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# A NOTE ON TYPES OF NOETHERIAN LOCAL RINGS

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Abstract. In this note we investigate some results which concern the types of local rings. In particular it is shown that if the type of a quasi-unmixed local ring $A$ is less than or equal to depth $A+1$, and $\hat{A}_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p} \neq \hat{\mathfrak{m}}$, then $A$ is Cohen-Macaulay. (This implies the previously known result: if $\hat{A}$ satisfies $(S_{n-1})$, where $n$ is the type of a ring $A$, then $A$ is Cohen-Macaulay.)

## 1. Backgrounds

Throughout this paper, we assume that $(A, \mathfrak{m})$ is a commutative Noetherian local ring of dimension $d$, and $M$ is a finitely generated $A$-module. We also assume that all modules are unitary.

For a prime ideal $\mathfrak{p}$ of $A$, the $i$-th Bass number of $M$ at $\mathfrak{p}$, denoted $\mu_i(\mathfrak{p}, M)$, is defined to be $\dim_{k(\mathfrak{p})}\mathrm{Ext}^i_{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$, where $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$: we set $\mu_i(A)=\mu_i(\mathfrak{m}, A)$ for brevity. The *type* of $A$, denoted by $r(A)$, is defined to be $\mu_d(A)$. One of the interesting studies on the types of rings is to investigate the conditions which make $A$ Cohen-Macaulay when a type of $A$ is known.

Gorenstein rings were characterized by Bass ([2]) as Cohen-Macaulay rings $A$ with $r(A) = 1$. Vasconcelos ([15]) conjectured that the condition $r(A) = 1$ is sufficient for $A$ to be Gorenstein, i.e., the condition "$A$ is Cohen-Macaulay" can be omitted. In [5], Foxby proved this conjecture for essentially equicharacteristic rings using a version of the Intersection Theorem. The conjecture was proven in general by Roberts ([14]): he showed that local rings of type one are Cohen-Macaulay, (and hence Gorenstein) using a minimal free resolution of a dualizing complex:

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THEOREM 1. ([14]) *Let $A$ be a Noetherian local ring of dimension $d$. If $r(A) = 1$, then $A$ is Gorenstein.*

By modifying Roberts' argument, Costa, Huneke and Miller showed the following:

THEOREM 2. ([4]) *Let $A$ be a local ring whose completion is a domain. If $r(A) = 2$, then $A$ is Cohen-Macaulay.*

Expecting the above might be the best possible, they gave two examples: a complete equidimensional local ring of type two that is not Cohen-Macaulay, and a complete reduced local ring of type two that is not Cohen-Macaulay. Afterward, the following question was posed:

*Does there exist a complete, equidimensional, reduced local ring $A$ with $r(A) = 2$ that is not Cohen-Macaulay?*

However, Marley ([9]) answered this question in negative by proving the following theorem: we note here that a complete equidimensional reduced local ring is unmixed since a ring is reduced if and only if it satisfies conditions $(R_0) + (S_1)$ ([10]).

THEOREM 3. ([9]) *Let $A$ be an unmixed local ring of type two. Then $A$ is Cohen-Macaulay.*

Marley also asked that if a complete local ring of type $n$ satisfies Serre's condition $(S_{n-1})$, then it is Cohen-Macaulay: if $n = 1$ (resp. $n = 2$), then the answer is 'yes' by Roberts' theorem (resp. by Marley's theorem). Kawasaki answered this question in the affirmative when rings contains a field and $n \geq 3$. He used a result of Brun's ([3]) which was proved using a big-Cohen-Macaulay module. Marley question was proved in general by Aoyama ([1]) using Kawasaki's idea.

THEOREM 4. ([1]) *Let $n \geq 3$ be an integer. If $r(A) \leq n$ and $\hat{A}$ is $(S_{n-1})$, then $A$ is Cohen-Macaulay.*

In [8], Kawasaki conjectured the following which is still open:

CONJECTURE. ([8]) Let $A$ be a complete unmixed local ring of type $n$. If $A_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p}$ in $Spec(A)$ such that $ht(\mathfrak{p}) < n$, then $A$ is Cohen-Macaulay.

We note that the condition "$A_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p}$ in $Spec(A)$ such that $ht(\mathfrak{p}) < n$" is weaker than $(S_{n-1})$, and that if $A$ is complete and $(S_2)$, then $A$ is unmixed; hence Conjecture implies Theorem 4.

If $r(A) > \dim A$ in the above conjecture, then the conjecture trivially holds. The other case, i.e., when $r(A) \leq \dim A$, seems difficult to be established. In Section 2, we consider the rings with $r(A) \leq \operatorname{depth} A + 1$, and extra conditions.

