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# FUZZY INTERIOR SPACES

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ABSTRACT. In this paper, we study some properties of fuzzy interior spaces. Also, we investigate the relations between fuzzy interior spaces and fuzzy topological spaces. In particular, we prove the existence of product fuzzy topological spaces and product fuzzy interior spaces. We investigate the relations between them.

## 1. Introduction

The concept of a fuzzy topology was first defined in 1968 by Chang [1] and later redefined in somewhat different way by Lowen [13] and by Hutton [9]. Šostak [16] introduced a new definition of fuzzy topology as the concept of the degree of the openness of a fuzzy set. Chattopadhyay et al. [4] introduced the fuzzy closure spaces in a Šostak's sense. Höhle et al. [7, 8] substituted a lattice $L$ ( GL-monoid, cqm-lattice) for the unit interval or the two-point lattice $2 = \{0, 1\}$ in the definitions of fuzzy topologies and fuzzy closure spaces in [1, 4, 11, 13, 16].

In this paper, we introduced the notion of fuzzy interior spaces in a sense [7, 8] and we study some properties of them. We investigate the relations between fuzzy interior spaces and fuzzy topological spaces. In particular, we prove the existence of product fuzzy topological spaces and product fuzzy interior spaces. We investigate the relations between them.

## 2. Preliminaries

In this paper, let $X$ be a nonempty set. Let $L = (L, \leq, \vee, \wedge)$ be a completely distributive lattice with the least element 0 and the greatest

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element 1 in $L$ (ref. [12]). For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for each $x \in X$.

DEFINITION 2.1. [7, 8, 18]. A triple $(L, \leq, \otimes)$ is called a *strictly two-sided, commutative quantale lattice* (scq-lattice, for short) iff it satisfies the following properties:

(L1) $(L, \otimes)$ is a commutative semigroup.

(L2) $x = x \otimes 1$, for each $x \in L$.

(L3) $\otimes$ is distributive over arbitrary joins, i.e.,

$$(\bigvee_{i \in \Gamma} r_i) \otimes s = \bigvee_{i \in \Gamma} (r_i \otimes s).$$

A scq-lattice $(L, \leq, \otimes)$ is called *idempotent* if $x \otimes x = x$ for each $x \in L$.

REMARK 2.2 [7, 8, 18]. (1) Every completely distributive lattice $(L, \leq, \wedge)$ coincided with $\otimes = \wedge$ is a scq-lattice. In particular, the unit interval $([0,1], \leq, \wedge)$ is a scq-lattice.

(2) Every continuous t-norm $([0,1], \leq, t)$ coincided with $\otimes = t$ is a scq-lattice.

(3) Let $(L, \leq, \otimes)$ be a scq-lattice. For each $x, y \in L$, we define

$$x \to y = \bigvee \{z \in L \mid x \otimes z \leq y\}.$$

Then it satisfies Galois correspondence, i.e.

$$(x \otimes y) \leq z \text{ iff } x \leq (y \to z).$$

(4) Let $(L, \leq, \otimes)$ be a scq-lattice. If $x \leq y$ for each $x, y \in L$, by (L3) of Definition 2.1, then $x \otimes z \leq y \otimes z$.

In a scq-lattice $L$, $x^* = (x \to 0)$ is called *complement* of $x \in L$.

DEFINITION 2.3. [7, 8, 18]. A scq-lattice $(L, \leq, \otimes)$ is called a *complete MV-algebra* iff it satisfies the following property:

(MV) $(x \to y) \to y = x \vee y, \ \ \forall x, y \in L.$

LEMMA 2.4 [7, 8, 18]. *Let $L$ be a complete MV-algebra. For each $x, y, z \in L$, $\{y_i \mid i \in \Gamma\} \subset L$, we have the following properties.*

(1) $(x^*)^* = x$.

(2) $x \wedge y = x \otimes (x \to y)$.

(3) $x \leq y$ *iff* $x^* \geq y^*$.

(4) *If* $y \leq z$, $(x \to y) \leq (x \to z)$.

(5) $x \le y$ *iff* $x^* \ge y^*$.
(6) $x \to y = y^* \to x^*$.
(7) $x \to \bigvee_{i\in\Gamma} y_i = \bigvee_{i\in\Gamma}(x \to y_i)$.
(8) $x \to \bigwedge_{i\in\Gamma} y_i = \bigwedge_{i\in\Gamma}(x \to y_i)$.
(9) $(\bigvee_{i\in\Gamma} y_i) \to x = \bigwedge_{i\in\Gamma}(y_i \to x)$.
(10) $\bigwedge_{i\in\Gamma} y_i \to x = \bigvee_{i\in\Gamma}(y_i \to x)$.
(11) *If* $L$ *is idempotent, then* $x \wedge y = x \otimes y$.

All algebraic operations on $L$ can be extended pointwise to the set $L^X$ as follows: for all $x \in X$,
(1) $f \le g$ iff $f(x) \le g(x)$;
(2) $(f \otimes g)(x) = f(x) \otimes g(x)$;
(3) $(f \to g)(x) = f(x) \to g(x)$.
We always assume that $(L, \le, \otimes)$ is a scq-lattice if we do not suggest the condition.

DEFINITION 2.5 [7, 8, 16]. A map $\mathcal{T} : L^X \to L$ is called a *fuzzy topology* on $X$ if it satisfies the following conditions:
(O1) $\mathcal{T}(\underline{0}) = \mathcal{T}(\underline{1}) = 1$,
(O2) $\mathcal{T}(\mu_1 \otimes \mu_2) \ge \mathcal{T}(\mu_1) \otimes \mathcal{T}(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$,
(O3) $\mathcal{T}(\bigvee_{i\in\Lambda} \mu_i) \ge \bigwedge_{i\in\Lambda} \mathcal{T}(\mu_i)$, for any $\{\mu_i\}_{i\in\Lambda} \subset L^X$.
The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be fuzzy topologies on $X$. We say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ ($\mathcal{T}_2$ is *coarser* than $\mathcal{T}_1$), denoted by $\mathcal{T}_2 \le \mathcal{T}_1$, if $\mathcal{T}_2(\lambda) \le \mathcal{T}_1(\lambda)$ for all $\lambda \in L^X$. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be fuzzy topological spaces. A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called *fuzzy continuous* if $\mathcal{T}_2(\lambda) \le \mathcal{T}_1(f^{-1}(\lambda))$, for all $\lambda \in L^Y$.

