T) + \bar{g}(\bar{\nabla}_X V^N, Y) + \bar{g}(X, \bar{\nabla}_Y V^N),\end{aligned}$$

we have

$$\mathfrak{L}_V \bar{g}(X,Y) = \mathfrak{L}_{V^T} g(X,Y) - 2\bar{g}(h(X,Y), V^N),$$

where h denotes the second fundamental form. Hence from the hypothesis we obtain that

$$\begin{aligned}\mathfrak{L}_{V^T} g(X,Y) &= \mathfrak{L}_V \bar{g}(X,Y) + 2\bar{g}(V,H) g(X,Y)\\ &= 2\{\sigma + \bar{g}(V,H)\} g(X,Y),\end{aligned}$$

where H denotes the mean curvature vector field. This completes the proof. □

Now we prove Theorem 1 as follows:

Proof of Theorem 1. Suppose that V and V^T satisfy $\mathfrak{L}_V \bar{g} = 2\sigma\bar{g}$ on $(\bar{M}_k^{n+1}(\bar{c}), \bar{g})$ and $\mathfrak{L}_{V^T} g = 2\tau g$ on M^n, respectively. From the proof of the above proposition, we obtain

$$\bar{g}(V, h(X,Y)) = (\tau - \sigma) g(X,Y), \quad X, Y \in TM^n, \tag{2.1}$$

where h denotes the second fundamental form. We let $U = \{p \in M | V^N(p) \neq 0\}$. From now on, we use $\langle \cdot, \cdot \rangle$ for the metrics $\bar{g}$ and g unless they are confused. Then (2.1) shows that U is totally umbilic in

$\bar{M}_k^{n+1}(\bar{c})$ with mean curvature vector field $H = \frac{(\tau-\sigma)}{\langle V,\xi\rangle}\xi$, where ξ is a unit normal vector field on U. By the Codazzi equation, we have

$$H = a_i\xi, \quad A_\xi = \epsilon a_i I \tag{2.2}$$

for some constant a_i on each connected component U_i of U where $\epsilon = \langle \xi, \xi\rangle = \pm 1$. Hence, the Gauss equation shows that each U_i has constant sectional curvature $c_i = \bar{c} + \epsilon a_i^2$. From (2.2) and the hypothesis we obtain

$$\nabla\langle V,\xi\rangle = -\{\bar{\nabla}_\xi V + \epsilon a_i V\}^T, \tag{2.3}$$

where $\nabla\langle V,\xi\rangle$ is the gradient vector of $\langle V,\xi\rangle$ on M^n. Furthermore, on each U_i, by differentiating both sides of (2.3), we have

$$\begin{aligned} &-\langle \nabla_X\nabla\langle V,\xi\rangle, Y\rangle \\ &= \{\langle \bar{\nabla}_X\bar{\nabla}_\xi V, Y\rangle + \epsilon a_i\langle \bar{\nabla}_X V, Y\rangle\} \\ &\quad + \{\epsilon a_i\sigma + \epsilon a_i^2\langle V,\xi\rangle\}\langle X, Y\rangle, \quad X, Y \in TM^n. \end{aligned} \tag{2.4}$$

Since $\bar{M}_k^{n+1}(\bar{c})$ has constant sectional curvature $\bar{c}$, for $X, Y \in TM^n$ the Riemann curvature tensor $\bar{R}$ of $\bar{M}_k^{n+1}(\bar{c})$ satisfies

$$\langle \bar{R}(X,\xi)V, Y\rangle = \bar{c}\langle V,\xi\rangle\langle X, Y\rangle. \tag{2.5}$$

Note that we can extend X, Y locally to $\bar{M}_k^{n+1}(\bar{c})$ so that

$$\bar{\nabla}_\xi X = \bar{\nabla}_\xi Y = 0. \tag{2.6}$$

Hence on U_i we have from (2.2) and (2.5)

$$\begin{aligned} &\langle \bar{\nabla}_X\bar{\nabla}_\xi V, Y\rangle + \epsilon a_i\langle \bar{\nabla}_X V, Y\rangle \\ &= \langle \bar{\nabla}_\xi\bar{\nabla}_X V, Y\rangle + \bar{c}\langle V,\xi\rangle\langle X, Y\rangle, \quad X, Y \in TM^n. \end{aligned} \tag{2.7}$$