## 2. Main theorem

We first pose the following question, which is motivated by the proof of Aoyama's theorem (Theorem 4).

PROBLEM 2.1. *Let $(A, \mathfrak{m})$ be a complete unmixed Noetherian local ring. If $r(A) \leq \operatorname{depth} A + 1$, where $r(A)$ is the type of $A$, then is $A$ Cohen-Macaulay?, or equivalently, if $A$ is not Cohen-Macaulay, then $r(A) \geq \operatorname{depth} A + 2$?*

We recall that $A$ is *equidimensional* if $\dim A/\mathfrak{p} = \dim A$ for every minimal prime $\mathfrak{p}$ of $A$. $A$ is said to be *quasi-unmixed* (or formally equidimensional) if its completion $\hat{A}$ is equidimensional, and to be *unmixed* if $\dim \hat{A}/\mathfrak{p} = \dim \hat{A}$ for all $\mathfrak{p} \in Ass(\hat{A})$.

A finite $A$-module $M$ is said to *satisfy a Serre's condition* $(S_t)$ if $\operatorname{depth} M_{\mathfrak{p}} \geq \min\{t, \dim M_{\mathfrak{p}}\}$ for every prime $\mathfrak{p}$ in $Supp(M)$.

PROPOSITION 2.2. *Assume that Problem 2.1 is true. Then the following hold:*
(1) (Aoyama's result) *Let $n \geq 3$ be an integer. If $r(A) \leq n$ and $\hat{A}$ is $(S_{n-1})$, then $A$ is Cohen-Macaulay.*
(2) (Marley's result) *Let $A$ be an unmixed local ring of type two. Then $A$ is Cohen-Macaulay.*

*Proof.* For (1), we may assume that $A$ is complete. Since $A$ is $(S_2)$, $A$ is unmixed. If $A$ is not Cohen-Macaulay, then $\operatorname{depth} A \leq r(A) - 2$ by assumption. Thus $\operatorname{depth} A < r(A) - 1 \leq n - 1$ which contradicts that $A$ is $(S_{n-1})$. Hence $A$ is Cohen-Macaulay.

For (2), we may also assume that $A$ is complete. Suppose that $A$ is not Cohen-Macaulay. If $r(A) = 2$, then $\operatorname{depth} A \leq r(A) - 2 = 0$, and so $\mathfrak{m} \in Ass(A)$. Since $A$ is unmixed, $\dim A = 0$, which implies that $A$ is Cohen-Macaulay. $\square$

With some extra conditions, we answer Problem 2.1 in the affirmative using the techniques of the proof of Aoyama's theorem.

THEOREM 2.3. *Let* $(A, \mathfrak{m})$ *be a quasi-unmixed Noetherian local ring of dimension* $d$ *such that* $r(A) \leq \operatorname{depth} A + 1$. *Suppose that* $\hat{A}_{\mathfrak{p}}$ *is Cohen-Macaulay for every prime* $\mathfrak{p} \neq \hat{\mathfrak{m}}$. *Then* $A$ *is Cohen-Macaulay.*

*Proof.* We may assume that $A$ is complete. Suppose that $A$ is not Cohen-Macaulay and $\operatorname{depth} A = t < d$. Let

$$(I^\bullet, \phi^\bullet) : 0 \to A \to I^0 \to I^1 \to \cdots \to I^i \to \cdots,$$

be a minimal injective resolution of $A$, where $I^i = \oplus_{\mathfrak{p} \in \operatorname{Spec}(A)} E(A/\mathfrak{p})^{\mu_i(\mathfrak{p})}$. By applying $\operatorname{Hom}_A(-, E(A/\mathfrak{m}))$ to $H^0_{\mathfrak{m}}(I^\bullet)$ $(= \varinjlim \operatorname{Hom}_A(A/\mathfrak{m}^r, I^\bullet))$, we have

$$(F_\bullet, f_\bullet) : \cdots \to A^{\mu_i(\mathfrak{m})} \xrightarrow{f_{i-1}} A^{\mu_{i-1}(\mathfrak{m})} \to \cdots \to A^{\mu_{t+1}(\mathfrak{m})} \to A^{\mu_t(\mathfrak{m})} \to 0.$$

We note that the $i$-th homology of $F_\bullet$ is $H^i_{\mathfrak{m}}(A)^\vee$, where $(-)^\vee$ denotes the Matlis dual.