DEFINITION 2.6 [7]. A map $I : L^X \times L \to L^X$ is called a *fuzzy interior operator* on $X$ iff $I$ satisfies the following conditions:
(I1) $I(\underline{1}, r) = \underline{1}$, $\forall r \in L$.
(I2) $I(\lambda, r) \le \lambda$, $\forall r \in L$.
(I3) If $\lambda \le \mu$ and $r \le s$ , then $I(\lambda, s) \le I(\mu, r)$.
(I4) $I(\lambda \otimes \mu, r \otimes s) \ge I(\lambda, r) \otimes I(\mu, s)$.
(I5) $I(\lambda, 0) = \lambda$.
The pair $(X, I)$ is called a *fuzzy interior space.*
A fuzzy interior space $(X, I)$ is called *topological* if

$$I(I(\lambda, r), r) = I(\lambda, r),\ \forall \lambda \in L^X,\ r \in L.$$

A fuzzy interior space $(X, I)$ is called *weakly stratified* if

$$I(\underline{\alpha}, r) \ge \underline{\alpha}, \quad \forall \underline{\alpha} \in L^X,\ \forall r,\ \alpha \in L.$$

Let $I_1$ and $I_2$ be fuzzy interior operators on $X$. We say that $I_1$ is *finer* than $I_2$ ($I_2$ is *coarser* than $I_1$), denoted by $I_2 \leq I_1$, if $I_2(\lambda, r) \leq I_1(\lambda, r)$ for all $\lambda \in L^X, r \in L$.

THEOREM 2.7 [7]. *Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in L, \lambda \in L^X$, we define a map $I_{\mathcal{T}} : L^X \times L \to L^X$ as follows:*

$$I_{\mathcal{T}}(\lambda, r) = \bigvee \{\mu \in L^X \mid \mu \leq \lambda, \mathcal{T}(\mu) \geq r\}.$$

*Then $I_{\mathcal{T}}$ satisfies the following properties:*
(1) $I_{\mathcal{T}}(\underline{1}, r) = \underline{1}, \forall r \in L$.
(2) $I_{\mathcal{T}}(\lambda, r) \leq \lambda, \forall r \in L$.
(3) *If $\lambda \leq \mu$ and $r \leq s$, then $I_{\mathcal{T}}(\lambda, s) \leq I_{\mathcal{T}}(\mu, r)$.*
(4) $I_{\mathcal{T}}(\lambda \otimes \mu, r \otimes s) \geq I_{\mathcal{T}}(\lambda, r) \otimes I_{\mathcal{T}}(\mu, s)$.
(5) $I(\lambda, 0) = \lambda$.
(6) $I_{\mathcal{T}}(I_{\mathcal{T}}(\lambda, r), r) = I_{\mathcal{T}}(\lambda, r)$.
(7) *If $I_{\mathcal{T}}(\lambda, s) = \mu, \forall s \in K \neq \emptyset$, then $I_{\mathcal{T}}(\lambda, \bigvee K) = \mu$.*

The following theorem is the similar result of Theorem 8.1.2 in [7].

THEOREM 2.8 [7]. *Let a map $I : L^X \times L \to L^X$ be a fuzzy interior operator. Define a map $\mathcal{T}_I : L^X \to L$ by*

$$\mathcal{T}_I(\lambda) = \bigvee \{r \in L \mid \lambda \leq I(\lambda, r)\}.$$

*Then it satisfies the following properties.*
(1) *$\mathcal{T}_I$ is a fuzzy topology on $X$.*
(2) *If $\mathcal{T}$ is a fuzzy topology on $X$, then $\mathcal{T}_{I_{\mathcal{T}}} = \mathcal{T}$.*
(3) *$I_{\mathcal{T}_I} = I$ iff $I$ is topological and $I(\lambda, s) = \mu, \forall s \in K \neq \emptyset$ implies $I(\lambda, \bigvee K) = \mu$.*

## 3. Fuzzy topological spaces and fuzzy interior spaces

DEFINITION 3.1. A map $\mathcal{B} : L^X \to L$ is called a *fuzzy base* on $X$ if it satisfies the following conditions:
(B1) $\mathcal{B}(\underline{1}) = \mathcal{B}(\underline{0}) = 1$
(B2) $\mathcal{B}(\mu_1 \otimes \mu_2) \geq \mathcal{B}(\mu_1) \otimes \mathcal{B}(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$.

A fuzzy base $\mathcal{B}$ always generates a fuzzy topology $\mathcal{T}_{\mathcal{B}}$ on $X$ in the following sense.

THEOREM 3.2. *Let $\mathcal{B}$ be a fuzzy base on $X$. Define a map $\mathcal{T}_{\mathcal{B}} : L^X \to L$ as follows:*

$$\mathcal{T}_{\mathcal{B}}(\mu) = \bigvee \{ \bigwedge_{j \in \Lambda} \mathcal{B}(\mu_j) \mid \mu = \bigvee_{j \in \Lambda} \mu_j \}.$$

*Then $\mathcal{T}_{\mathcal{B}}$ is the coarsest fuzzy topology on $X$ such that $\mathcal{T}_{\mathcal{B}}(\lambda) \geq \mathcal{B}(\lambda)$, for all $\lambda \in L^X$.*

*Proof.* (O1) It is trivial from the definition of $\mathcal{T}_{\mathcal{B}}$.

