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STRONG LAWS FOR WEIGHTED SUMS OF I.I.D. RANDOM VARIABLES (II)

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ABSTRACT. Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants. Let $\phi(x)$ be a positive increasing function on $(0, \infty)$ satisfying $\phi(x) \uparrow \infty$ and $\phi(Cx) = O(\phi(x))$ for any $C > 0$. When $EX = 0$ and $E[\phi(|X|)] < \infty$, some conditions on ϕ and $\{a_{ni}\}$ are given under which $\sum_{i=1}^n a_{ni}X_i \to 0$ a.s.

1. Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ an array of constants. Throughout this paper, we assume that $\phi(x)$ is a positive increasing function on $(0, \infty)$ satisfying

$$\phi(x) \uparrow \infty \quad \text{and} \quad \phi(Cx) = O(\phi(x)), \ \forall C > 0. \tag{1}$$

We also assume that $EX = 0$ and $E[\phi(|X|)] < \infty$.

When $\phi(x) = x^p (p \geq 1)$, ϕ satisfies (1). In this case, the a.s. (almost sure) limiting behavior for the weighted sums $\sum_{i=1}^n a_{ni}X_i$ was studied by many authors (see, Bai and Cheng [1], Choi and Sung [2], Cuzick [3], and Li et al. [4]). We recommend the paper of Rosalsky and Sreehari [5] for more information.

However, when $\phi(x) = e^{h|x|^\gamma} (h > 0, \gamma > 0)$, ϕ does not satisfy (1). In this case, the a.s. limiting behavior for the weighted sums was studied by Bai and Cheng [1], Sung [7], and Wu [8].

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The purpose of this work is to present various conditions on ϕ and $\{a_{ni}\}$ under which $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s. Our result extends that of Bai and Cheng [1], Cuzick [3], and Li et al. [4].

Throughout this paper, C denotes a positive constant which may be different in various places.

2. Main results

Let $\psi(x)$ be the inverse function of $\phi(x)$. Since $\phi(x) \uparrow \infty$, it follows that $\psi(x) \uparrow \infty$. For easy notation, we let $\phi(0) = 0$ and $\psi(0) = 0$.

To prove our main results, we will need the following lemma.

LEMMA 1. *Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies*

$$\psi(n)\sum_{i=1}^{n}\frac{1}{\psi(i)} = O(n),\ \psi^2(n)\sum_{i=n}^{\infty}\frac{1}{\psi^2(i)} = O(n). \tag{2}$$

If $E[\phi(|X|)] < \infty$, then the followings hold.

(i) $\sum_{n=1}^{\infty}\frac{1}{\psi(n)}E|X|I(|X| > \psi(n)) < \infty$,

(ii) $\sum_{n=1}^{\infty}\frac{1}{\psi^2(n)}EX^2I(|X| \le \psi(n)) < \infty$.

Proof. Since $\psi(x)$ is increasing function, we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{\psi(n)}E|X|I(|X| > \psi(n)) \\
&= \sum_{n=1}^{\infty}\frac{1}{\psi(n)}\sum_{i=n}^{\infty}E|X|I(\psi(i) < |X| \le \psi(i+1)) \\
&= \sum_{i=1}^{\infty}E|X|I(\psi(i) < |X| \le \psi(i+1))\sum_{n=1}^{i}\frac{1}{\psi(n)} \\
&\le \sum_{i=1}^{\infty}P(\psi(i) < |X| \le \psi(i+1))\psi(i+1)\sum_{n=1}^{i}\frac{1}{\psi(n)} \\
&\le C\sum_{i=1}^{\infty}P(\psi(i) < |X| \le \psi(i+1))i \\
&\le CE[\phi(|X|)] < \infty.
\end{aligned}$$

So (i) is proved. The proof of (ii) is similar to that of (i) and is omitted. $\square$

Now, we state and prove one of our main results.

THEOREM 1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0$ and $E[\phi(|X|)] < \infty$. Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies (2). Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants such that*

(i) $\max_{1\leq i\leq n} |a_{ni}| = O(\frac{1}{\psi(n)})$,

(ii) $\max_{1\leq j\leq n} \frac{\psi^2(j)}{j} \sum_{i=j}^n a_{ni}^2 = O(\frac{1}{n^\alpha})$ *for some* $\alpha > 0$.

Then $\sum_{i=1}^n a_{ni}X_i \to 0$ a.s.