Since the left hand side of (2.4) and the second term of right hand side of (2.4) are symmetric in $X, Y \in TM^n$, respectively, we easily see that $\langle \bar{\nabla}_\xi\bar{\nabla}_X V, Y\rangle$ is symmetric in $X, Y \in TM^n$ on each on U_i. Thus we have from (2.6)

$$\xi\langle \bar{\nabla}_X V, Y\rangle = \xi\langle \bar{\nabla}_Y V, X\rangle, X, Y \in TM^n. \tag{2.8}$$

Using the conformality of V, from (2.8) we obtain

$$\langle \bar{\nabla}_\xi \bar{\nabla}_X V, Y\rangle = \xi(\sigma)\langle X, Y\rangle, \quad X, Y \in TM^n. \tag{2.9}$$

On the other hand, equations (2.7) and (2.9) imply that for $X, Y \in TM^n$

$$\begin{aligned} &\langle \bar{\nabla}_X \bar{\nabla}_\xi V, Y\rangle \\ &= \xi(\sigma)\langle X, Y\rangle - \epsilon a_i \langle \bar{\nabla}_X V, Y\rangle + \bar{c}\langle V, \xi\rangle\langle X, Y\rangle. \end{aligned} \tag{2.10}$$

Hence (2.4) gives

$$\langle \nabla_X \nabla \langle V, \xi\rangle, Y\rangle = -\{c_i\langle V, \xi\rangle + \xi(\sigma) + \epsilon a_i \sigma\}\langle X, Y\rangle, \tag{2.11}$$

on each U_i where $X, Y \in TM^n$ and c_i denotes the sectional curvature $\bar{c} + \epsilon a_i^2$ of U_i. Since V satisfies $\mathfrak{L}_V \bar{g} = 2\sigma\bar{g}$ on $\bar{M}_k^{n+1}(\bar{c})$, σ satisfies the following ([19] or [13]; Corollary 2.2):

$$\bar{\nabla}_X \bar{\nabla}\sigma = -\bar{c}\sigma X, \quad X \in T\bar{M}. \tag{2.12}$$

This together with (2.2) implies that $\xi(\sigma) + \epsilon a_i \sigma$ is a constant b_i on each U_i. Thus from (2.11) we obtain on each U_i

$$\nabla_X \nabla\langle V, \xi\rangle = -\{c_i\langle V, \xi\rangle + b_i\}X, \quad X \in TM^n, \tag{2.13}$$

which implies that $\nabla\langle V, \xi\rangle$ is a closed conformal vector field on each U_i, and hence on the closure of U.

If the complement of U has nonempty interior, then $\nabla\langle V, \xi\rangle$ is a trivial closed conformal vector field on it. Thus, by continuity, $\nabla\langle V, \xi\rangle$ is a closed conformal vector field on M^n. Therefore Proposition 2.3 in [12] shows that either U is dense in M^n, or $V^N = 0$ identically on M^n. This completes the proof. □

Now we characterize conformal vector fields on totally umbilic hypersurface of semi-Riemannian space form $\bar{M}_k^{n+1}(\bar{c})$ as follows:

THEOREM 3. *Let M^n be a connected totally umbilic hypersurface of a semi-Riemannian space form $\bar{M}_k^{n+1}(\bar{c})$. Then any conformal vector field on M^n can be obtained as the tangential part V^T of a conformal vector field V on $\bar{M}_k^{n+1}(\bar{c})$. Furthermore, for any conformal vector field W on M^n there exists a unique conformal vector field V on $\bar{M}_k^{n+1}(\bar{c})$ which satisfies $V|_{M^n} = W$.*

Proof. Let $C(\bar{M})$ and $C(M)$ denote the space of conformal vector fields on $\bar{M}_k^{n+1}(\bar{c})$ and on M^n respectively. Define a map $\psi : C(\bar{M}) \to C(M)$ by $\psi(V) = V^T$. Then Proposition 2 shows that ψ is a well-defined linear map.

Suppose that $V \in Ker\psi$. Then we have $V = f\xi$ on M^n, where $f = \epsilon\langle V, \xi\rangle$, ξ is a unit normal vector field on M^n, and ϵ denotes $\langle \xi, \xi\rangle = \pm 1$. Since M^n is totally umbilic, (2.13) shows that f satisfies

$$\nabla_X \nabla f = -(cf + b)X, \quad X \in TM, \tag{2.14}$$

where c denotes the constant sectional curvature of M^n and b is a constant.