(O2) For all families $\{\lambda_j \mid \lambda = \bigvee_{j \in \Lambda} \lambda_j\}$ and $\{\mu_k \mid \mu = \bigvee_{k \in \Gamma} \mu_k\}$, there exists a family $\{\lambda_j \otimes \mu_k\}$ such that, by Definition 2.1 (L3),

$$\lambda \otimes \mu = (\bigvee_{j \in \Lambda} \lambda_j) \otimes (\bigvee_{k \in \Lambda} \mu_k) = \bigvee_{j \in \Lambda, k \in \Gamma} (\lambda_j \otimes \mu_k).$$

It implies

$$\begin{aligned} \mathcal{T}_{\mathcal{B}}(\lambda \otimes \mu) &\geq \bigwedge_{j \in \Lambda, k \in \Gamma} \mathcal{B}(\lambda_j \otimes \mu_k) \\ &\geq \bigwedge_{j \in \Lambda, k \in \Gamma} \Big(\mathcal{B}(\lambda_j) \otimes \mathcal{B}(\mu_k)\Big) \text{ ( by Definition 3.1 (B2))} \\ &\geq \Big( \bigwedge_{j \in \Lambda} \mathcal{B}(\lambda_j)\Big) \otimes \Big( \bigwedge_{k \in \Lambda} \mathcal{B}(\mu_k)\Big) \text{ ( by Remark 2.2(4)).} \end{aligned}$$

By Definition 2.1 (L3), $\mathcal{T}_{\mathcal{B}}(\lambda \otimes \mu) \geq \mathcal{T}_{\mathcal{B}}(\lambda) \otimes \mathcal{T}_{\mathcal{B}}(\mu)$.

(O3) Let $\mathcal{J}_i$ be the collection of all index sets $K_i$ such that $\{\lambda_{i_k} \in L^X \mid \lambda_i = \bigvee_{k \in K_i} \lambda_{i_k}\}$ with $\lambda = \bigvee_{i \in \Gamma} \lambda_i = \bigvee_{i \in \Gamma} \bigvee_{k \in K_i} \lambda_{i_k}$. For each $i \in \Gamma$ and each $\psi \in \Pi_{i \in \Gamma} \mathcal{J}_i$ with $\psi(i) = K_i$, we have

$$(A) \qquad \mathcal{T}_{\mathcal{B}}(\lambda) \geq \bigwedge_{i \in \Gamma} \Big( \bigwedge_{k \in K_i} \mathcal{B}(\lambda_{i_k})\Big).$$

Put $a_{i,\psi(i)} = \bigwedge_{k \in K_i} \mathcal{B}(\lambda_{i_k})$. From $(A)$,

$$\begin{aligned} \mathcal{T}_{\mathcal{B}}(\lambda) &\geq \bigvee_{\psi \in \Pi_{i \in \Gamma} \mathcal{J}_i} (\bigwedge_{i \in \Gamma} a_{i,\psi(i)}) \\ &\quad \text{(Since } L \text{ is a completely distributive lattice)} \\ &= \bigwedge_{i \in \Gamma} ( \bigvee_{M_i \in \mathcal{J}_i} a_{i,M_i}) \\ &= \bigwedge_{i \in \Gamma} \Big( \bigvee_{M_i \in \mathcal{J}_i} ( \bigwedge_{m \in M_i} \mathcal{B}(\lambda_{i_m}))\Big) \\ &= \bigwedge_{i \in \Gamma} \mathcal{T}_{\mathcal{B}}(\lambda_i). \end{aligned}$$

Thus $\mathcal{T}_{\mathcal{B}}$ is a fuzzy topology on $X$.

If $\mathcal{T} \geq \mathcal{B}$, for every $\lambda = \bigvee_{j\in\Lambda} \lambda_j$,

$$\mathcal{T}(\lambda) \geq \bigwedge_{j\in\Lambda} \mathcal{T}(\lambda_j) \geq \bigwedge_{j\in\Lambda} \mathcal{B}(\lambda_j).$$

Thus $\mathcal{T} \geq \mathcal{T}_{\mathcal{B}}$. □

From Theorem 3.2, we easily prove the following lemma.

LEMMA 3.3. *Let $\mathcal{T}$ be a fuzzy topology on $X$ and $\mathcal{B}$ a fuzzy base on $Y$. Let $f : X \to Y$ be a map. Then $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_{\mathcal{B}})$ is fuzzy continuous iff $\mathcal{T}(f^{-1}(\lambda)) \geq \mathcal{B}(\lambda)$, for each $\lambda \in L^Y$.*

DEFINITION 3.4. Let $(X, I_1)$ and $(Y, I_2)$ be two fuzzy interior spaces. A map $f : (X, I_1) \longrightarrow (Y, I_2)$ is called

(1) a *fuzzy $I$-map* if $f^{-1}(I_2(\lambda, r)) \leq I_1(f^{-1}(\lambda), r), \forall r \in L, \forall \lambda \in L^Y$.

(2) a *fuzzy $I$-open map* if $f(I_1(\lambda, r)) \leq I_2(f(\lambda), r)), \forall r \in L, \forall \lambda \in L^X$.

DEFINITION 3.5. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be fuzzy topological spaces. A map $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ is *fuzzy open* if $\mathcal{T}_1(\lambda) \leq \mathcal{T}_2(f(\lambda)), \forall \lambda \in L^X$.

THEOREM 3.6. *Let $(X, \mathcal{T}_1), (Y, \mathcal{T}_2)$ be fuzzy topological spaces. Then:*

*(1) $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is fuzzy continuous iff $f : (X, I_{\mathcal{T}_1}) \to (Y, I_{\mathcal{T}_2})$ is a fuzzy $I$-map.*

*(2) $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is fuzzy open iff $f : (X, I_{\mathcal{T}_1}) \to (Y, I_{\mathcal{T}_2})$ is fuzzy $I$-open.*

*Proof.* (1) Let $f$ be fuzzy continuous. For all $\mu \in L^Y, r \in L$,

$$\begin{aligned} f^{-1}(I_{\mathcal{T}_2}(\mu, r)) &= f^{-1}(\bigvee\{\rho \in L^Y \mid \rho \leq \mu, \mathcal{T}_2(\rho) \geq r\}) \\ &\leq f^{-1}(\bigvee\{\rho \in L^Y \mid f^{-1}(\rho) \leq f^{-1}(\mu), \mathcal{T}_1(f^{-1}(\rho)) \geq r\}) \\ &= \bigvee\{f^{-1}(\rho) \in L^X \mid f^{-1}(\rho) \leq f^{-1}(\mu), \mathcal{T}_1(f^{-1}(\rho)) \geq r\} \\ &\leq \bigvee\{\nu \in L^X \mid \nu \leq f^{-1}(\mu), \mathcal{T}_1(\nu) \geq r\} \\ &= I_{\mathcal{T}_1}(f^{-1}(\mu), r). \end{aligned}$$

Conversely, since $\mu \leq I_{\mathcal{T}_2}(\mu, r)$ implies $f^{-1}(\mu) \leq I_{\mathcal{T}_1}(f^{-1}(\mu), r)$, by Theorem 2.8 , it is easily proved.