Since $A_{\mathfrak{p}}$ is Cohen-Macaulay for all primes $\mathfrak{p} \neq \mathfrak{m}$, we can show that $H_i(F_\bullet)_{\mathfrak{p}} = 0$ for $i \neq d$. Indeed, since $A$ is complete, $A \cong S/J$ such that $(S, \mathfrak{m}_S)$ is a Gorenstein local ring, $J$ is an ideal of $S$ and $\dim S = \dim A$. Hence by local duality

$$H^i_{\mathfrak{m}}(A)^\vee \cong H^i_{\mathfrak{m}_S}(A)^\vee \cong \operatorname{Ext}^{d-i}_S(A, S).$$

For a prime ideal $\mathfrak{p} (\neq \mathfrak{m})$ of $A$, since $A_{\mathfrak{p}}$ is Cohen-Macaulay and $S_{\mathfrak{p}_s}$ is Gorenstein, it follows that

$$\operatorname{Ext}^j_S(A, S) \otimes S_{\mathfrak{p}_s} \cong \operatorname{Ext}^j_{S_{\mathfrak{p}_s}}(A_{\mathfrak{p}}, S_{\mathfrak{p}_s}) = 0 \text{ if } j \neq \dim S_{\mathfrak{p}_s} - \dim A_{\mathfrak{p}}.$$

We note that $\dim S_{\mathfrak{p}_s} = \dim A_{\mathfrak{p}}$ since $A$ is quasi-unmixed and $\dim S = \dim A$. Thus it is obtained that $H_i(F_\bullet \otimes A_{\mathfrak{p}}) = 0$ if $i \neq d$ since

$$\begin{aligned} H_i(F_\bullet \otimes A_{\mathfrak{p}}) \cong H_i(F_\bullet) \otimes A_{\mathfrak{p}} &\cong H^i_{\mathfrak{m}}(A)^\vee \otimes A_{\mathfrak{p}} \\ &\cong \operatorname{Ext}^{d-i}_S(A, S) \otimes S_{\mathfrak{p}_s} \cong \operatorname{Ext}^{d-i}_{S_{\mathfrak{p}_s}}(A_{\mathfrak{p}}, S_{\mathfrak{p}_s}). \end{aligned}$$

Thus $(F_d \to F_{d-1} \to \cdots \to F_t \to 0) \otimes A_{\mathfrak{p}}$ is exact and so split. Now we assume that $\operatorname{depth} A = t < d - 1$ (see Lemma 2.4 below for the case of $t = d - 1$: we know $\ell(H^{d-1}_{\mathfrak{m}}(A)) < \infty$ since $Supp(H_{d-1}(F_\bullet)) = \{\mathfrak{m}\}$, and $H_{d-1}(F_\bullet)$ is finitely generated), and follow the procedure of Aoyama's proof. Let

$$G_\bullet : 0 \to G_{d-t} \xrightarrow{g_{d-t}} G_{d-t-1} \to \cdots \to G_1 \xrightarrow{g_1} G_0,$$

where $G_i = \mathrm{Hom}_A(F_{d-i}, A)$, and $g_i = \mathrm{Hom}_A(f_{d-i}, A)$. For $j = 1, \cdots, d-t$, let $r_j = \sum_{i=j}^{d-t} rank(G_i)$ and $I_j$ the ideal generated by the $r_j$-minors of $g_j$. Then it can be shown $rank(F_d) - r_1 > 1$, and $r_1 \geq t$. Thus we have

$$r(A) = \mu_d(\mathfrak{m}) = rank(F_d) > r_1 + 1 \geq t + 1 = \operatorname{depth} A + 1 \geq r(A),$$

which is a contradiction. Hence $A$ is Cohen-Macaulay. □

PROPOSITION 2.4. (Lemma 2.4) *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d$, and* $\operatorname{depth} A = d - 1$ *(and so $A$ is not Cohen-Macaulay). Suppose $\ell(H_{\mathfrak{m}}^{d-1}(A)) < \infty$. Then $r(A) \geq$* $\operatorname{depth} A + 2$.