We denote by $GC(M^n)$ the space of all functions f on M^n satisfying (2.14) for some constant $b \in R$. Then $GC(M^n)$ is of $(n+2)$-dimensional ([18]; pp.336–337). If we define $\varphi : ker\psi \to GC(M^n)$ by $\varphi(V) = \epsilon\langle V, \xi\rangle$, then (2.14) shows that φ is a well-defined linear map. From the fact that the codimension of the zero set of a nontrivial conformal vector field is greater than 1, we see that φ is injective. Hence we have

$$\dim Ker\psi \leq \dim GC(M) = n + 2. \tag{2.15}$$

Since $\dim C(\bar{M}) = \frac{(n+2)(n+3)}{2}$ and $\dim C(M) = \frac{(n+1)(n+2)}{2}$ ([11]), by counting dimensions, we see that the inequality in (2.15) becomes an equality, and $\dim Im\psi = \dim C(M)$, which implies that φ is bijective and ψ is surjective.

For any fixed $W \in C(M)$ we choose a $V_0 \in C(\bar{M})$ such that $\psi(V_0) = W$. If we denote by $f\xi$ the normal part V_0^N of V_0 on M^n, then (2.13) shows that f and $-f$ belong to $GC(M^n)$. Thus it follows from the bijectivity of φ that there exists a unique V_1 in $Ker\psi$ which satisfies $V_1|_{M^n} = -f\xi$. Therefore $V = V_0 + V_1$ is the desired conformal vector field in $C(\bar{M})$. This completes the proof. □

3. Conformal vector fields on space forms

In this section, first of all, we give a complete description of conformal vector fields on semi-Euclidean space R_k^{n+1}, which might be well-known but we could not find a reference for it (cf. [9]; pp. 25–26). Then theorem 3 gives a complete description of conformal vector fields on non-flat semi-Riemannian space forms (cf. [18]; pp. 336–337).

Consider the semi-Euclidean space $(R_k^{n+1}, \bar{g})$ with metric tensor $ds^2 = \sum_{i=1}^{n+1} \epsilon_i dx_i^2$, where $\epsilon_1 = \cdots = \epsilon_k = -1, \epsilon_{k+1} = \cdots = \epsilon_{n+1} = 1$. If $V = (V_1, \cdots, V_{n+1})$ is a conformal vector field on R_k^{n+1} which satisfies

$$\mathfrak{L}_V \bar{g} = 2\sigma \bar{g}, \tag{3.1}$$

then σ satisfies $\bar{\nabla}_X \bar{\nabla}\sigma = 0$ ([19] or [13]; Corollary 2.2), so that we have for some constants $a_1, \cdots, a_{n+1}, b$

$$\sigma(x_1, \cdots, x_{n+1}) = \sum_{j=1}^{n+1} a_j x_j + b. \tag{3.2}$$

From (3.1) and (3.2) we obtain the following:

$$V_{j,j} = \sigma, \quad j \in \{1, 2, \cdots, n+1\}, \tag{3.3}$$

$$\epsilon_j V_{j,k} + \epsilon_k V_{k,j} = 0 \quad \text{for distinct} \quad j, k \in \{1, 2, \cdots, n+1\}, \tag{3.4}$$

where $V_{j,k}$ denotes the k-th partial derivative $\frac{\partial V_j}{\partial x_k}$ of V_j. If we also denote by $V_{j,kl}$ the l-th partial derivative of $V_{j,k}$, then (3.4) implies that for all distinct $j, k, l \in \{1, 2, \cdots, n+1\}$

$$\begin{aligned} V_{j,k\ell} &= -(\epsilon_j \epsilon_k) V_{k,\ell j} = (-\epsilon_j \epsilon_k)(-\epsilon_k \epsilon_\ell) V_{\ell, jk} \\ &= (-\epsilon_j \epsilon_k)(-\epsilon_k \epsilon_\ell)(-\epsilon_\ell \epsilon_j) V_{j,k\ell} = -V_{j,k\ell}, \end{aligned}$$

so that we have

$$V_{j,k\ell} = 0 \quad \text{for distinct} \quad j, k, \ell \in \{1, \cdots, n+1\}. \tag{3.5}$$

From (3.2), (3.3) and (3.4) we also have the following:

$$V_{j,kj} = V_{j,jk} = a_k, \quad j, k \in \{1, \cdots, n+1\}, \tag{3.6}$$

$$\begin{aligned} V_{j,kk} &= (-\epsilon_j \epsilon_k) V_{k,jk} = -\epsilon_j \epsilon_k V_{k,kj} \\ &= -\epsilon_j \epsilon_k a_j \quad \text{for distinct} \quad j, k \in \{1, \cdots, n+1\}. \end{aligned} \tag{3.7}$$