(2) It is similarly proved as in (1). □

LEMMA 3.7. *Let $(L, \leq, \otimes)$ be an MV-algebra. Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in L, \lambda \in L^X$, we define a fuzzy closure operator $C_{\mathcal{T}} : L^X \times L \to L^X$ as follows:*

$$C_{\mathcal{T}}(\lambda, r) = \bigwedge\{\rho \in L^Y \mid \lambda \leq \rho, \mathcal{T}(\rho \to \underline{0}) \geq r\}.$$

*Then*

$$C_{\mathcal{T}}(\lambda \to \underline{0}, r) = I_{\mathcal{T}}(\lambda, r) \to \underline{0}.$$

*Proof.* Since $\bigvee_{i\in\Gamma} a_i \to 0 = \bigwedge_{i\in\Gamma}(a_i \to 0)$ and $(a \to 0) \to 0 = a$ from Lemma 2.4(1,9), we have

$$\begin{aligned}
I_{\mathcal{T}}(\lambda, r) \to \underline{0} &= \bigvee\{\mu \mid \mu \leq \lambda, \mathcal{T}(\mu) \geq r\} \to \underline{0} \\
&= \bigwedge\{\mu \to \underline{0} \mid \mu \leq \lambda, \mathcal{T}(\mu) \geq r\} \\
&= \bigwedge\{\mu \to \underline{0} \mid (\mu \to \underline{0}) \geq (\lambda \to \underline{0}), \mathcal{T}\Big((\mu \to \underline{0}) \to \underline{0}\Big) \geq r\} \\
&= \bigwedge\{\rho \mid (\lambda \to \underline{0}) \leq \rho, \mathcal{T}(\rho \to \underline{0}) \geq r\} \\
&= C_{\mathcal{T}}(\lambda \to \underline{0}, r).
\end{aligned}$$

□

THEOREM 3.8. *Let $(L, \leq, \otimes)$ be an MV-algebra, $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ fuzzy topological spaces. Then the following statement are equivalent:*
*(1) $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is fuzzy continuous.*
*(2) $f^{-1}(I_{\mathcal{T}_2}(\mu, r)) \leq I_{\mathcal{T}_1}(f^{-1}(\mu), r)$, for each $\mu \in L^Y$ and $r \in L$.*
*(3) $f(C_{\mathcal{T}_1}(\lambda, r)) \leq C_{\mathcal{T}_2}(f(\lambda), r)$, for each $\lambda \in L^X$ and $r \in L$.*

*Proof.* (1) $\Leftrightarrow$ (2) It is proved in Theorem 3.6(1).
(2)$\Rightarrow$(3) From Lemma 3.7, we have

$$\begin{aligned}
& f^{-1}\Big(C_{\mathcal{T}_2}(f(\lambda), r)\Big) \\
&= f^{-1}\Big(I_{\mathcal{T}_2}(f(\lambda) \to \underline{0}, r) \to \underline{0}\Big) \quad \text{(by Lemma 3.7)} \\
&= f^{-1}\Big(I_{\mathcal{T}_2}(f(\lambda) \to \underline{0}, r)\Big) \to \underline{0} \\
&\geq I_{\mathcal{T}_1}\Big(f^{-1}(f(\lambda) \to \underline{0}), r\Big) \to \underline{0} \quad \text{(by (2) and Lemma 2.4(5))} \\
&= I_{\mathcal{T}_1}\Big(f^{-1}(f(\lambda)) \to \underline{0}, r\Big) \to \underline{0} \\
&\geq I_{\mathcal{T}_1}(\lambda \to \underline{0}, r) \to \underline{0} \qquad (\lambda \leq f^{-1}(f(\lambda))) \\
&= C_{\mathcal{T}_1}(\lambda, r).
\end{aligned}$$

It implies

$$\begin{aligned}C_{\mathcal{T}_2}(f(\lambda), r) &\geq f\Big(f^{-1}(C_{\mathcal{T}_2}(f(\lambda), r))\Big)\\ &\geq f(C_{\mathcal{T}_1}(\lambda, r)).\end{aligned}$$

(3)⇒(2) Similar to (2)⇒(3). □

EXAMPLE 3.9. Let $(L, \leq, \otimes) = ([0,1], \leq, \wedge)$ be given where $[0,1]$ is the unit interval. For $\mu \notin \{\underline{1}, \underline{0}\}$, we define fuzzy interior operators $I_1, I_2 : [0,1]^X \times [0,1] \to [0,1]^X$ as follows:

$$I_1(\lambda, r) = \begin{cases} \underline{1} & \text{if } \lambda = \underline{1},\ \forall r \in [0,1], \\ \mu & \text{if } \underline{1} \neq \lambda \geq \mu,\ 0 < r < \frac{1}{2}, \\ \lambda & \text{if } r = 0, \\ \underline{0} & \text{otherwise.} \end{cases}$$

$$I_2(\lambda, r) = \begin{cases} \underline{1} & \text{if } \lambda = \underline{1},\ \forall r \in [0,1], \\ \mu & \text{if } \underline{1} \neq \lambda \geq \mu,\ 0 < r \leq \frac{1}{2}, \\ \lambda & \text{if } r = 0, \\ \underline{0} & \text{otherwise.} \end{cases}$$

Then the identity map $id_X : (X, I_1) \to (X, I_2)$ is not a fuzzy $I$-map because

$$\mu = I_2(\mu, \frac{1}{2}) > I_1(\mu, \frac{1}{2}) = \underline{0}.$$

On the other hand, from Theorem 2.8, we can obtain fuzzy topologies $\mathcal{T}_{I_1} = \mathcal{T}_{I_2} : [0,1]^X \to [0,1]$ as follows:

$$(\mathcal{T}_{I_1} = \mathcal{T}_{I_2})(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0} \text{ or } \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the identity map $id_X : (X, \mathcal{T}_{I_1}) \to (X, \mathcal{T}_{I_2})$ is a fuzzy continuous map. From above facts, the identity map $id_X : (X, I_2) \to (X, I_1)$ is not a fuzzy $I$-open map and the identity map $id_X : (X, \mathcal{T}_{I_2}) \to (X, \mathcal{T}_{I_1})$ is a fuzzy open map.