*Proof.* We may assume that $A$ is complete. Suppose to the contrary that $r(A) < \operatorname{depth} A + 2$. As in the proof of Theorem 2.3, let

$$(I^\bullet, \phi^\bullet) : 0 \to A \to I^0 \to I^1 \to \cdots \to I^i \to \cdots$$

be a minimal injective resolution of $A$. By applying $\mathrm{Hom}_A(-, E(A/\mathfrak{m}))$ to $H_{\mathfrak{m}}^0(I^\bullet)$ $(= \varinjlim \mathrm{Hom}_A(A/\mathfrak{m}^r, I^\bullet))$, we have

$$(F_\bullet, f_\bullet) : \cdots \to A^{\mu_d(\mathfrak{m})} \xrightarrow{f_{d-1}} A^{\mu_{d-1}(\mathfrak{m})} \to 0.$$

We note that $Supp(H_{d-1}(F_\bullet)) = \{\mathfrak{m}\}$ since

$$\ell(H_{d-1}(F_\bullet)) = \ell((H_{\mathfrak{m}}^{d-1}(A))^\vee) = \ell(H_{\mathfrak{m}}^{d-1}(A)) < \infty.$$

Thus if $I_1$ is an ideal generated by the maximal minors of $f_{d-1}$, then $I_1$ is $\mathfrak{m}$-primary since $(F_d \to F_{d-1} \to 0) \otimes_A A_{\mathfrak{p}}$ is exact for all primes $\mathfrak{p} \neq \mathfrak{m}$. By Theorem 13.10 in [10] and the assumption, we know that

$$d = \operatorname{ht} I_1 \leq rank(F_d) - rank(F_{d-1}) + 1 \leq r(A) \leq \operatorname{depth} A + 1 = d.$$

Hence $d = r(A) = rank(F_d)$, and $rank(F_{d-1}) = 1$. Thus $H_{d-1}(F_\bullet) = A/(x_1, \cdots, x_d)A$, where $x_1, \cdots, x_d \in \mathfrak{m}$. Since $\ell(H_{d-1}(F_\bullet)) < \infty$, $x_1, \cdots, x_d$ is a system of parameters of $A$. Note that $(x_1, \cdots, x_d) \subseteq Ann\,(H_{\mathfrak{m}}^{d-1}(A))$ since $H_{\mathfrak{m}}^{d-1}(A) = (H_{d-1}(F_\bullet))^\vee = (A/(x_1, \cdots, x_d)A)^\vee$. Thus by Lemma 2.(c) in [11], $H_{\mathfrak{m}}^{d-1}(A) \cong H_1(K_\bullet(\underline{x}))$ since $\ell(H_{\mathfrak{m}}^{d-1}(A)) < \infty$. Since $\operatorname{depth} A = d - 1$, we have $H_i(K_\bullet(\underline{x})) = 0$ for all $i > 1$. Hence

$$e(\underline{x}; A) = \ell(A/\underline{x}) - \ell(H_1(K_\bullet(\underline{x}))) = \ell(A/\underline{x}) - \ell((A/\underline{x})^\vee) = 0,$$

which is a contradiction. Hence $r(A) \geq \operatorname{depth} A + 2$. □

REMARK 2.5. In the above lemma, the condition $\ell(H_{\mathfrak{m}}^{d-1}(A)) < \infty$ can be replaced by '$\hat{A}$ satisfies Serre condition $(S_{d-1})$' since $\ell(H_{\mathfrak{m}}^{d-1}(A)) < \infty$ if and only if $\hat{A}$ is equidimensional and $(S_{d-1})$ ([7]).

REMARK 2.6. The author later learned that Theorem 2.3 was shown by Kawasaki in the case when $A$ contains a field ([8, Proposition 3.3 (3)]). A finitely generated module $M$ is called *a module with finite local cohomologies* if $H^i_{\mathfrak{m}}(M)$ has a finite length for every $i \neq \dim M$. It is known ([6 or 8]) that if $M$ is a module with finite local cohomologies, then $M$ is equidimensional and $M_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p} \neq \mathfrak{m}$ (the converse also holds if $A$ is a homomorphic image of a Cohen-Macaulay ring). This fact and Theorem 2.3 give us the following corollary:

COROLLARY 2.7. *Let $A$ be a Noetherian local ring (not necessarily containing a field). If $A$ is a non-Cohen-Macaulay ring with finite local cohomologies, then $r(A) \geq \operatorname{depth} A + 2$.*

*Proof.* We may assume that $A$ is complete. Then $A$ is quasi-unmixed, and for a prime $\mathfrak{p} \neq \hat{\mathfrak{m}}$ $\hat{A}_{\mathfrak{p}}$ is Cohen-Macaulay. Thus by Theorem 2.3, the conclusion follows. □

Theorem 4 (Aoyama) is also obtained from Theorem 2.3. We first remind that if $A$ is quasi-unmixed, then (i) $A_{\mathfrak{p}}$ is quasi-unmixed for every prime $\mathfrak{p}$ of $A$, and (ii) $A/I$ is equidimensional if and only if $A/I$ is quasi-unmixed for an ideal $I$ of $A$ ([10, Theorem 31.6]). Then we have the following:

LEMMA 2.8. *Let $(A, \mathfrak{m})$ be a quasi-unmixed Noetherian local ring of dimension $d$. Suppose $A$ is a homomorphic image of a Cohen-Macaulay ring. Then $A_{\mathfrak{p}}$ is Cohen-Macaulay for every prime $\mathfrak{p} \neq \mathfrak{m}$ if and only if $\hat{A}_{\mathfrak{p}}$ is Cohen-Macaulay for every $\mathfrak{p} \neq \hat{\mathfrak{m}}$.*

*Proof.* If $A$ is quasi-unmixed, then $A$ is equidimensional by the above note, and so together with the assumptions $A$ is a ring with finite local cohomologies by Remark 2.6. Thus $\hat{A}$ is also a ring with finite local cohomologies since $H^i_{\mathfrak{m}}(A) \cong H^i_{\hat{\mathfrak{m}}}(\hat{A})$ for each $i$. Hence $\hat{A}_{\mathfrak{p}}$ is Cohen-Macaulay for every $\mathfrak{p} \neq \hat{\mathfrak{m}}$.

The converse also holds similarly again using Remark 2.6 since $\hat{A}$ is equidimensional and $\hat{A}$ is a homomorphic image of a Cohen-Macaulay ring. □

For a finitely generated $A$-module $M$, it is known ([13]) that if $ht(\mathfrak{q}/\mathfrak{p}) = t$ for primes $\mathfrak{p}$ and $\mathfrak{q}$, then $\mu_i(\mathfrak{p}, M) \leq \mu_{i+t}(\mathfrak{q}, M)$. Using this fact, it is easy to obtain that if $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$, then $r(M_{\mathfrak{p}}) \leq r(M)$. (Let $\dim M = s$ and $\dim A/\mathfrak{p} = t$, i.e., $ht(\mathfrak{m}/\mathfrak{p}) = t$. Then $\dim M_{\mathfrak{p}} =$

$s - t$, and $r(M_{\mathfrak{p}}) = \mu_{s-t}(\mathfrak{p}, M) \leq \mu_s(\mathfrak{m}, M) = r(M)$.) In particular, $r(A_{\mathfrak{p}}) \leq r(A)$ for a prime $\mathfrak{p}$ if $A$ is unmixed and catenary.

COROLLARY 2.9. ([1]) *Let $n \geq 3$ be an integer. If $r(A) \leq n$ and $\hat{A}$ is $(S_{n-1})$, then $A$ is Cohen-Macaulay.*

*Proof.* We may assume that $A$ is complete. Since $A$ is $(S_2)$ and catenary, $A$ is unmixed ([12]). Suppose that $A$ is not Cohen-Macaulay. If $A_{\mathfrak{p}}$ is Cohen-Macaulay for each prime $\mathfrak{p} \neq \mathfrak{m}$, then by Theorem 2.3, $\operatorname{depth} A < r(A) - 1$, which implies $\operatorname{depth} A < n - 1$. This contradicts that $A$ is $(S_{n-1})$, and hence $A$ is Cohen-Macaulay.

Suppose that $A_{\mathfrak{p}}$ is not Cohen-Macaulay for some prime $\mathfrak{p} \neq \mathfrak{m}$, but $A_{\mathfrak{q}}$ is Cohen-Macaulay for every prime $\mathfrak{q} \subsetneqq \mathfrak{p}$. Since $A$ is quasi-unmixed, $A_{\mathfrak{p}}$ is also quasi-unmixed by Theorem 31.6 in [10]. We note that since $A$ is complete, $A$ is a homomorphic image of a Cohen-Macaulay local ring, and so is $A_{\mathfrak{p}}$ (since $R \to A \to 0$ implies $R_{\mathfrak{p}} \to A_{\mathfrak{p}} \to 0$). Thus $\hat{A}_{\mathfrak{p}}\mathfrak{q}'$ is Cohen-Macaulay for every prime $\mathfrak{q}'$ which is not maximal in $\hat{A}_{\mathfrak{p}}$ by Lemma 2.8. Since $A$ is $(S_{n-1})$, unmixed, and catenary, we have

$$r(A_{\mathfrak{p}}) \leq r(A) \leq n \leq \operatorname{depth} A_{\mathfrak{p}} + 1.$$

Thus $A_{\mathfrak{p}}$ is Cohen-Macaulay by Theorem 2.3, which is a contradiction. This completes the proof. □

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