Hence (3.5) together with (3.6) and (3.7) implies for distinct $j, k \in \{1, \cdots, n+1\}$

$$V_{j,k} = a_k x_j - \epsilon_j \epsilon_k a_j x_k + b_{jk}, \quad b_{jk} \in R. \tag{3.8}$$

Furthermore, (3.4) and (3.8) show that b_{jk} satisfies

$$\epsilon_j b_{jk} + \epsilon_k b_{kj} = 0 \quad \text{for distinct} \quad j, k \in \{1, \cdots, n+1\}. \tag{3.9}$$

Thus from (3.3) and (3.8) we have for some constants c_j, $j \in \{1, 2, \cdots, n+1\}$

$$V_j(x_1, \cdots, x_{n+1}) = \sigma x_j - \frac{1}{2}\epsilon_j a_j \langle x, x\rangle + \sum_{i \neq j} b_{ji} x_i + \frac{1}{2} c_j, \tag{3.10}$$

or, equivalently we have

$$V(x_1, \cdots, x_{n+1}) = \sigma x - \frac{1}{2}\langle x, x\rangle \bar{a} + Bx + \frac{1}{2}C, \tag{3.11}$$

where $\bar{a} = (\epsilon_1 a_1, \cdots, \epsilon_{n+1} a_{n+1})$, $C = (c_1, \cdots, c_{n+1})$, σ is given by (3.2) and B denotes an $(n+1) \times (n+1)$ matrix (b_{ij}) which satisfies (3.9) and $b_{jj} = 0$. Note that $Bx + \frac{1}{2}C$ is the Killing part of V on R_k^{n+1} ([14]; p. 253).

Now consider a non-flat semi-Riemannian space form $M^n(\epsilon)$ with constant sectional curvature $\epsilon = \pm 1$ which is given by as a totally umbilic hypersurface of R_k^{n+1} for some suitable k;

$$M^n(\epsilon) = \{x \in R_k^{n+1} | \langle x, x\rangle = \epsilon\}.$$

Then Theorem 3 shows that the space $C(M)$ of conformal vector fields on $M^n(\epsilon)$ is given by

$$C(M) = \{V|_{M^n} | V \in C(R_k^{n+1}), \quad \langle V, x\rangle \equiv 0 \text{ on } M^n(\epsilon)\}. \tag{3.12}$$

Hence from (3.11) we see that on $M^n(\epsilon)$

$$\langle V, x\rangle = \frac{\epsilon}{2}\sum_{j=1}^{n+1}(a_j + \epsilon\epsilon_j c_j)x_j + b\epsilon = 0, \tag{3.13}$$

which shows that

$$\bar{a} = -\epsilon C, \quad b = 0. \tag{3.14}$$

Thus (3.11), (3.12) and (3.14) imply that every element of $C(M)$ is of the following form:

$$V|_{M^n} = \{C - \epsilon\langle C, x\rangle x + Bx\}|_{M^n}, \tag{3.15}$$

where $C = (c_1, \cdots, c_{n+1})$, and B is an $(n+1)\times(n+1)$ matrix (b_{ij}) which satisfies $\epsilon_j b_{jk} + \epsilon_k b_{kj} = 0$ for all $j, k \in \{1, \cdots, n+1\}$. The restriction of $C - \epsilon\langle C, x\rangle x$ to M^n is nothing but the tangential part C^T of a constant vector $C \in R^{n+1}_k$, which is also the gradient vector field $\nabla\sigma_C(x)$ of a linear function $\sigma_C(x) = \langle C, x\rangle$ on $M^n(\epsilon)$ ([9, 10]). And $Bx|_{M^n}$ is the Killing part of $V|_{M^n}$ on $M^n(\epsilon)$.

From (2.1), (3.1), (3.2), (3.12) and (3.14) we also see that $\mathfrak{L}_{V|_{M^n}} g = 2\tau g$ on $M^n(\epsilon)$, where τ is the restriction to $M^n(\epsilon)$ of a linear function $\sigma = -\epsilon\langle C, x\rangle$ on R^{n+1}_k (cf. Lemma in [3]).

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DONG-SOO KIM, SEON-BU KIM, AND SEONG-HEE PARK, DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, CHONNAM NATIONAL UNIVERSITY, KWANGJU 500-757, KOREA
E-mail: dosokim@chonnam.chonnam.ac.kr
sbk@chonnam.chonnam.ac.kr

YOUNG HO KIM, DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA
E-mail: yhkim@knu.ac.kr