Proof. First we prove that

$$\sum_{i=1}^n a_{ni}X_i \to 0 \text{ in probability.} \tag{3}$$

Define $Y_n' = X_n I(|X_n| \leq \psi(n))$ and $Y_n'' = X_n - Y_n'$ for $n \geq 1$. Then for any $\epsilon > 0$

$$\begin{aligned}
&P(|\sum_{i=1}^n a_{ni}X_i| > \epsilon) \\
&\leq P(|\sum_{i=1}^n a_{ni}(Y_i' - EY_i')| > \frac{\epsilon}{2}) + P(|\sum_{i=1}^n a_{ni}(Y_i'' - EY_i'')| > \frac{\epsilon}{2}) \\
&\leq \frac{4}{\epsilon^2} E|\sum_{i=1}^n a_{ni}(Y_i' - EY_i')|^2 + \frac{2}{\epsilon} E|\sum_{i=1}^n a_{ni}(Y_i'' - EY_i'')| \\
&\leq \frac{C}{\psi^2(n)} \sum_{i=1}^n EY_i'^2 + \frac{C}{\psi(n)} \sum_{i=1}^n E|Y_i''|
\end{aligned}$$

by (i). From Lemma 1 and the Kronecker lemma, the two terms on the last expression converge to 0. Thus (3) is proved.

From (3) it follows at once that $\mu(\sum_{i=1}^n a_{ni}X_i) \to 0$, where $\mu(Y)$ is a median of Y. Hence, by Theorem 3.2.1 in Stout [6], it suffices to prove that

$$\sum_{i=1}^n a_{ni}X_i^s \to 0 \text{ a.s.,} \tag{4}$$

where $\{X_n^s\}$ is a symmetrized version of $\{X_n\}$. So we need only to prove the result for $\{X_n\}$ symmetric. Since $E[\phi(|X|)] < \infty$ and $\phi(Cx) = O(\phi(x))$ for any $C > 0$, $\sum_{n=1}^\infty P(|X_n| > \epsilon\psi(n)) < \infty$ for any $\epsilon > 0$.

Thus it is possible to construct a sequence $\{b_n\}$ of real numbers such that $0 < b_n \le 1, b_n \downarrow 0$, and

$$\sum_{n=1}^{\infty} P(|X_n| > b_n\psi(n)) < \infty. \tag{5}$$

Define $X_n' = X_n I(|X_n| \le b_n\psi(n))$ and $X_n'' = X_n - X_n'$ for $n \ge 1$. Then we have by (5) and the Borel-Cantelli lemma that

$$|\sum_{i=1}^{n} a_{ni}X_i''| \le C\frac{\sum_{i=1}^{n}|X_i''|}{\psi(n)} \to 0 \text{ a.s.}$$

Thus it is enough to show that

$$\sum_{i=1}^{n} a_{ni}X_i' \to \text{ a.s.} \tag{6}$$

From an inequality $e^x \le 1 + x + \frac{1}{2}x^2e^{|x|}$ for all $x \in R$, we have

$$\begin{aligned} E[e^{ta_{ni}X_i'}] &\le 1 + \frac{1}{2}t^2a_{ni}^2E[X_i'^2e^{t|a_{ni}||X_i'|}] \\ &\le 1 + \frac{1}{2}t^2a_{ni}^2e^{t|a_{ni}|b_i\psi(i)}EX_i'^2 \\ &\le \exp\{\frac{1}{2}t^2a_{ni}^2e^{t|a_{ni}|b_i\psi(i)}EX_i'^2\} \end{aligned}$$

for any $t > 0$. Let $u_n = \max_{1\le i\le n}|a_{ni}|b_i\psi(i)$. Then it follows by (i), $\psi(x) \uparrow \infty$, and $b_n \downarrow 0$ that $u_n \to 0$. From (ii), we obtain that

$$\begin{aligned} \sum_{i=1}^{n} a_{ni}^2EX_i'^2 &\le \sum_{i=1}^{n} a_{ni}^2EX^2I(|X| \le \psi(i)) \ (\because b_n \le 1) \\ &= \sum_{j=1}^{n} EX^2I(\psi(j-1) < |X| \le \psi(j))\sum_{i=j}^{n} a_{ni}^2 \\ &\le \sum_{j=1}^{n} P(\psi(j-1) < |X| \le \psi(j))\psi^2(j)\sum_{i=j}^{n} a_{ni}^2 \qquad (7) \\ &\le C\frac{1}{n^\alpha}\sum_{j=1}^{n} P(\psi(j-1) < |X| \le \psi(j))j \\ &\le C\frac{1}{n^\alpha}E[\phi(|X|)]. \end{aligned}$$