THEOREM 3.10. *Let $\{(X_i, \mathcal{T}_i) \mid i \in \Gamma\}$ be a family of fuzzy topological spaces, $X$ a set and for each $i \in \Gamma$, $f_i : X \to X_i$ a map. Define a map $\mathcal{B} : L^X \to L$ on $X$ by*

$$\mathcal{B}(\mu) = \bigvee\{\otimes_{j=1}^{n} \mathcal{T}_{k_j}(\nu_{k_j}) \mid \mu = \otimes_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j})\},$$

*where $\bigvee$ is taken over all finite subsets $K = \{k_1, ..., k_n\} \subset \Gamma$.*

*Then:*

*(1) $\mathcal{B}$ is a fuzzy base on $X$.*

*(2) The fuzzy topology $\mathcal{T}_{\mathcal{B}}$ generated by $\mathcal{B}$ is the coarsest fuzzy topology on $X$ for which all $f_i$, $i \in \Gamma$, are fuzzy continuous maps.*

*(3) A map $f : (Y, \mathcal{T}') \to (X, \mathcal{T}_{\mathcal{B}})$ is fuzzy continuous iff for each $i \in \Gamma$, $f_i \circ f : (Y, \mathcal{T}') \to (X_i, \mathcal{T}_i)$ is a fuzzy continuous map.*

*Proof.* (1) Since $\lambda = f_i^{-1}(\lambda)$ for each $\lambda \in \{\underline{0}, \underline{1}\}$, $\mathcal{B}(\underline{1}) = \mathcal{B}(\underline{0}) = 1$.

For all finite subsets $K = \{k_1, ..., k_p\}$ and $J = \{j_1, ..., j_q\}$ of $\Gamma$ such that

$$\lambda = \otimes_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i}), \quad \mu = \otimes_{i=1}^{q} f_{j_i}^{-1}(\mu_{j_i}),$$

we have

$$\lambda \otimes \mu = \Big(\otimes_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i})\Big) \otimes \Big(\otimes_{i=1}^{q} f_{j_i}^{-1}(\mu_{j_i})\Big).$$

Furthermore, we have for each $k \in K \cap J$,

$$f_k^{-1}(\lambda_k) \otimes f_k^{-1}(\mu_k) = f_k^{-1}(\lambda_k \otimes \mu_k).$$

Put $\lambda \otimes \mu = \otimes_{m_i \in K \cup J} f_{m_i}^{-1}(\rho_{m_i})$ where

$$\rho_{m_i} = \begin{cases} \lambda_{m_i} & \text{if } m_i \in K - (K \cap J) \\ \mu_{m_i} & \text{if } m_i \in J - (K \cap J) \\ \lambda_{m_i} \otimes \mu_{m_i} & \text{if } m_i \in K \cap J. \end{cases}$$

We have

$$\begin{aligned} \mathcal{B}(\lambda \otimes \mu) &\geq \otimes_{j \in K \cup J} \mathcal{T}_j(\rho_j) \\ &\geq \Big(\otimes_{i=1}^{p} \mathcal{T}_{k_i}(\lambda_{k_i})\Big) \otimes \Big(\otimes_{i=1}^{q} \mathcal{T}_{j_i}(\mu_{j_i})\Big). \end{aligned}$$

By Definition 2.1(L3), $\mathcal{B}(\lambda \otimes \mu) \geq \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$.

(2) For each $\lambda_i \in L^{X_i}$, one family $\{f_i^{-1}(\lambda_i)\}$ and $i \in \Gamma$, we have

$$\mathcal{T}_{\mathcal{B}}\Big(f_i^{-1}(\lambda_i)\Big) \geq \mathcal{B}\Big(f_i^{-1}(\lambda_i)\Big) \geq \mathcal{T}_i(\lambda_i).$$

Thus, for each $i \in \Gamma$, $f_i : (X, \mathcal{T}_{\mathcal{B}}) \to (X_i, \mathcal{T}_i)$ is fuzzy continuous. Let $f_i : (X, \mathcal{T}^0) \to (X_i, \mathcal{T}_i)$ be fuzzy continuous, that is, for each $i \in \Gamma$ and $\lambda_i \in L^{X_i}$, $\mathcal{T}^0(f_i^{-1}(\lambda_i)) \geq \mathcal{T}_i(\lambda_i)$. For all finite subsets $K = \{k_1, ..., k_p\}$ of $\Gamma$ such that $\lambda = \otimes_{i=1}^p f_{k_i}^{-1}(\lambda_{k_i})$, we have

$$\begin{aligned}\mathcal{T}^0(\lambda) &\geq \otimes_{i=1}^p \mathcal{T}^0\Big(f_{k_i}^{-1}(\lambda_{k_i})\Big)\\ &\geq \otimes_{i=1}^p \mathcal{T}_{k_i}(\lambda_{k_i}).\end{aligned}$$

It implies $\mathcal{T}^0(\lambda) \geq \mathcal{B}(\lambda)$ for each $\lambda \in L^X$. By Theorem 3.2, $\mathcal{T}^0 \geq \mathcal{T}_{\mathcal{B}}$.

(3)($\Rightarrow$) Let $f : (Y, \mathcal{T}') \to (X, \mathcal{T}_{\mathcal{B}})$ be fuzzy continuous. For each $i \in \Gamma$ and $\lambda_i \in L^{X_i}$, we have

$$\mathcal{T}'\Big((f_i \circ f)^{-1}(\lambda_i)\Big) = \mathcal{T}'\Big(f^{-1}(f_i^{-1}(\lambda_i))\Big) \geq \mathcal{T}_{\mathcal{B}}\Big(f_i^{-1}(\lambda_i)\Big) \geq \mathcal{T}_i(\lambda_i).$$

Hence $f_i \circ f : (Y, \mathcal{T}') \to (X_i, \mathcal{T}_i)$ is fuzzy continuous.