Now, let $\epsilon > 0$ be given. By putting $t = 2\log n/\epsilon$, we have that

$$\begin{aligned}
P(\sum_{i=1}^{n} a_{ni}X_i' > \epsilon) &\leq e^{-t\epsilon}E[e^{t\sum_{i=1}^{n} a_{ni}X_i'}] \\
&\leq e^{-t\epsilon}\exp\{\frac{1}{2}t^2 e^{tu_n}\sum_{i=1}^{n} a_{ni}^2 EX_i'^2\} \\
&\leq e^{-2\log n}\exp\{\frac{2(\log n)^2}{\epsilon^2} n^{2u_n/\epsilon}\sum_{i=1}^{n} a_{ni}^2 EX_i'^2\} \\
&\leq e^{-2\log n}\exp\{C(\log n)^2 n^{-\alpha+2u_n/\epsilon}\} \\
&\leq Cn^{-2}
\end{aligned}$$

for all sufficiently large n. Hence

$$\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}X_i' > \epsilon) < \infty.$$

By the Borel-Cantelli lemma, we have

$$\limsup_{n\to\infty}\sum_{i=1}^{n} a_{ni}X_i' \leq 0 \text{ a.s.}$$

By replacing X_i' by $-X_i'$ from the above statement, we obtain

$$\liminf_{n\to\infty}\sum_{i=1}^{n} a_{ni}X_i' \geq 0 \text{ a.s.}$$

Thus (6) is proved. □

The following theorem shows that if the variance of X exists, then the conditions of Theorem 1 can be replaced by more simple conditions. In particular, the additional condition (2) on ψ (and ϕ) is not necessary.

THEOREM 2. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0, Var(X) < \infty$, and $E[\phi(|X|)] < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants such that*

(i) $\max_{1\leq i\leq n}|a_{ni}| = O(\frac{1}{\psi(n)})$,

(ii) $\sum_{i=1}^{n} a_{ni}^2 = O(\frac{1}{n^\alpha})$ *for some* $\alpha > 0$.

Then $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s.

Proof. We first observe that

$$E(\sum_{i=1}^{n} a_{ni}X_i)^2 = EX^2 \sum_{i=1}^{n} a_{ni}^2 \to 0$$

as $n \to \infty$. It follows that $\sum_{i=1}^{n} a_{ni}X_i \to 0$ in probability. The rest of the proof is similar to that of Theorem 1 except that (7) is replaced by

$$\sum_{i=1}^{n} a_{ni}^2 EX_i'^2 \le EX^2 \sum_{i=1}^{n} a_{ni}^2 \le C\frac{1}{n^\alpha}.$$

□

COROLLARY 1. *Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables satisfying $EX = 0$ and $E|X|^p < \infty$ for some $p \ge 1$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants such that*

(i) $\max_{1 \le i \le n} |a_{ni}| = O(1/n^{1/p})$,

(ii) $\sum_{i=1}^{n} a_{ni}^2 = \begin{cases} O(1/n^{2/p-1+\alpha}) \text{ for some } \alpha > 0, & \text{if } 1 < p < 2, \\ O(1/n^\alpha) \text{ for some } \alpha > 0, & \text{if } p \ge 2. \end{cases}$

Then $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s.

Proof. When $p = 1$, the result is the content of Theorem 5 in Choi and Sung [2]. Let $\phi(x) = x^p (p > 1)$. Then $\phi^{-1}(x) = \psi(x) = x^{1/p}$. When $1 < p < 2$, ψ satisfies (2), and

$$\max_{1 \le j \le n} \frac{\psi^2(j)}{j} \sum_{i=j}^{n} a_{ni}^2 \le \max_{1 \le j \le n} \frac{\psi^2(j)}{j} \sum_{i=1}^{n} a_{ni}^2 \le C\frac{1}{n^\alpha}.$$

So the result follows by Theorem 1. When $p \ge 2$, the result follows by Theorem 2. □

REMARK 1. Li et al. [4] proved Corollary 1 when $p > 1$.

In some cases, it is not easy to check the condition (ii) of Corollary 1. To solve this problem, we need the following lemma.

LEMMA 2. *Let $p > 0$ and $0 < r < s$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying $\max_{1 \le i \le n} |a_{ni}| = O(1/n^{1/p})$. Then the following statements are equivalent.*

(i) $\sum_{i=1}^{n} |a_{ni}|^r = O(1/n^{r/p-1+\alpha})$ *for some* $\alpha > 0$,

(ii) $\sum_{i=1}^{n} |a_{ni}|^s = O(1/n^{s/p-1+\beta})$ *for some* $\beta > 0$.

Proof. The implication (i) $\Longrightarrow$ (ii) follows by

$$\sum_{i=1}^{n}|a_{ni}|^s \le \max_{1\le i\le n}|a_{ni}|^{s-r}\sum_{i=1}^{n}|a_{ni}|^r = O(\frac{1}{n^{s/p-1+\alpha}}).$$

To prove the converse, we take $t>0$ such that $\beta - t(s-r) > 0$. Define $A = \{1 \le i \le n : |a_{ni}| \le 1/n^{t+1/p}\}$ and $B = \{1, \cdots, n\} \setminus A$. Then we have by (ii) that

$$\begin{aligned}\sum_{i=1}^{n}|a_{ni}|^r &= \sum_{i\in A}|a_{ni}|^r + \sum_{i\in B}|a_{ni}|^r \\ &\le n(\frac{1}{n^{t+1/p}})^r + (n^{t+1/p})^{s-r}\sum_{i\in B}|a_{ni}|^s \\ &= O(\frac{1}{n^{r/p-1+\min\{tr,\beta-t(s-r)\}}}).\end{aligned}$$

Thus, the converse is proved. □

From Corollary 1 and Lemma 2, we can obtain the following theorem.

THEOREM 3. *Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $EX = 0$ and $E|X|^p < \infty$ for some $1 < p < 2$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants such that*

(i) $\max_{1\le i\le n}|a_{ni}| = O(1/n^{1/p})$,

(ii) $\sum_{i=1}^{n}|a_{ni}|^r = O(1/n^{r/p-1+\alpha})$ *for some $r>0$ and $\alpha>0$.*

Then $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s.

REMARK 2. When $1<p<2$, Corollary 1 follows by Theorem 3 with $r=2$.

The following corollary is due to Cuzick [3].

COROLLARY 2. *Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $EX = 0$ and $E|X|^p < \infty$ for some $p>1$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants such that*

$$\sum_{i=1}^{n}|a_{ni}|^q = O(\frac{1}{n^{q/p}}), \tag{8}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Then $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s.

Proof. By (8), $\max_{1\le i\le n}|a_{ni}|^q = O(1/n^{q/p})$, which implies that $\max_{1\le i\le n}|a_{ni}| = O(1/n^{1/p})$. So when $1 < p < 2$, the result follows by Theorem 3 with $r = q$ and $\alpha = 1$.

Now let $p \ge 2$. Since $q \le 2$, it follows that

$$\sum_{i=1}^{n} a_{ni}^2 \le (\sum_{i=1}^{n} |a_{ni}|^q)^{2/q} = O(\frac{1}{n^{2/p}}).$$

So the result follows by Corollary 1. □

The following corollary is due to Bai and Cheng [1]

COROLLARY 3. *Let $1 < p < 2$ and $\frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta}$ for $1 < \alpha, \beta < \infty$. Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $EX = 0$ and $E|X|^\beta < \infty$. Let $\{b_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants such that*

$$\sum_{i=1}^{n} |b_{ni}|^\alpha = O(n). \tag{9}$$

Then $\sum_{i=1}^n b_{ni}X_i/n^{1/p} \to 0$ a.s.

Proof. Define $a_{ni} = b_{ni}/n^{1/p}$ for $1 \le i \le n$ and $n \ge 1$. By (9), $\max_{1\le i\le n}|b_{ni}|^\alpha = O(n)$, which implies that

$$\max_{1\le i\le n}|a_{ni}| = O(\frac{n^{1/\alpha}}{n^{1/p}}) = O(\frac{1}{n^{1/\beta}}).$$

We also have that $\sum_{i=1}^n |a_{ni}|^\alpha = O(1/n^{\alpha/\beta})$. Thus, when $1 < \beta < 2$, the result follows by Theorem 3.

We now let $\beta \ge 2$. If $\alpha \le 2$, we have that

$$\sum_{i=1}^{n} a_{ni}^2 \le \frac{1}{n^{2/p}}(\sum_{i=1}^{n} |b_{ni}|^\alpha)^{2/\alpha} = O(\frac{n^{2/\alpha}}{n^{2/p}}) = O(\frac{1}{n^{2/\beta}}).$$

If $\alpha > 2$, we obtain that

$$\sum_{i=1}^{n} a_{ni}^2 = \frac{1}{n^{2/p}}\sum_{i=1}^{n} b_{ni}^2 \le \frac{1}{n^{2/p}}(n + \sum_{i=1}^{n}|b_{ni}|^\alpha) = O(\frac{1}{n^{2/p-1}}).$$

Thus, when $\beta \ge 2$, the result follows by Corollary 1. □

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