($\Leftarrow$) For all finite subsets $K = \{k_1, ..., k_p\}$ of $\Gamma$ such that

$$\lambda = \otimes_{i=1}^p f_{k_i}^{-1}(\lambda_{k_i}),$$

since $f_{k_i} \circ f : (Y, \mathcal{T}') \to (X_{k_i}, \mathcal{T}_{k_i})$ is fuzzy continuous,

$$(B) \qquad \mathcal{T}'\Big(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\Big) \geq \mathcal{T}_{k_i}(\lambda_{k_i}).$$

Hence we have

$$\begin{aligned}\mathcal{T}'(f^{-1}(\lambda)) &= \mathcal{T}'\Big(f^{-1}(\otimes_{i=1}^p f_{k_i}^{-1}(\lambda_{k_i}))\Big)\\ &= \mathcal{T}'\Big(\otimes_{i=1}^p f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\Big)\\ &\geq \otimes_{i=1}^p \mathcal{T}'\Big(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\Big)\\ &\geq \otimes_{i=1}^p \mathcal{T}_{k_i}(\lambda_{k_i}). \quad \text{(by } (B))\end{aligned}$$

It implies $\mathcal{T}'(f^{-1}(\lambda)) \geq \mathcal{B}(\lambda)$ for all $\lambda \in L^X$. By Lemma 3.3, $f : (Y, \mathcal{T}') \to (X, \mathcal{T}_{\mathcal{B}})$ is fuzzy continuous. □

From Theorem 3.10, we can define a product fuzzy topology.

DEFINITION 3.11. Let $\{(X_i, \mathcal{T}_i)\}_{i\in\Gamma}$ be a family of fuzzy topological spaces, $X = \Pi_{i\in\Gamma} X_i$ a product set and for each $i \in \Gamma$, $\pi_i : X \to X_i$ a projection map. The *product fuzzy topology* is the coarsest fuzzy topology on $X$ for which all $\pi_i$, $i \in \Gamma$, are fuzzy continuous maps.

THEOREM 3.12. *Let $(L, \leq, \otimes)$ be an idempotent scq-lattice, $\{(X_i, I_i) \mid i \in \Gamma\}$ a family of fuzzy interior spaces and $f_i : X \to X_i$ a map. Define a map $I^* : L^X \times L \to L^X$ by*

$$I^*(\lambda, r) = \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(\lambda_{i_k}, r)\Big)\Big\}$$

*for all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$. Then $I^*$ is the coarsest fuzzy interior operator on $X$ for which each $i \in \Gamma$, $f_i : X \to X_i$ is a fuzzy $I$-map.*

*Proof.* (I1) Since $I^*(\underline{1}, r) \geq f_i^{-1}\Big(I(\underline{1}, r)\Big) = \underline{1}$, we have $I^*(\underline{1}, r) = \underline{1}$.

(I2) For all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$, we have

$$\begin{aligned} I^*(\lambda, r) &= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(\lambda_{i_k}, r)\Big)\Big\} \\ &\leq \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ \otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k})\Big\} \\ &\leq \lambda. \end{aligned}$$

(I3) and (I5) are easily proved from the definition of $I^*$.

(I4) For all finite subsets $K = \{k_1, ..., k_p\}$ and $J = \{j_1, ..., j_q\}$ of $\Gamma$ such that

$$\otimes_{i=1}^p f_{k_i}^{-1}(\lambda_{k_i}) \leq \lambda, \ \ \otimes_{i=1}^q f_{j_i}^{-1}(\mu_{j_i}) \leq \mu,$$

we have

$$\Big( \otimes_{i=1}^p f_{k_i}^{-1}(\lambda_{k_i})\Big) \otimes \Big( \otimes_{i=1}^q f_{j_i}^{-1}(\mu_{j_i})\Big) \leq (\lambda \otimes \mu).$$

Furthermore, we have for each $k \in K \cap J$,

$$f_k^{-1}(\lambda_k) \otimes f_k^{-1}(\mu_k) = f_k^{-1}(\lambda_k \otimes \mu_k),$$

$$f_k^{-1}\Big(I_k(\lambda_k, r)\Big) \otimes f_k^{-1}\Big(I_k(\mu_k, s)\Big) \leq f_k^{-1}\Big(I_k(\lambda_k \otimes \mu_k, r \otimes s)\Big).$$

Put $M = K \cup J = \{m_1, ..., m_r\}$ with

$$\rho_{m_i} = \begin{cases} \lambda_{m_i} \otimes \underline{1} & \text{if } m_i \in K - (K \cap J) \\ \mu_{m_i} \otimes \underline{1} & \text{if } m_i \in J - (K \cap J) \\ \lambda_{m_i} \otimes \mu_{m_i} & \text{if } m_i \in K \cap J. \end{cases}$$

If $m_i \in K - (K \cap J)$,

$$\begin{aligned} & f_{m_i}^{-1}\Big(I_{m_i}(\lambda_{m_i}, r)\Big) \\ &= f_{m_i}^{-1}\Big(I_{m_i}(\lambda_{m_i}, r)\Big) \otimes \underline{1} \\ &= f_{m_i}^{-1}\Big(I_{m_i}(\lambda_{m_i}, r)\Big) \otimes f_{m_i}^{-1}\Big(I_{m_i}(\underline{1}, s)\Big) \\ &\leq f_{m_i}^{-1}\Big(I_{m_i}(\lambda_{m_i} \otimes \underline{1}, r \otimes s)\Big) \\ &= f_{m_i}^{-1}\Big(I_{m_i}(\rho_{m_i}, r \otimes s)\Big). \end{aligned}$$

Similarly, if $m_i \in J - (K \cap J)$,

$$f_{m_i}^{-1}\Big(I_{m_i}(\mu_{m_i}, s)\Big) \leq f_{m_i}^{-1}\Big(I_{m_i}(\rho_{m_i}, r \otimes s)\Big).$$

Hence

$$\begin{aligned} & \Big( \bigvee_{\otimes_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i}) \leq \lambda} \Big\{ \otimes_{i=1}^{p} f_{k_i}^{-1}\Big(I_{k_i}(\lambda_{k_i}, r)\Big) \Big\} \Big) \\ & \otimes \Big( \bigvee_{\otimes_{i=1}^{q} f_{j_i}^{-1}(\mu_{j_i}) \leq \mu} \Big\{ \otimes_{i=1}^{q} f_{j_i}^{-1}\Big(I_{j_i}(\mu_{j_i}, s)\Big) \Big\} \Big) \\ \leq & \bigvee_{\otimes_{i=1}^{r} f_{m_i}^{-1}(\rho_{m_i}) \leq \lambda \otimes \mu} \Big\{ \otimes_{i=1}^{r} f_{m_i}^{-1}\Big(I_{m_i}(\rho_{m_i}, r \otimes s)\Big) \Big\} \\ \leq & I^*(\lambda \otimes \mu, r \otimes s). \end{aligned}$$

Thus,

$$I^*(\lambda, r) \otimes I^*(\mu, s) \leq I^*(\lambda \otimes \mu, r \otimes s).$$

For each $\lambda_i \in L^{X_i}$, one family $\{f_i^{-1}(\lambda_i)\}$ and $i \in \Gamma$, we have

$$I^*(f_i^{-1}(\lambda_i), r) \geq f_i^{-1}(I_i(\lambda_i, r)).$$

Thus, each $i \in \Gamma$, $f_i : (X, I^*) \to (X_i, I_i)$ is a fuzzy $I$-map.

Let $f_i : (X, I^0) \to (X_i, I_i)$ is a fuzzy $I$-map for each $i \in \Gamma$. Since for each $i \in \Gamma$ and $\lambda_i \in L^{X_i}$,

$$I^0(f_i^{-1}(\lambda_i), r) \geq f_i^{-1}(I_i(\lambda_i, r)),$$

for all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$, we have

$$\begin{aligned}
I^*(\lambda, r) &= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(\lambda_{i_k}, r)\Big)\Big\} \\
&\leq \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ \otimes_{k=1}^n I^0\Big(f_{i_k}^{-1}(\lambda_{i_k}), r\Big)\Big\} \\
&\leq \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} I^0\Big( \otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}), \otimes_{k=1}^n r\Big) \quad \text{(by (I4))} \\
&= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} I^0\Big( \otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}), r\Big) \quad (\text{since } \otimes_{k=1}^n r = r) \\
&\leq I^0(\lambda, r).
\end{aligned}$$

□

From Theorem 3.12, we can define product fuzzy interior spaces for an idempotent scq-lattice $(L, \leq, \otimes)$.

DEFINITION 3.13. Let $\{(X_i, I_i) \mid i \in \Gamma\}$ be a family of fuzzy interior spaces, $X = \Pi_{i\in\Gamma} X_i$ a product set and for each $i \in \Gamma$, $\pi_i : X \to X_i$ a projection map. The *product fuzzy interior operator* is the coarsest fuzzy interior operator on $X$ for which all $\pi_i, i \in \Gamma$, are fuzzy I-maps.

THEOREM 3.14. *Let $(L, \leq, \otimes)$ be an idempotent scq-lattice, $\{(X_i, \mathcal{T}_i) \mid i \in \Gamma\}$ a family of fuzzy topological spaces, $X$ a set and for each $i \in \Gamma$, $f_i : X \to X_i$ a map. Define the map $I : L^X \times L \to L^X$ on $X$ by*

$$I(\lambda, r) = \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{\mathcal{T}_{i_k}}(\lambda_{i_k}, r)\Big)\Big\}$$

*for all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$. Then we have*

$$\mathcal{T}_{\mathcal{B}} = \mathcal{T}_I,$$

*where the fuzzy topology $\mathcal{T}_I$ is induced by $I$ and $\mathcal{T}_{\mathcal{B}}$ is defined by Theorem 3.10.*

*Proof.* Suppose there exists $\lambda \in L^X$ such that $\mathcal{T}_{\mathcal{B}}(\lambda) \not\geq \mathcal{T}_I(\lambda)$. By the definition of $\mathcal{T}_I$ from Theorem 2.8, there exists $r \in L$ such that $I(\lambda, r) = \lambda$ and $\mathcal{T}_{\mathcal{B}}(\lambda) \not\geq r$. It implies

$$\begin{aligned} \lambda &= I(\lambda, r) \\ &= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \left\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{\mathcal{T}_{i_k}}(\lambda_{i_k}, r)\Big) \right\} \end{aligned}$$

for all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$. From Theorem 2.7(6), since $I_{\mathcal{T}_{i_k}}(\lambda_{i_k}, r) = I_{\mathcal{T}_{i_k}}(I_{\mathcal{T}_{i_k}}(\lambda_{i_k}, r), r)$, using the fact $\mathcal{T}_i = \mathcal{T}_{I_{\mathcal{T}_i}}$ from Theorem 2.8, we have

$$\mathcal{T}_{i_k}\Big(I_{\mathcal{T}_{i_k}}(\lambda_{i_k}, r)\Big) \geq r.$$

Put $\mu_k = f_{i_k}^{-1}\Big(I_{\mathcal{T}_{i_k}}(\lambda_{i_k}, r)\Big)$. From Theorem 3.10, we have

$$\begin{aligned} \mathcal{B}(\otimes_{k=1}^n \mu_k) &\geq \otimes_{k=1}^n \mathcal{T}_{i_k}\Big(I_{\mathcal{T}_{i_k}}(\lambda_{i_k}, r)\Big) \\ &\geq \otimes_{k=1}^n r \\ &= r. \end{aligned}$$

Put $\mu_K = \otimes_{k=1}^n \mu_k$. For all finite sets $\{K \subset \Gamma \mid \otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda\}$, by the definition of $\mathcal{T}_{\mathcal{B}}$ in Theorem 3.10, we have

$$\mathcal{T}_{\mathcal{B}}(\lambda) = \mathcal{T}_{\mathcal{B}}(\bigvee_{K \subset \Gamma} \mu_K) \geq \bigwedge_{K \subset \Gamma} \mathcal{B}(\mu_K) \geq r.$$

It is a contradiction. Therefore $\mathcal{T}_{\mathcal{B}}(\mu) \geq \mathcal{T}_I(\mu)$ for all $\mu \in L^X$.

We show that $\mathcal{T}_{\mathcal{B}}(\mu) \leq \mathcal{T}_I(\mu)$ for every $\mu \in L^X$, equivalently, the identity map $id_X : (X, \mathcal{T}_I) \to (X, \mathcal{T}_{\mathcal{B}})$ is fuzzy continuous. From Theorem 3.10 (3), we only show that $f_i \circ id_X : (X, \mathcal{T}_I) \to (X_i, \mathcal{T}_i)$ is fuzzy continuous. If $\mathcal{T}_i(\nu_i) \geq r$ for $r \in L$, then, by Theorem 2.7, we have

$$I_{\mathcal{T}_i}(\nu_i, r) = \nu_i.$$

From the definition of $I$, it follows that

$$I(f_i^{-1}(\nu_i), r) \geq f_i^{-1}(I_{\mathcal{T}_i}(\nu_i, r)) = f_i^{-1}(\nu_i).$$

By Theorem 2.8, $\mathcal{T}_I(f_i^{-1}(\nu_i)) \geq r$. Hence $\mathcal{T}_i(\nu_i) \leq \mathcal{T}_I(f_i^{-1}(\nu_i))$, for all $\nu_i \in I^{X_i}$. $\square$

THEOREM 3.15. *Let $(L, \leq, \otimes)$ be an idempotent scq-lattice, $\{(X_i, I_i) | i \in \Gamma\}$ a family of fuzzy interior spaces, $f_i : X \to X_i$ a map and $I^*$ the fuzzy interior operator as in Theorem 3.10, then:*

*(1) If there exists $i \in \Gamma$ such that $I_i$ is weakly stratified, then $I^*$ is also weakly stratified on $X$.*

*(2) If $\{(X_i, I_i)\}_{i \in \Gamma}$ is a family of topological fuzzy interior spaces, $(X, I^*)$ is a topological fuzzy interior space.*

*(3) A map $f : (Y, I') \to (X, I^*)$ is a fuzzy $I$-map iff for each $i \in \Gamma$, $f_i \circ f : (Y, I') \to (X_i, I_i)$ is a fuzzy $I$-map.*

*Proof.* (1) Let $I_i$ be weakly stratified on $X_i$. Since $I_i(\underline{\alpha}, r) \geq \underline{\alpha}$ for each $\underline{\alpha} \in L^{X_i}$, $r \in L$ and $f_i^{-1}(\underline{\alpha}) = \underline{\alpha}$, we have, for each $\underline{\alpha} \in L^X$, $r \in L$,

$$\begin{aligned} I^*(\underline{\alpha}, r) &= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \underline{\alpha}} \left\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(\lambda_{i_k}, r)\Big) \right\} \\ &\geq f_i^{-1}\Big(I_i(\underline{\alpha}, r)\Big) \\ &\geq \underline{\alpha}. \end{aligned}$$

Thus $I^*$ is also weakly stratified on $X$.

(2) For all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$, we have

$$\begin{aligned} I^*(\lambda, r) &= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \left\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(\lambda_{i_k}, r)\Big) \right\} \\ &= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \left\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(I_{i_k}(\lambda_{i_k}, r), r)\Big) \right\} \\ &\leq \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(I_{i_k}(\lambda_{i_k}, r)) \leq I^*(\lambda, r)} \left\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(I_{i_k}(\lambda_{i_k}, r), r)\Big) \right\} \\ &\leq I^*\Big(I^*(\lambda, r), r\Big) \end{aligned}$$

because $\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda$ implies $\otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(\lambda_{i_k}, r)\Big) \leq I^*(\lambda, r)$.

(3) For all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$, we have

$$\begin{aligned}
f^{-1}(I^*(\lambda, r)) &= f^{-1}\Big( \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(\lambda_{i_k}, r)\Big)\Big\}\Big) \\
&= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ f^{-1}\Big( \otimes_{k=1}^n f_{i_k}^{-1}\Big(I_{i_k}(\lambda_{i_k}, r)\Big)\Big)\Big\} \\
&= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ \otimes_{k=1}^n f^{-1}\Big( f_{i_k}^{-1}\Big(I_{i_k}(\lambda_{i_k}, r)\Big)\Big)\Big\} \\
&\leq \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ \otimes_{k=1}^n I'\Big(f^{-1}(f_{i_k}^{-1}(\lambda_{i_k})), r\Big)\Big\} \\
&\leq \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ I'\Big(f^{-1}(\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k})), \otimes_{k=1}^n r\Big)\Big\} \\
&= \bigvee_{\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k}) \leq \lambda} \Big\{ I'\Big(f^{-1}(\otimes_{k=1}^n f_{i_k}^{-1}(\lambda_{i_k})), r\Big)\Big\} \\
&\leq I'(f^{-1}(\lambda), r).
\end{aligned}$$

□

From Theorem 3.12, we can prove the following corollaries.

COROLLARY 3.16. *Let $(L, \leq, \otimes)$ be an idempotent scq-lattice and $I$ a fuzzy interior operator on $Y$. Let $f : X \to Y$ a map. We define a map $I^f : L^X \times L \to L^X$ as*

$$I^f(\lambda, r) = \bigvee_{f^{-1}(\mu) \leq \lambda} f^{-1}\Big(I(\mu, r)\Big).$$

*Then $I^f$ is the coarsest fuzzy interior operator on $X$ for which $f$ is a fuzzy $I$-map.*

COROLLARY 3.17. *Let $(L, \leq, \otimes)$ be an idempotent scq-lattice. Let $\{I_i \mid i \in \Gamma\}$ be a family of fuzzy interior operators on $X$. We define a map $I^* : L^X \times L \to L^X$ defined by*

$$I^*(\lambda, r) = \bigvee_{\lambda_{i_1} \otimes \lambda_{i_2} \otimes ... \otimes \lambda_{i_n} \leq \lambda} \Big(I_{i_1}(\lambda_{i_1}, r) \otimes ... \otimes I_{i_n}(\lambda_{i_n}, r)\Big)$$

*for all finite subsets $K = \{i_1, ..., i_n\}$ of $\Gamma$. Then $I^*$ is the coarsest fuzzy interior operator finer than $I_i$ for each $i \in \Gamma$.*

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